Mean curvature flow and geometric inequalities

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Abstract. We explore the link between the mean curvature flow and inequalities relating geometric quantities such as area, volume, diameter and Willmore energy. On one hand we observe how isoperimetric inequalities can be extracted from the theory of curve shortening on surfaces; in particular we give a new inequality which simultaneously generalises the inequalities of Alexandrov, Huber, Bol and others. On the other hand, we find geometric inequalities which must be satisfied for the mean curvature flow in higher dimensions to evolve smoothly before disappearing at a point; in particular we give a lower bound for the area of a dumbbell in terms of its length which must be satisfied for its neck not to pinch off.

1. A new isoperimetric inequality

In this article, we will both develop and apply the theory of mean curvature flow. The main application will be a new isoperimetric inequality on surfaces, and we begin by stating the inequality and discussing its consequences independently of curvature flow theory.

Let (D, g) be the closed 2-disc equipped with some metric g. Given a domain $\Omega \subset D$, we denote the area of Ω with respect to g by $A(\Omega)$. Similarly, the length of a curve γ, with respect to g will be written $L(\gamma)$. We use the shorthand $A = A(D)$ and $L = L(\partial D)$. The Gauss curvature with respect to g, denoted K, induces the rearrangement $K^* : (0, A) \to \mathbb{R}$ defined to be the unique decreasing function with the property that

$$\text{Area}\left(\{x \in D \mid K(x) \geqq s\}\right) = \left|\{y \in (0, A) \mid K^*(y) \geqq s\}\right|,$$

for all $s \in \mathbb{R}$. We also write $K_+(x) = \max\{K(x), 0\}$.

Theorem 1. *Suppose the surface (D, g) satisfies*

$$(1) \qquad\qquad \int_D K_+ < 2\pi,$$

and that the boundary ∂D is convex in the sense that the curvature vector of ∂D is inward pointing. Then there holds the isoperimetric inequality

$$(2) \qquad 4\pi A \leqq L^2 + 2 \int_0^A (A - x) K^*(x)\, dx .$$

A proof of this theorem will be offered in the next section, once we have introduced the notion of curve shortening on surfaces.

Equality is achieved in (2) for *any* rotationally symmetric g which satisfies $K(x) \geqq K(y)$ whenever $|x| \leqq |y|$, and of course for any other metric having this property after being pulled back by a diffeomorphism of D.

The theorem as stated is no longer true if we remove the hypothesis that the surface is simply connected (i.e. a disc). As counterexamples we propose a long but thin cylinder, and a flat torus with a small disc removed. Both of these have $K \equiv 0$ and small boundary length but with potentially large area. Of course we may derive an inequality for multiply connected subdomains of a simply connected domain by filling in any holes (which increases A and decreases L) before applying Theorem 1. However, the Gauss curvature term in (2) will necessarily have to be calculated over the whole outer domain.

The condition (1) and the boundary convexity hypothesis are used to enable a concise curve shortening proof; in fact, the result remains true without them[1]).

Our inequality generalises a variety of classical inequalities, as we shall now demonstrate. A discussion of these, together with comprehensive bibliographic details, may be found in the book of Bandle [3] and the article of Osserman [12].

Corollary 1. *With the hypotheses of Theorem* 1 *we have the following special cases*:

(i)   (Alexandrov)   *Given $K_0 \in \mathbb{R}$,*

$$4\pi A \leqq L^2 + K_0 A^2 + 2A \int_D (K - K_0)_+ .$$

(ii)   (Fiala-Huber)

$$4\pi A \leqq L^2 + 2A \int_D K_+ .$$

(iii)   (Bol)   *If $K \leqq K_0 \in \mathbb{R}$,*

$$4\pi A \leqq L^2 + K_0 A^2 .$$

(iv)   (Bernstein-Schmidt)   *If K is constant,*

$$4\pi A \leqq L^2 + K A^2 .$$

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[1]) In [18] we adapt the method of polyhedral approximations of Alexandrov, Burago, Zalgaller and others to give a complete proof without additional hypotheses. This result also provides information for non-simply connected surfaces. We also have a proof using potential theory, without these conditions but with the assumption that $K \geqq 0$.

Each of the special cases is sharp. In particular, equality is achieved in (iv) for geodesic discs on a round sphere, and (ii) is seen to be optimal by considering a long cylinder capped at one end by a hemisphere (and perturbed slightly to satisfy the hypotheses of Theorem 1). The latter example justifies the 'loss of a factor of two' in (ii) compared with (iv).

