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Local monotonicity and mean value formulas for evolving Riemannian manifolds

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Abstract. We derive identities for general flows of Riemannian metrics that may be regarded as local mean-value, monotonicity, or Lyapunov formulae. These generalize previous work of the first author for mean curvature flow and other nonlinear diffusions. Our results apply in particular to Ricci flow, where they yield a local monotone quantity directly analogous to Perelman's reduced volume $\tilde{V}$ and a local identity related to Perelman's average energy $\mathscr{F}$.

1. Introduction

To motivate the local formulas we derive in this paper, consider the following simple but quite general strategy for finding monotone quantities in geometric flows, whose core idea is simply integration by parts. Let $\big(\mathscr{M}^n, g(t)\big)$ be a smooth one-parameter family of complete Riemannian manifolds evolving for $t \in [a, b]$ by

$$\frac{\partial}{\partial t} g = 2h. \tag{1.1}$$

Observe that the formal conjugate of the time-dependent heat operator $\dfrac{\partial}{\partial t} - \Delta$ on the evolving manifold $\big(\mathscr{M}^n, g(t)\big)$ is $-\left(\dfrac{\partial}{\partial t} + \Delta + \operatorname{tr}_g h\right)$. If $\varphi, \psi : \mathscr{M}^n \times [a, b] \to \mathbb{R}$ are smooth functions for which the divergence theorem is valid (e.g. if $\mathscr{M}^n$ is compact or if φ and ψ and their derivatives decay rapidly enough at infinity), one has[1)]

$$\frac{d}{dt} \int_{\mathscr{M}^n} \varphi\psi \, d\mu = \int_{\mathscr{M}^n} \left\{ \psi \left[\left(\frac{\partial}{\partial t} - \Delta \right) \varphi \right] + \varphi \left[\left(\frac{\partial}{\partial t} + \Delta + \operatorname{tr}_g h \right) \psi \right] \right\} d\mu. \tag{1.2}$$

[1)] Here and throughout this paper, $d\mu$ denotes the volume form associated to $g(t)$.

If φ solves the heat equation and ψ solves the adjoint heat equation, it follows that the integral $\int_{\mathscr{M}^n} \varphi\psi \, d\mu$ is independent of time. More generally, if $\psi\left[\left(\frac{\partial}{\partial t} - \Delta\right)\varphi\right]$ and $\varphi\left[\left(\frac{\partial}{\partial t} + \Delta + \mathrm{tr}_g h\right)\psi\right]$ both have the same sign, then $\int_{\mathscr{M}^n} \varphi\psi \, d\mu$ will be monotone in t. If the product $\varphi\psi$ is geometrically meaningful, this can yield useful results. Here are but a few examples.

Example 1. The simplest example uses the heat equation on Euclidean space. Let

$$\psi(x,t) = \frac{1}{[4\pi(s-t)]^{n/2}} e^{-\frac{|y-x|^2}{4(s-t)}} \quad (x \in \mathbb{R}^n, t < s) \tag{1.3}$$

denote the backward heat kernel with singularity at $(y,s) \in \mathbb{R}^n \times \mathbb{R}$. If φ solves the heat equation and neither it nor its derivatives grow too fast at infinity, then

$$\varphi(y,s) = \lim_{t \nearrow s} \int_{\mathbb{R}^n} \varphi(x,t)\psi(x,t) \, dx.$$

Because $\frac{d}{dt} \int_{\mathbb{R}^n} \varphi(x,t)\psi(x,t) \, dx = 0$, one has $\varphi(y,s) = \int_{\mathbb{R}^n} \varphi(x,t)\psi(x,t) \, dx$ for all $y \in \mathbb{R}^n$ and $t < s$, which illustrates the averaging property of the heat operator.