Proof of Corollary 1.   We could simply prove part (i) and then deduce the other parts as consequences. Instead, we broaden the discussion in order to indicate the behaviour of the K^* term of (2) in different special cases. To prove (iii), we simply observe that if $K \leqq K_0$ then $K^* \leqq K_0$, so that

$$2 \int_0^A (A - x) K^*(x) \, dx \leqq 2 \int_0^A (A - x) K_0 \, dx = K_0 A^2 \, .$$

Part (iv) follows as a special case, or directly by observing that $K^* = K$ if K is constant. To prove (i), we define

$$\Omega_+ = \{ x \in D \, | \, K(x) \geqq K_0 \} \, ,$$

and set $m = A(\Omega_+)$ so that $K^*(x) \geqq K_0$ for any $x < m$, but $K^*(x) \leqq K_0$ for any $x > m$. We may then decompose

$$2 \int_0^A (A - x) K^*(x) \, dx = 2 \int_0^m (A - x)(K^*(x) - K_0) \, dx + 2 \int_m^A (A - x)(K^*(x) - K_0) \, dx$$

$$+ 2 \int_0^A (A - x) K_0 \, dx \, .$$

Abandoning the negative second term, crudely estimating the first, and calculating the third, we see that

$$2 \int_0^A (A - x) K^*(x) \, dx \leqq 2 A \int_0^m (K^*(x) - K_0) \, dx + K_0 A^2 = K_0 A^2 + 2 A \int_{\Omega_+} (K - K_0)$$

$$= K_0 A^2 + 2 A \int_D (K - K_0)_+ \, ,$$

at which point Alexandrov's inequality follows from Theorem 1. Part (ii) is then just the special case $K_0 = 0$, or may be proved directly with a simplified calculation.   □

2. Curve shortening on surfaces and a proof of Theorem 1

Let Σ be a Riemannian surface. Given a smooth embedded closed curve γ_0 in Σ, we call a family γ_t ($0 < t < T$) of other such curves the 'curve shortening flow' of γ_0 until time T if there exists a smooth map $F : S^1 \times [0, T) \to \Sigma$ such that $F(\cdot, t) : S^1 \to \Sigma$ is an embedding representing γ_t for each $t \in [0, T)$, and F satisfies the equation

$$\frac{\partial}{\partial t} F(\theta, t) = -\kappa_t(F(\theta, t)) \, \nu_t(F(\theta, t)) \, ,$$

where κ_t is the curvature of γ_t corresponding to a choice v_t of normal vector, so that $-\kappa_t v_t$ gives the curvature vector.

Under the curve shortening flow, the length L_t of γ_t decreases monotonically according to

$$(3) \qquad\qquad \frac{dL_t}{dt} = -\int_{\gamma_t} \kappa_t^2 \, .$$

The theory of curve shortening was originally developed for curves in the plane ($\Sigma = \mathbb{R}^2$). In this case, any embedded closed curve has a corresponding flow which evolves smoothly before shrinking to a point at some time $t = T$; this was proved by Grayson [6] and Gage-Hamilton [5]. During its evolution, the area A_t enclosed by the curve decreases at constant rate

$$\frac{dA_t}{dt} = -\int_{\gamma_t} \kappa_t = -2\pi \, .$$

A standard tool in the development of curve shortening theory is the isoperimetric inequality. However, we observe that we may also often recover inequalities from the basic flow theory. (This has already been observed by Ben Andrews in the context of affine curvature flow and affine isoperimetric inequalities [1].) For example, given a closed curve γ_0 in $\mathbb{R}^2$ of length $L = L_0$ which encloses an area $A = A_0$, let us move the curve via the curve shortening flow until time $t = T$ when it disappears at a point. We may use the formulae for the evolution of A_t and L_t to calculate

$$-4\pi \frac{dA_t}{dt} = 4\pi \int_{\gamma_t} \kappa_t = 2\left(\int_{\gamma_t} \kappa_t\right)^2 \leqq 2L_t \int_{\gamma_t} \kappa_t^2 = -2L_t \frac{dL_t}{dt} = -\frac{dL_t^2}{dt} \, .$$

Integrating between times $t = 0$ and $t = T$, we recover the isoperimetric inequality for simply connected domains in the plane

$$(4) \qquad\qquad\qquad 4\pi A \leqq L^2 \, .$$

This example serves merely to emphasise a link – in fact, in [17] we have given a one line proof of (4) for *any* domain in the plane, using only Stokes' theorem. However, as we shall shortly see, the same ideas may be used to prove our new isoperimetric inequality (2).

Let us now drop the restriction that $\Sigma = \mathbb{R}^2$. Instead, we allow Σ to be any compact Riemannian surface, though we insist that any boundary is convex. Grayson has developed a theory of curve shortening for curves on such surfaces [8]. He proves that given an initial embedded closed curve, the curve shortening flow exists either for a finite time T when the curve shrinks to a point, or for infinite time, in which case the curvature κ_t converges to zero in C^∞. In the latter case, there exists a sequence of times $t_i \to \infty$ such that γ_{t_i} converges to a geodesic in C^∞. It is unknown whether this convergence is uniform in time, though for curve shortening in manifolds of dimension larger than two, we have an example of where it is not uniform [16]. We remark that the condition that the boundary is convex is used to prevent the evolving curve from escaping outside Σ.