Example 2. Let $F_t : \mathscr{M}^n \hookrightarrow \mathscr{M}_t^n \subset \mathbb{R}^{n+1}$ be a one-parameter family of hypersurfaces evolving by mean curvature flow, $\frac{\partial}{\partial t} F_t = -Hv$, where H is the mean curvature and v the outward unit normal of the hypersurface $\mathscr{M}_t^n$. This corresponds to $h = -HA$ in (1.1), where A is the second fundamental form. Define ψ by formula (1.3) applied to $x \in \mathbb{R}^{n+1}$ and $t < s$. Using $\mathrm{tr}_g h = -H^2$, one calculates that

$$\left(\frac{\partial}{\partial t} + \Delta - H^2\right)\psi = -\left|\frac{(x-y)^\perp}{2(s-t)} - Hv\right|^2 \psi.$$

Hence by (1.2),

$$\frac{d}{dt} \int_{\mathscr{M}_t^n} \varphi\psi \, d\mu = \int_{\mathscr{M}_t^n} \left[\left(\frac{\partial}{\partial t} - \Delta\right)\varphi\right]\psi \, d\mu - \int_{\mathscr{M}_t^n} \left|\frac{(x-y)^\perp}{2(s-t)} - Hv\right|^2 \varphi\psi \, d\mu.$$

This is established for $\varphi \equiv 1$ by Huisken [20], Theorem 3.1, and generalized by Huisken and the first author [8], §1, to any smooth φ for which the integrals are finite and integration by parts is permissible.

Hence $\int_{\mathscr{M}_t^n} \psi \, d\mu$ is monotone nonincreasing in time and is constant precisely on homothetically shrinking solutions. The monotonicity implies that the density

$$\Theta_{\mathscr{O}}^{\mathrm{MCF}} := \lim_{t \nearrow 0} \int_{\mathscr{M}_t^n} \psi \, d\mu$$

of the limit point $\mathcal{O} = (0,0)$ is well defined. Another consequence is that $\sup_{\mathcal{M}_b^n} \varphi \leqq \sup_{\mathcal{M}_a^n} \varphi$ if $\left(\frac{\partial}{\partial t} - \Delta\right)\varphi \leqq 0$ for $t \in [a,b]$.

Example 3. A compact Riemannian manifold $\big(\mathcal{M}^n, g(t)\big)$ evolving by Ricci flow corresponds to $h = -\mathrm{Rc}$ in (1.1), so that $\mathrm{tr}_g h = -R$. If[2)]

$$\varphi \equiv 1 \quad \text{and} \quad \psi = [\tau(2\Delta f - |\nabla f|^2 + R) + f - n](4\pi\tau)^{-n/2} e^{-f},$$

then Perelman's entropy may be written as $\mathcal{W}\big(g(t), f(t), \tau(t)\big) = \int_{\mathcal{M}^n} \varphi\psi \, d\mu$. If $d\tau/dt = -1$ and $\left(\frac{\partial}{\partial t} + \Delta\right) f = |\nabla f|^2 - R - \frac{n}{2\tau}$, then

$$\left(\frac{\partial}{\partial t} + \Delta - R\right)\psi = 2\tau \left|\mathrm{Rc} + \nabla\nabla f - \frac{1}{2\tau} g\right|^2 (4\pi\tau)^{-n/2} e^{-f}.$$

In this case, (1.2) becomes

$$\frac{d}{dt}\mathcal{W}\big(g(t), f(t), \tau(t)\big) = \int_{\mathcal{M}^n} 2\tau \left|\mathrm{Rc} + \nabla\nabla f - \frac{1}{2\tau} g\right|^2 (4\pi\tau)^{-n/2} e^{-f} \, d\mu,$$

which is formula (3.4) of [28]. In particular, $\mathcal{W}$ is monotone increasing and is constant precisely on compact shrinking gradient solitons.

Example 4. Again for $\big(\mathcal{M}^n, g(t)\big)$ evolving smoothly by Ricci flow for $t \in [a,b]$, let ℓ denote Perelman's reduced distance [28] from an origin (y,b). Take $\varphi \equiv 1$ and choose $\psi \equiv v$ to be the reduced-volume density[3)]

$$v(x,t) = \frac{1}{[4\pi(b-t)]^{n/2}} e^{-\ell(x,b-t)} \quad (x \in \mathcal{M}^n, t < b).$$

Then Perelman's reduced volume is given by $\tilde{V}(t) = \int_{\mathcal{M}^n} \varphi\psi \, d\mu$. By [28], §7, $\left(\frac{\partial}{\partial t} + \Delta - R\right) v \geqq 0$ holds in the barrier sense, hence in the distributional sense.[4)] Thus one obtains monotonicity of the reduced volume if $\mathcal{M}^n$ is compact or if its Ricci curvature is bounded.

More generally, one gets monotonicity of $\int_{\mathcal{M}^n} \varphi\psi \, d\mu$ for any nonnegative supersolution φ of the heat equation. In particular, taking $\varphi(x,t) = R(x,t) - R_{\min}(0)$ on a

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[2)] Throughout this paper, ∇ represents the *spatial* covariant derivative, and $\Delta = \mathrm{tr}_g \nabla\nabla$.

[3)] The formula used here and throughout this paper differs from Perelman's by the constant factor $(4\pi)^{-n/2}$. This normalization is more convenient for our applications.

[4)] It is a standard fact that a suitable barrier inequality implies a distributional inequality. See [4] for relevant definitions and a proof. A direct proof for v is found in [38], Lemma 1.12.

compact manifold and noting that $\left(\frac{\partial}{\partial t} - \Delta\right)\varphi \geqq 0$ holds pointwise, one verifies that $\int_{\mathscr{M}^n} [R - R_{\min}(0)]v \, d\mu$ is nondecreasing in time.

In [12], Feldman, Ilmanen, and the third author introduce an expanding entropy and a forward reduced volume for compact manifolds evolving by Ricci flow. Monotonicity of these quantities may also be derived from (1.2) with $\varphi \equiv 1$.

Similar ideas play important roles in Perelman's proofs of differential Harnack estimates [28], §9, and pseudolocality [28], §10.

The strategy of integration by parts can be adapted to yield *local* monotone quantities for geometric flows. We shall present a rigorous derivation in Section 2 when we prove our main result, Theorem 7. Before doing so, however, we will explain the underlying motivations by a purely formal argument. Suppose for the purposes of this argument that $\Omega = \bigcup_{a \leq t \leq b} \Omega_t$ is a smooth, precompact subset of $\mathscr{M}^n \times [a, b]$. Assume that $\partial\Omega_t$ is smooth with outward unit normal v, and let $d\sigma$ denote the measure on $\partial\Omega_t$ induced by $g(t)$. If the product $\varphi\psi$ vanishes on $\partial\Omega$, then

$$\int_\Omega \left\{ \psi \left[\left(\frac{\partial}{\partial t} - \Delta \right) \varphi \right] + \varphi \left[\left(\frac{\partial}{\partial t} + \Delta + \operatorname{tr}_g h \right) \psi \right] \right\} d\mu \, dt = \int_a^b \left(\frac{d}{dt} \int_{\Omega_t} \varphi\psi \, d\mu \right) dt + \int_{\partial\Omega} (\varphi \langle \nabla\psi, v \rangle - \psi \langle \nabla\varphi, v \rangle) \, d\sigma \, dt. \tag{1.4}$$

This formula may be regarded as a space-time analog of Green's second identity. In the special case that Ω is the super-level set $\{(x, t) : \psi(x, t) > 0\}$ and both Ω_a and Ω_b are empty, then $v = -|\nabla\psi|^{-1}\nabla\psi$, whence (1.4) reduces to

$$\int_{\{\psi > 0\}} \left\{ \psi \left[\left(\frac{\partial}{\partial t} - \Delta \right) \varphi \right] + \varphi \left[\left(\frac{\partial}{\partial t} + \Delta + \operatorname{tr}_g h \right) \psi \right] \right\} d\mu \, dt + \int_{\{\psi = 0\}} \varphi |\nabla\psi| \, d\sigma \, dt = 0. \tag{1.5}$$

Formula (1.5) enables a strategy for the construction of local monotone quantities.