In the case that Σ is simply connected, we again have the notion of the subdomain Ω_t enclosed by γ_t. As before, its area A_t, evolves according to

$$\frac{dA_t}{dt} = - \int_{\gamma_t} \kappa_t \,,$$

though it is no longer forced to decrease at constant rate – indeed it may increase in general.

We are now in a position to prove our main result.

Proof of Theorem 1.   If γ is a closed curve in D with interior Ω, the Gauss-Bonnet theorem states that

(5) $$2\pi - \int_{\Omega} K = \int_{\gamma} \kappa \,,$$

where κ is the curvature of γ. An immediate consequence is that the hypothesis (1) bars the existence of a closed geodesic in D.

Our strategy will be to let $\gamma_0 = \partial D$ flow by curvature, just as we did when recovering the isoperimetric inequality in the plane. Appealing to Grayson's theory as described above, either the curve shrinks to a point in a finite time T, or it subconverges to a closed geodesic at $t \to \infty$. Since we have just ruled out the existence of a closed geodesic in D, we are left only with the former option.

By using (5) and (1), we see that the area A_t of Ω_t decreases according to

$$\frac{dA_t}{dt} = - \int_{\gamma_t} \kappa_t = - \left(2\pi - \int_{\Omega_t} K \right) < 0 \,.$$

Since the length L_t of γ_t decreases according to (3), we may calculate

$$-2\frac{dA_t}{dt}\left(2\pi - \int_{\Omega_t} K \right) = 2 \left(\int_{\gamma_t} \kappa_t \right)^2 \leqq 2 L_t \int_{\gamma_t} \kappa_t^2 = -2 L_t \frac{dL_t}{dt} = - \frac{dL_t^2}{dt} \,.$$

We now return to the definition of K^* to find that

$$\int_{\Omega_t} K \leqq \int_0^{A_t} K^* \,.$$

Therefore we have

$$-4\pi \frac{dA_t}{dt} + \frac{dL_t^2}{dt} \leqq -2\frac{dA_t}{dt} \int_0^{A_t} K^* = -2 \frac{d}{dt} \int_0^{A_t} \int_0^s K^*(x)\,dx\,ds \,,$$

which we may integrate between times $t = 0$ and $t = T$ to give

$$4\pi A - L^2 \leqq 2 \int_0^A \int_0^s K^*(x)\,dx\,ds = 2 \int_0^A \int_0^A \chi_{x<s} K^*(x)\,dx\,ds \,,$$

where $\chi_{x<s}$ is defined to be 1 if $x < s$, but 0 otherwise. Evaluating the integral with respect to s yields the conclusion

$$4\pi A \leqq L^2 + 2 \int_0^A (A - x)\, K^*(x)\, dx\,. \quad \square$$

3. The mean curvature flow in higher dimensions

In the remaining part of this paper, the main theme will be singularity formation in the mean curvature flow – the higher dimensional analogue of the curve shortening flow. The central idea of comparing the evolution of different geometric quantities during the flow will persist. However our motivation will not be to obtain geometrically significant facts by using the flow; instead we will be deriving geometric conditions on a surface which must be satisfied in order for its flow to evolve smoothly before disappearing at a point. When we then produce surfaces which violate these conditions, we may deduce the existence of singularities in their flows. The advantages of our methods are that they are suitable for generalisation to, for example, the flow of hypersurfaces in curved ambient manifolds, and that we may easily get estimates on the time taken for singularities to develop.

Let us begin by introducing the mean curvature flow with traditional notation and appropriate normalisations. Further details may be found in [14].

Let $\mathscr{M}$ be a complete oriented m-dimensional manifold without boundary. Given a smooth embedding $\mathscr{M}_0$ of $\mathscr{M}$ in $\mathbb{R}^{m+1}$, a family $\mathscr{M}_t$ ($0 < t < T$) of other embeddings of $\mathscr{M}$ is said to be the 'mean curvature flow' of $\mathscr{M}_0$ if each $\mathscr{M}_t$ is represented by an embedding $F(\cdot, t): \mathscr{M} \to \mathbb{R}^{m+1}$ such that

$$\frac{\partial}{\partial t} F(x, t) = \boldsymbol{H}(x, t)\,,$$

on $\mathscr{M}_0 \times [0, T)$, where $\boldsymbol{H}(x, t)$ is the mean curvature vector of $\mathscr{M}_t$ at the point $F(x, t)$. We adopt the normalisation for the mean curvature vector which makes it the *unit* normal vector for the unit sphere (pointing inwards) so $|\boldsymbol{H}|$ is the magnitude of the *mean* of the principal curvatures $\{\kappa_i\}$.

The mean curvature flow decreases the area $\mathscr{A}(\mathscr{M}_t)$ of $\mathscr{M}_t$ according to

$$(6) \qquad\qquad \frac{d}{dt}\mathscr{A}(\mathscr{M}_t) = -m\mathscr{W}(\mathscr{M}_t)\,,$$

where

$$\mathscr{W}(\mathscr{M}) := \int_{\mathscr{M}} |\boldsymbol{H}|^2$$

is the 'Willmore energy.' We will need the fact that for any surface $\mathscr{M}$ in $\mathbb{R}^3$, there holds the estimate

$$(7) \qquad\qquad \mathscr{W}(\mathscr{M}) \geqq 4\pi\,,$$

with equality if and only if $\mathcal{M}$ is a round sphere. This follows from the pointwise identity $|\boldsymbol{H}|^2 - K = \frac{1}{4}(\kappa_1 - \kappa_2)^2$ (where K is the Gauss curvature) which we may integrate over the set $\mathcal{M}_+ = \{x \in \mathcal{M} \mid K(x) \geqq 0\}$ to obtain

$$\mathcal{W}(\mathcal{M}) \geqq \int_{\mathcal{M}_+} |\boldsymbol{H}|^2 \geqq \int_{\mathcal{M}_+} K \geqq 4\pi.$$

An alternative proof is a bi-product of Section 6.

A crucial property enjoyed by the mean curvature flow is the so-called 'comparison principle.' Loosely speaking this says that two nonintersecting hypersurfaces will remain nonintersecting when each is evolved simultaneously.

As we shall see, the comparison principle may be used to restrict the movement of an evolving hypersurface, by comparing it to a 'barrier' – a known solution of the mean curvature flow which is initially disjoint. Typically these known solutions are 'self-shrinking' solutions which evolve simply under homothety $\mathcal{M}_t = \sqrt{(1 - t/T)}\,\mathcal{M}_0$ for an appropriate origin. The simplest examples are the round spheres whose radii evolve according to

$$(8) \qquad\qquad r(t) = \sqrt{r_0^2 - 2t},$$

where r_0 is the initial radius. Note that the sphere exists precisely up to time $t = \dfrac{r_0^2}{2}$. However, there is another extremely useful example, of a 2-torus $\mathcal{T} \hookrightarrow \mathbb{R}^3$ discovered by Angenent [2] which evolves in the same self-shrinking manner.

In [10], Gerhard Huisken proved that convex hypersurfaces flow smoothly until shrinking to a point at a time $t = T$. In particular, the area of the hypersurface and the volume it encloses in the ambient $\mathbb{R}^{m+1}$ both converge to zero as $t \uparrow T$. In contrast, non-convex hypersurfaces may develop singularities before shrinking to a point. Intuitively, two round spheres connected by a thin neck (a 'dumbbell') should develop a singularity rapidly because the thin neck has a tendency to collapse much faster than the two spheres. A rigorous proof of this idea was first given by Grayson [7]. However, a more elegant proof resulted from Angenent's discovery of a self-shrinking torus, used in conjuction with the comparison principle. As described with pictures in [2], the self-shrinking torus may be used to collapse around the neck, whilst the bells of the dumbbell are prevented from collapsing by two self-shrinking spheres in the interior. We call this the 'noose' method.

Klaus Ecker has shown that a neck may be forced to collapse by comparison with a family of hyperboloids which acts as a subsolution for the mean curvature flow. The argument may be found in [4] together with a survey of Brakke's 'clearing out lemma' which will also prevent a thin neck from persisting and force a singularity when applied appropriately.

In the sequel, we will give three new methods to prove the existence of singularities in the flow of dumbbells. The first two, below, will use geometric inequalities and be particularly suitable for generalisations to situations where self-shrinking solutions are no longer known, or make no sense – for example to the situation of curved ambient manifolds.

The third method, which we call the 'guillotine' method, will use only the comparison principle, and only with explicitly defined barriers. This final method is detailed in Section 5.

4. The formation of singularities

The use of the mean curvature flow to prove isoperimetric inequalities in higher dimensions is severely limited by the possibility of singularity formation in flows. However, for any compact surface $\mathcal{M}_0$ in $\mathbb{R}^3$ with mean curvature flow $\mathcal{M}_t$ for $t \in [0, T)$ such that the volume $\mathcal{V}(\mathcal{M}_t)$ enclosed by $\mathcal{M}_t$ converges to zero as $t \uparrow T$, we may use (6) and (7) to calculate

$$(9) \qquad -\frac{d}{dt}\mathcal{V}(\mathcal{M}_t) = -\int_{\mathcal{M}_t} \boldsymbol{H}.v \leqq \int_{\mathcal{M}_t} |\boldsymbol{H}| \leqq \mathcal{A}(\mathcal{M}_t)^{\frac{1}{2}} \mathcal{W}(\mathcal{M}_t)^{\frac{1}{2}}$$

$$(10) \qquad \leqq \mathcal{A}(\mathcal{M}_t)^{\frac{1}{2}} \mathcal{W}(\mathcal{M}_t)^{\frac{1}{2}} \left(\frac{\mathcal{W}(\mathcal{M}_t)}{4\pi}\right)^{\frac{1}{2}} = -\frac{1}{2\sqrt{\pi}} \mathcal{A}(\mathcal{M}_t)^{\frac{1}{2}} \frac{1}{2} \frac{d}{dt} \mathcal{A}(\mathcal{M}_t)$$

$$(11) \qquad = -\frac{1}{6\sqrt{\pi}} \frac{d}{dt} \mathcal{A}(\mathcal{M}_t)^{\frac{3}{2}},$$

where v is the outward pointing unit normal. Integrating over the range $t = 0$ to $t = T$, and squaring, we recover the isoperimetric inequality

$$\mathcal{V}(\mathcal{M}_0)^2 \leqq \frac{1}{36\pi} \mathcal{A}(\mathcal{M}_0)^3 \,.$$

In particular, this argument works for any convex $\mathcal{M}_0$ by the result of Huisken mentioned in Section 3.