Here is the strategy, again presented as a purely formal argument. Let φ and $\Psi > 0$ be given. Define $\psi = \log \Psi$, and for $r > 0$, let $\psi_{(r)} = \log(r^n \Psi)$. Notice that $\nabla\psi_{(r)} = \nabla\psi$ for all $r > 0$. Take Ω to be the set E_r defined for $r > 0$ by

$$E_r := \{(x, t) : \Psi(x, t) > r^{-n}\} = \{(x, t) : \psi_{(r)} > 0\}. \tag{1.6}$$

(When Ψ is a fundamental solution[5)] of a backward heat equation, the set E_r is often called a 'heatball'.) Assume for the sake of this formal argument that the outward unit normal to the time slice $E_r(t) := E_r \cap (\mathscr{M}^n \times \{t\})$ is $v = -|\nabla\psi|^{-1}\nabla\psi$. Observe that

[5)] See Section 5 below.

$$\left(\frac{\partial}{\partial t}+\Delta\right)\psi=\Psi^{-1}\left(\frac{\partial}{\partial t}+\Delta+\mathrm{tr}_g\, h\right)\Psi-|\nabla\psi|^2-\mathrm{tr}_g\, h. \tag{1.7}$$

Applying the coarea formula to each time slice $E_r(t)$, followed by an integration in t, one obtains

$$\frac{d}{dr}\int_{E_r}|\nabla\psi|^2\varphi\,d\mu\,dt=\frac{n}{r}\int_{\partial E_r}|\nabla\psi|\varphi\,d\sigma\,dt. \tag{1.8}$$

Similarly, one has

$$\frac{d}{dr}\int_{E_r}(\mathrm{tr}_g\, h)\psi_{(r)}\varphi\,d\mu\,dt=\int_{E_r}(\mathrm{tr}_g\, h)\left[\frac{\partial}{\partial r}\log(r^n\Psi)\right]\varphi\,d\mu\,dt \tag{1.9a}$$

$$=\frac{n}{r}\int_{E_r}(\mathrm{tr}_g\, h)\varphi\,d\mu\,dt, \tag{1.9b}$$

because the boundary integral vanishes in this case. Now by rearranging (1.5) and using (1.7)–(1.9), one gets

$$\begin{aligned}&\int_{E_r}\left\{\psi_{(r)}\left[\left(\frac{\partial}{\partial t}-\Delta\right)\varphi\right]+\Psi^{-1}\left[\left(\frac{\partial}{\partial t}+\Delta+\mathrm{tr}_g\, h\right)\Psi\right]\varphi\right\}d\mu\,dt\\&\quad=\int_{E_r}\{|\nabla\psi|^2-(\mathrm{tr}_g\, h)\psi_{(r)}\}\varphi\,d\mu\,dt-\int_{\partial E_r}\varphi|\nabla\psi|\,d\sigma\,dt+\int_{E_r}(\mathrm{tr}_g\, h)\varphi\,d\mu\,dt\\&\quad=\int_{E_r}\{|\nabla\psi|^2-(\mathrm{tr}_g\, h)\psi_{(r)}\}\varphi\,d\mu\,dt-\frac{r}{n}\frac{d}{dr}\int_{E_r}\{|\nabla\psi|^2-(\mathrm{tr}_g\, h)\psi_{(r)}\}\varphi\,d\mu\,dt.\end{aligned}$$

Defining

$$P_{\varphi,\Psi}(r):=\int_{E_r}\{|\nabla\log\Psi|^2-(\mathrm{tr}_g\, h)\log(r^n\Psi)\}\varphi\,d\mu\,dt \tag{1.10}$$

and applying an integrating factor, one obtains the following formal identity. Since $\log(r^n\Psi)=\psi_{(r)}>0$ in E_r, this identity produces a local monotone quantity whenever $\left(\frac{\partial}{\partial t}-\Delta\right)\varphi$ and $\varphi\left(\frac{\partial}{\partial t}+\Delta+\mathrm{tr}_g\, h\right)\Psi$ have the same sign.