We may use the above argument to prove the existence of singularities in the flow of certain 2-tori, before the volume they enclose can shrink to zero. Let us define

$$\Theta := \inf_{T^2 \hookrightarrow \mathbb{R}^3} \mathcal{W}(T^2)$$

to be the greatest lower bound for the Willmore energy of each torus embedded in $\mathbb{R}^3$. By (7) we know that $\Theta \geqq 4\pi$. However, by appealing to the existence of a torus minimising the Willmore energy over all tori – a fact proved by Leon Simon in [13] – we may deduce that $\Theta > 4\pi$. (The Willmore conjecture states that $\Theta = 2\pi^2$.) Now let us carry out the above argument for a torus $\mathcal{M}_0$ which evolves under mean curvature flow such that $\mathcal{V}(\mathcal{M}_t) \to 0$ as $t \uparrow T$. Using the definition of Θ in place of (7), we see that

$$(12) \qquad \mathcal{V}(\mathcal{M}_0)^2 \leqq \frac{1}{9\Theta} \mathcal{A}(\mathcal{M}_0)^3 \,.$$

However, as $\Theta > 4\pi$, we can find a torus $\mathcal{M}_0$ which violates (12) – by taking the boundary of a ball with a thin hole drilled through it, for example. Such a torus *must* then develop a singularity under the mean curvature flow before the volume it encloses can shrink to zero.

If we assume that such a 'fat' torus develops its singularity by the small hole closing up, then we have an alternative barrier for the 'noose' method of Angenent mentioned in Section 3. By passing a thin neck of a sphere through the small hole in the torus, the neck would be strangled before the sphere could shrink to nothing, just as described in Section 3. In fact, we have a better theoretical reason for why a singularity must develop, based on the violation of a geometric inequality, as we now describe.

Theorem 2. *Suppose $\mathcal{M}_0$ is a 2-sphere embedded in $\mathbb{R}^3$ which encloses two round 2-spheres, both of radius r, with centres separated by a distance $d > 0$.*

Then a necessary condition for $\mathscr{A}(\mathcal{M}_t)$ to decrease to zero under the mean curvature flow of $\mathcal{M}_0$ is that

$$\text{(13)} \qquad \mathscr{A}(\mathcal{M}_0)^2 > \left[\frac{d^2 r^2}{2} + \frac{4 d r^3}{3} + r^4 \right] \pi^2$$

and in particular that

$$\text{(14)} \qquad \mathscr{A}(\mathcal{M}_0) > \frac{d r \pi}{\sqrt{2}}.$$

If (14) *fails to hold, then a singularity must develop in the flow before time*

$$\text{(15)} \qquad t = \frac{\mathscr{A}(\mathcal{M}_0)^2}{d^2 \pi^2}.$$

Corollary 2. *There exists an embedded two sphere $\mathcal{M}_0$ in $\mathbb{R}^3$ enclosing two unit spheres with centres separated by a distance* 10, *such that $\mathcal{M}_0$ must develop a singularity under the mean curvature flow before $\mathscr{A}(\mathcal{M}_t)$ can decrease to zero.*

The corollary follows by observing that if $r = 1$ then we can choose $\mathcal{M}_0$ to have area arbitrarily close to 8π, at which point the condition (13) is violated for $d = 10$.

Remark 1. We conjecture that we may strengthen the condition (13) by a factor of two. This will be discussed in Remark 2 below.

It is interesting to observe to what extent the necessary conditions of Theorem 2 sharpen the sufficient condition of Huisken [10] that $\mathcal{M}_0$ is convex. In particular, whilst $\mathscr{A}(\mathcal{M}_0) > \frac{r d \pi}{\sqrt{2}}$ is necessary (and we expect $\mathscr{A}(\mathcal{M}_0) > \sqrt{2} r d \pi$ to be necessary – see Remark 1) any *convex* sphere satisfying the hypotheses of Theorem 2 must clearly satisfy

$$\mathscr{A}(\mathcal{M}_0) > 2\pi r d + 4\pi r^2 \sim 2 r d \pi \quad \text{for large } d.$$

To prove Theorem 2 we will need an estimate of the external diameter

$$\text{diam}(\mathcal{M}) = \max_{x,\, y \in \mathcal{M}} |x - y|$$

of a surface in $\mathbb{R}^3$ in terms of its area and Willmore energy.

Lemma 1.   *For any compact connected orientable surface $\mathcal{M}$ immersed in $\mathbb{R}^3$, we have the estimate*

$$(16) \qquad\qquad \text{diam}(\mathcal{M}) < \frac{2}{\pi} \left(\mathcal{A}(\mathcal{M}) \mathcal{W}(\mathcal{M}) \right)^{\frac{1}{2}}.$$

The estimate was obtained by Leon Simon in [13] with some universal constant in place of the constant $\frac{2}{\pi}$. We will give a proof (following Simon's strategy) in Section 6.

Remark 2.   It seems likely that the constant $\frac{2}{\pi}$ in (16) could be halved. Certainly no further improvement is possible in general, as we may see by considering a long cylinder with the ends capped by hemispheres. This improvement would translate directly into an improvement of Theorem 2 as mentioned in Remark 1.

In fact, for any immersed surface $\mathcal{M}$, we conjecture that

$$(17) \qquad\qquad \text{diam}(\mathcal{M}) < \frac{1}{\pi} \int_{\mathcal{M}} |\boldsymbol{H}|,$$

to which we could apply the Cauchy-Schwartz inequality.

In the case that $\mathcal{M}$ is a surface of constant mean curvature immersed in $\mathbb{R}^3$, we established in [17] that

$$\text{diam}(\mathcal{M}) \leqq \frac{\mathcal{A}(\mathcal{M})|\boldsymbol{H}|}{2\pi},$$

which is twice as strong as (17) and *four* times as strong as (16). Equality is then achieved when $\mathcal{M}$ is a sphere.

Proof of Theorem 2.   We will use (16) as a lower bound on the Willmore energy rather than an upper bound on the diameter. Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be the spheres of radius r with centres separated by a distance d. Under the mean curvature flow, both spheres shrink in a self-similar manner, possessing radii $\sqrt{r^2 - 2t}$ at time t. Therefore, by comparing the flow of $\mathcal{M}_0$ with the flow of $\mathcal{S}_1$ and $\mathcal{S}_2$, we see that

$$\text{diam}(\mathcal{M}_t) \geqq d + 2\sqrt{r^2 - 2t}$$

for times $t \in \left[0, \dfrac{r^2}{2}\right)$. Applying Lemma 1 and (6) we then find that

$$d + 2\sqrt{r^2 - 2t} < \frac{2}{\pi} \left(\mathscr{A}(\mathscr{M}_t) \mathscr{W}(\mathscr{M}_t)\right)^{\frac{1}{2}} = \frac{2}{\pi} \left(-\mathscr{A}(\mathscr{M}_t) \frac{1}{2} \frac{d}{dt} \mathscr{A}(\mathscr{M}_t)\right)^{\frac{1}{2}}$$

$$= \frac{1}{\pi} \left(-\frac{d}{dt} (\mathscr{A}(\mathscr{M}_t))^2\right)^{\frac{1}{2}}.$$

Taking the square, and integrating between times $t = 0$ and $t = \dfrac{r^2}{2}$, we arrive at the estimate

$$\mathscr{A}(\mathscr{M}_0)^2 > \mathscr{A}(\mathscr{M}_{\frac{r^2}{2}})^2 + \pi^2 \int_0^{\frac{r^2}{2}} (d + 2\sqrt{r^2 - 2t})^2 \, dt \geqq \pi^2 \left[\frac{d^2 r^2}{2} + \frac{4 dr^3}{3} + r^4\right].$$

The time limit (15) is obtained by integrating as before and observing that in this time, the area must have decreased by more than its original value. This estimate may clearly be improved, but at the expense of simplicity. □

5. The guillotine

We now describe our final method to prove that a sphere evolving under mean curvature flow may develop a singularity as a neck pinches off. The idea is less simple than that of Angenent as described in Section 3, but more elementary in the sense that we do not have to appeal to an existence theorem for a self-shrinking torus.

Defining $f : \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right) \to \mathbb{R}$ by

$$f(x) = -\ln\left(\cos(x)\right),$$

we recall that the graph of f – the so-called 'grim reaper' – evolves under curvature flow in the plane by vertical translation at unit speed.

Let us consider the graph surfaces $\mathscr{M}_0$ and $\mathscr{N}_0$ in $\mathbb{R}^3$ defined by

$$\mathscr{M}_0 = \left\{(x, y, f(x)) \mid x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \, y \in \mathbb{R}\right\},$$

and

$$\mathscr{N}_0 = \left\{\left(x, y, 2f\left(\frac{y}{2}\right) + \varepsilon\right) \mid x \in \mathbb{R}, \, y \in (-\pi, \pi)\right\},$$

where $\varepsilon \in \left(0, \dfrac{1}{16}\right)$ is some small constant. These are troughs, of different sizes, which lie perpendicular to each other. Under mean curvature flow, $\mathscr{M}_0$ and $\mathscr{N}_0$ evolve 'upwards' (i.e. in the direction of the third axis) at speeds 1 and $\dfrac{1}{2}$ respectively. Note that for times

$t < 2\varepsilon$, we may join the points $p_1 = (0, 10, 10)$ and $p_2 = (0, -10, 10)$ by a path lying entirely above $\mathcal{M}_t$ and only below $\mathcal{N}_t$, but for times immediately after $t = 2\varepsilon$, this is impossible. Note also that the spheres $\mathcal{S}_1$ and $\mathcal{S}_2$ of radius $\frac{1}{2}$ centred at p_1 and p_2 respectively, intersect neither $\mathcal{M}_0$ nor $\mathcal{N}_0$.