Proto-theorem. *Whenever the steps above can be rigorously justified and all integrals in sight make sense, the identity*

$$\begin{aligned}&\frac{d}{dr}\left(\frac{P_{\varphi,\Psi}(r)}{r^n}\right)\\&\quad=-\frac{n}{r^{n+1}}\int_{E_r}\left\{\log(r^n\Psi)\left(\frac{\partial}{\partial t}-\Delta\right)\varphi+\Psi^{-1}\left[\left(\frac{\partial}{\partial t}+\Delta+\mathrm{tr}_g\, h\right)\Psi\right]\varphi\right\}d\mu\,dt\end{aligned} \tag{1.11}$$

will hold in an appropriate sense.

In spirit, (1.11) is a parabolic analogue of the formula

$$\frac{d}{dr}\left(\frac{1}{r^n}\int_{|x-y|<r}\varphi(x)\,d\mu\right)=\frac{1}{2r^{n+1}}\int_{|x-y|<r}(r^2-|x-y|^2)\Delta\varphi\,d\mu,$$

which for harmonic φ (i.e. $\Delta\varphi=0$) leads to the classical local mean-value representation formulae

$$\varphi(y)=\frac{1}{\omega_n r^n}\int_{|x-y|<r}\varphi(x)\,d\mu=\frac{1}{n\omega_n r^{n-1}}\int_{|x-y|=r}\varphi(x)\,d\sigma.$$

The main result of this paper, Theorem 7, is a rigorous version of the motivational proto-theorem above. We establish Theorem 7 in a sufficiently robust framework to provide new proofs of some classical mean-value formulae (Examples 5–8), to generate several new results (Corollaries 13, 15, 18, 19, 23, 26, 27) and to permit generalizations for future applications. Our immediate original results are organized as follows: in Section 4, we study Perelman's reduced volume for manifolds evolving by Ricci flow; in Section 5, we discuss heat kernels on evolving Riemannian manifolds (including fixed manifolds as an interesting special case); and in Section 6, we consider Perelman's average energy for manifolds evolving by Ricci flow. In [27], the third author applies some of these results to obtain local regularity theorems for Ricci flow. Potential future generalizations that we have in mind concern varifold (Brakke) solutions of mean curvature flow, solutions of Ricci flow with surgery, and fundamental solutions in the context of 'weak' (Bakry-Émery) Ricci curvature, e.g. [23].

As noted above, Theorem 7 allows new proofs of several previously known local monotonicity formulae, all of which should be compared with (1.11). To wit:

Example 5. Consider the Euclidean metric on $\mathscr{M}^n=\mathbb{R}^n$ with $h=0$. If Ψ is the backwards heat kernel (1.3) centered at (y,s) and the heatball $E_r\equiv E_r(y,s)$ is defined by (1.6), then (1.10) becomes

$$P_{\varphi,\Psi}(r)=\int_{E_r(y,s)}\varphi(x,t)\frac{|y-x|^2}{4(s-t)^2}d\mu\,dt.$$

Thus (1.11) reduces to

$$\frac{d}{dr}\left(\frac{P_{\varphi,\Psi}(r)}{r^n}\right)=-\frac{n}{r^{n+1}}\int_{E_r(y,s)}\log(r^n\Psi)\left(\frac{\partial}{\partial t}-\Delta\right)\varphi\,d\mu\,dt. \tag{1.12}$$

Since $\int_{E_r(y,s)}\frac{|y-x|^2}{4(s-t)^2}d\mu\,dt=1$, this implies the mean value identity

$$\varphi(y,s)=\frac{1}{r^n}\int_{E_r(y,s)}\varphi(x,t)\frac{|y-x|^2}{4(s-t)^2}d\mu\,dt \tag{1.13}$$

for all φ satisfying $\left(\frac{\partial}{\partial t}-\Delta\right)\varphi=0$. This localizes Example 1.

To our knowledge, Pini [29], [30], [31] was the first to prove (1.13) in the case $n = 1$. This was later generalized to $n > 1$ by Watson [35]. The general formula (1.12) appears in Evans-Gariepy [9]. There are many similar mean-value representation formulae for more general parabolic operators. For example, see Fabes-Garofalo [10] and Garofalo-Lanconelli [14]. (Also see Corollaries 23 and 26, below.)