It is now not hard to see that any embedded sphere in $\mathbb{R}^3$ which lies entirely above $\mathcal{M}_0$, only below $\mathcal{N}_0$, and which encloses both $\mathcal{S}_1$ and $\mathcal{S}_2$, must have a neck pinched off within time $t = 2\varepsilon$ when evolved under the mean curvature flow. (Such a sphere must already have a thin neck to slip between $\mathcal{M}_0$ and $\mathcal{N}_0$.) Indeed, by comparing the evolving surface with the self-shrinking spheres starting at $\mathcal{S}_1$ and $\mathcal{S}_2$ (which survive until time $t = \frac{1}{8}$) we see that it continues to enclose p_1 and p_2, at least up to time $t = \frac{1}{8}$. However, by comparing the evolving surface with $\mathcal{M}_t$ and $\mathcal{N}_t$ between which it is sandwiched, we see that it must be guillotined before the time $t = 2\varepsilon < \frac{1}{8}$.

We remark that many sandwiching techniques using the maximum principle were developed in [15] to provide new examples of singularity formation in the harmonic map flow.

6. Diameter, area and Willmore energy

In this section we give a proof of Lemma 1. We follow the strategy of Leon Simon given in [13], with modifications reflecting our desire for an explicit constant.

Proof.   Let us fix, for the moment, $y \in \mathcal{M} \hookrightarrow \mathbb{R}^3$ and $0 < \sigma < \varrho$. Writing $X(x) = x - y$ and $|X|_\sigma = \max(|X|, \sigma)$ we define a Lipschitz vector field on $\mathbb{R}^3$ by

$$\Phi(x) = (|X|_\sigma^{-2} - \varrho^{-2})_+ X.$$

To Φ we will apply the first variation identity

$$\int_{\mathcal{M}} \mathrm{div}_{\mathcal{M}} \Phi = -2 \int_{\mathcal{M}} \Phi . \boldsymbol{H},$$

where for any orthonormal basis $\{e_1, e_2\}$ of $T_x \mathcal{M}$ we define

$$\mathrm{div}_{\mathcal{M}} \Phi(x) = \sum_{i=1}^{2} e_i . \nabla\left(\Phi(x) . e_i\right).$$

A short calculation reveals that

$$(18) \qquad \mathrm{div}_{\mathcal{M}} \Phi = \begin{cases} 2(\sigma^{-2} - \varrho^{-2}) & \text{if } |X| < \sigma, \\ 2\left(\dfrac{|X^{\perp}|^2}{|X|^4} - \varrho^{-2}\right) & \text{if } \sigma < |X| < \varrho, \\ 0 & \text{if } \varrho < |X|, \end{cases}$$

where $X^\perp = (X.v)v$, and v is a choice of normal vector. Hence we find the identity

$$(19) \quad \sigma^{-2}\mathscr{A}(\mathscr{M}_\sigma) = \varrho^{-2}\mathscr{A}(\mathscr{M}_\varrho) - \int_{\mathscr{M}_{\sigma,\varrho}} \frac{|X^\perp|^2}{|X|^4} - \int_{\mathscr{M}_\varrho} (|X|_\sigma^{-2} - \varrho^{-2})X.\boldsymbol{H},$$

where $\mathscr{M}_\sigma = \mathscr{M} \cap \mathbb{B}_\sigma(y)$ and $\mathscr{M}_{\sigma,\varrho} = \mathscr{M} \cap \big(\mathbb{B}_\varrho(y)\backslash\mathbb{B}_\sigma(y)\big)$. We now claim the pointwise estimate

$$(20) \qquad -\frac{|X^\perp|^2}{|X|^4} - (|X|_\sigma^{-2} - \varrho^{-2})X.\boldsymbol{H} \leqq \frac{1}{4}|\boldsymbol{H}|^2$$

for $x \in \mathscr{M}_{\sigma,\varrho}$. If $X.\boldsymbol{H} \geqq 0$ then the left-hand side of (20) is nonpositive, so the claim is trivial. If, on the other hand, $X.\boldsymbol{H} < 0$, then we may estimate

$$-\frac{|X^\perp|^2}{|X|^4} - (|X|_\sigma^{-2} - \varrho^{-2})X.\boldsymbol{H} = -\left|\frac{\boldsymbol{H}}{2} + \frac{X^\perp}{|X|^2}\right|^2 + \frac{1}{4}|\boldsymbol{H}|^2 + \varrho^{-2}X.\boldsymbol{H} \leqq \frac{1}{4}|\boldsymbol{H}|^2.$$

Adding the integral of (20) over $\mathscr{M}_{\sigma,\varrho}$ to (19) we find that

$$(21) \qquad \sigma^{-2}\mathscr{A}(\mathscr{M}_\sigma) \leqq \varrho^{-2}\mathscr{A}(\mathscr{M}_\varrho) + \int_{\mathscr{M}_{\sigma,\varrho}} \frac{1}{4}|\boldsymbol{H}|^2 - \int_{\mathscr{M}_\sigma} (\sigma^{-2} - \varrho^{-2})X.\boldsymbol{H}.$$

Since we chose $y \in \mathscr{M}$, the left-hand side of (21) has a nonzero limit as $\sigma \to 0$. In fact, the limit is equal to $k\pi$, where $k \geqq 1$ is the number of times y is covered by the immersion. Moreover, $|X.\boldsymbol{H}| = \mathcal{O}(|X|^2)$ as $|X| \to 0$, so by taking the limit $\sigma \to 0$ we find the key fact that

$$(22) \qquad k\pi \leqq \varrho^{-2}\mathscr{A}(\mathscr{M}_\varrho) + \frac{1}{4}\mathscr{W}(\mathscr{M}_\varrho).$$

We observe that (22) is sharp in the sense that we may not improve either coefficient on the right-hand side. Indeed, in the limit $\varrho \to 0$ we have equality, which pins the first coefficient, whilst the limit $\varrho \to \infty$ implies

$$(23) \qquad k\pi \leqq \frac{1}{4}\mathscr{W}(\mathscr{M})$$

in which we have equality when $\mathscr{M}$ is a round sphere. The inequality (23) was first obtained by Li and Yau in their work on conformal volume [11]. In particular it implies that the Willmore energy of any immersed surface with a self-intersection must be at least 8π.

We now proceed to use (22) to estimate the diameter of $\mathscr{M}$ (cf. [13]). Let us fix $\varrho = 2\left(\dfrac{\mathscr{A}(\mathscr{M})}{\mathscr{W}(\mathscr{M})}\right)^{\frac{1}{2}}$, and choose $N \geqq 0$ to be the integer such that

$$(24) \qquad \frac{\text{diam}(\mathscr{M})}{2\varrho} - 1 < N \leqq \frac{\text{diam}(\mathscr{M})}{2\varrho}.$$

Then let us fix $y_0 \in \mathcal{M}$ at an extremity of $\mathcal{M}$ – in other words so that

$$\mathrm{diam}(\mathcal{M}) = \max_{x \in \mathcal{M}} |x - y_0|$$

– and choose, for each j satisfying $1 \leqq j \leqq N$ (if $N \geqq 1$) a point $y_j \in \mathcal{M}$ such that $|y_j - y_0| = 2j\varrho$.

Now we apply (22) for each $y \in \{y_0, \ldots, y_N\}$ and take the sum. By virtue of the facts that the subsets $\mathbb{B}_\varrho(y_j)$ are disjoint, and that $k \geqq 1$, we obtain

$$(N+1)\pi \leqq \varrho^{-2} \mathcal{A}(\mathcal{M}) + \frac{1}{4}\mathcal{W}(\mathcal{M}) = \frac{1}{2}\mathcal{W}(\mathcal{M}).$$

Combining with the first inequality of (24) we conclude

$$\mathrm{diam}(\mathcal{M}) < 2\varrho(N+1) \leqq 4\left(\frac{\mathcal{A}(\mathcal{M})}{\mathcal{W}(\mathcal{M})}\right)^{\frac{1}{2}} \frac{1}{2\pi}\mathcal{W}(\mathcal{M}) = \frac{2}{\pi}\left(\mathcal{A}(\mathcal{M})\mathcal{W}(\mathcal{M})\right)^{\frac{1}{2}}.$$

We remark that although we do not insist that $N > 0$, we may in fact assume that $\mathrm{diam}(\mathcal{M}) \geqq 4\varrho$ and hence that $N \geqq 2$. This is because otherwise the lemma follows automatically from the observation

$$\mathrm{diam}(\mathcal{M}) < 4\varrho = 8\left(\frac{\mathcal{A}(\mathcal{M})}{\mathcal{W}(\mathcal{M})}\right)^{\frac{1}{2}} \leqq 8\left(\frac{\mathcal{A}(\mathcal{M})}{\mathcal{W}(\mathcal{M})}\right)^{\frac{1}{2}} \frac{\mathcal{W}(\mathcal{M})}{4\pi} = \frac{2}{\pi}\left(\mathcal{A}(\mathcal{M})\mathcal{W}(\mathcal{M})\right)^{\frac{1}{2}}. \quad \square$$

7. Acknowledgements

8. Addendum

Since the completion and acceptance of this work, Frank Morgan has informed us that a similar result to our Theorem 1 has been obtained independently by Morgan-Hutchings-Howards and is to be found in [9].

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