Example 6. Surface integrals over heatballs first appear in the work of Fulks [13], who proves that a continuous function φ on $\mathbb{R}^n \times (a, b)$ satisfies

$$\varphi(y,s) = \frac{1}{r^n} \int\limits_{\partial E_r} \varphi(x,t) \frac{|y-x|^2}{\sqrt{4|y-x|^2(s-t)^2 + [|y-x|^2 - 2n(s-t)]^2}} d\sigma$$

for all sufficiently small $r > 0$ if and only if φ is a solution of the heat equation. (Compare with Corollary 25 below.)

Example 7. Previous results of the first author [5] localize Example 2 for mean curvature flow. On $\mathbb{R}^{n+1} \times (-\infty, 0)$, define $\Psi(x,t) := (-4\pi t)^{-n/2} e^{|x|^2/4t}$. Substitute

$$\Psi^{-1}\left(\frac{\partial}{\partial t} + \Delta + \operatorname{tr}_g h\right)\Psi = -|\nabla^{\perp}\psi + Hv|^2$$

and $\operatorname{tr}_g h = -H^2$ into (1.10) and (1.11). If the space-time track $\mathscr{M} = \bigcup\limits_{t<0} \mathscr{M}_t^n$ of a solution to mean curvature flow is well defined in the cylinder $B(0, \sqrt{2n\bar{r}^2/\pi}) \times (-\bar{r}^2/4\pi, 0)$, then [5] proves that formula (1.11), with the integrals taken over $E_r \cap \mathscr{M}$, holds in the distributional sense for any $r \in (0, \bar{r})$ and any φ for which all integral expressions are finite. In particular, $P_{1,\Psi}(r)/r^n$ is monotone increasing in r. The density $\Theta_{\mathscr{O}}^{\mathrm{MCF}} := \lim\limits_{t \nearrow 0} \int\limits_{\mathscr{M}_t^n} \Psi(x,t)\, d\mu$ of the limit point $\mathscr{O} = (0,0)$ can thus be calculated locally by

$$\Theta_{\mathscr{O}}^{\mathrm{MCF}} = \lim_{r \searrow 0} \frac{P_{1,\Psi}(r)}{r^n}.$$

(Compare with Corollary 18 below.) Related work of the first author for other nonlinear diffusions is found in [7].

Example 8. Perelman's scaled entropy $\mathscr{W}$ and the forward reduced volume θ_+ are localized by the third author [25], Propositions 5.2, 5.3, 5.4. Although only stated there for Kähler-Ricci flow, these localizations remain valid for Ricci flow in general. They are motivated by the first author's work on mean curvature flow [6] and arise from (1.2) by taking φ to be a suitable cutoff function defined with respect to $\tau\ell$ and $\tau\ell_+$, respectively.

The remainder of this paper is organized as follows. In Section 2, we rigorously derive Theorem 7: the general local monotonicity formula motivated by formula (1.11) above. In Section 3, we derive a local gradient estimate for solutions of the conjugate heat equation. In Sections 4–5, we apply this machinery to obtain new results in some special cases where our assumptions can be checked and in which (1.11) simplifies and becomes more familiar. The Appendix (Section 7) reviews some relevant properties of Perelman's reduced distance and volume.

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2. The rigorous derivation

Let $-\infty < a < b < \infty$, and let $\left(\mathscr{M}^n, g(t)\right)$ be a smooth one-parameter family of complete Riemannian manifolds evolving by (1.1) for $t \in [a,b]$. As noted above, the formal conjugate of the heat operator $\frac{\partial}{\partial t} - \Delta$ on $\left(\mathscr{M}^n, g(t)\right)$ is $-\left(\frac{\partial}{\partial t} + \Delta + \operatorname{tr}_g h\right)$. For $\alpha \in \mathbb{R}$, we adopt the standard notation $[\alpha]_+ := \max\{\alpha, 0\}$.

Let Ψ be a given positive function on $\mathscr{M}^n \times [a,b)$. As in Section 1, it is convenient to work with

$$\psi := \log \Psi \tag{2.1}$$

and the function defined for each $r > 0$ by

$$\psi_{(r)} := \psi + n \log r. \tag{2.2}$$

For $r > 0$, we define the space-time super-level set ('heatball')

$$E_r = \{(x,t) \in \mathscr{M}^n \times [a,b) : \Psi > r^{-n}\} \tag{2.3a}$$

$$= \{(x,t) \in \mathscr{M}^n \times [a,b) : \psi_{(r)} > 0\}. \tag{2.3b}$$

We would like to allow Ψ to blow up as we approach time $t = b$; in particular, we have in mind various functions which have a singularity that agrees asymptotically with a (backwards) heat kernel centered at some point in $\mathscr{M}^n$ at time $t = b$. (See Sections 6–5.) In this context, we make, for the moment, the following three assumptions about Ψ.

Assumption 1. The function Ψ is locally Lipschitz on $\mathscr{M}^n \times [a,s]$ for any $s \in (a,b)$.

Assumption 2. There exists a compact subset $\Omega \subseteq \mathscr{M}^n$ such that Ψ is bounded outside $\Omega \times [a,b)$.

Assumption 3. There exists $\bar{r} > 0$ such that

$$\lim_{s \nearrow b} \left(\int_{E_{\bar{r}} \cap (\mathscr{M}^n \times \{s\})} |\psi| \, d\mu \right) = 0$$

and

$$\int_{E_{\bar{r}}} |\nabla\psi|^2 \, d\mu \, dt < \infty.$$

Remark 4. By the continuity of Ψ from Assumption 1 and its boundedness from Assumption 2, we can be sure, after reducing $\bar{r} > 0$ if necessary, that $\Psi \leqq \bar{r}^{-n}$ outside some compact subset of $\mathscr{M}^n \times (a, b]$. In particular, we then know that the super-level sets E_r lie inside this compact subset for $r \in (0, \bar{r}]$.

Remark 5. By Assumption 3 and compactness of $E_{\bar{r}}$, one has $\int_{E_{\bar{r}}} |\psi| \, d\mu \, dt < \infty$.

Remark 6. We make no direct assumptions about the regularity of the sets E_r themselves.

Let φ be an arbitrary smooth function on $\mathscr{M}^n \times (a, b]$. By Assumption 3, the quantity

$$P_{\varphi,\Psi}(r) := \int_{E_r} [|\nabla\psi|^2 - \psi_{(r)}(\mathrm{tr}_g h)]\varphi \, d\mu \, dt \tag{2.4}$$

is finite for $r \in (0, \bar{r}]$. Our main result is as follows:

Theorem 7. *Suppose that $\left(\mathscr{M}^n, g(t)\right)$ is a smooth one-parameter family of complete Riemannian manifolds evolving by (1.1) for $t \in [a, b]$, that $\Psi : \mathscr{M}^n \times [a, b) \to (0, \infty)$ satisfies Assumptions 1–3, that $\bar{r} > 0$ is chosen according to Assumption 3 and Remark 4, and that $0 < r_0 < r_1 \leqq \bar{r}$.*

If Ψ is smooth and the function

$$\frac{\left(\frac{\partial}{\partial t} + \Delta + \mathrm{tr}_g h\right)\Psi}{\Psi} \equiv \frac{\partial\psi}{\partial t} + \Delta\psi + |\nabla\psi|^2 + \mathrm{tr}_g h$$

belongs to $L^1(E_{\bar{r}})$, then

$$\begin{aligned} \frac{P_{\varphi,\Psi}(r_1)}{r_1^n} - \frac{P_{\varphi,\Psi}(r_0)}{r_0^n} &= \int_{r_0}^{r_1} \frac{n}{r^{n+1}} \int_{E_r} \left[-\left(\frac{\partial\psi}{\partial t} + \Delta\psi + |\nabla\psi|^2 + \mathrm{tr}_g h\right)\varphi \right. \\ &\qquad \left. - (\psi + n\log r)\left(\frac{\partial\varphi}{\partial t} - \Delta\varphi\right) \right] d\mu \, dt \, dr. \end{aligned} \tag{2.5}$$

If, instead, Ψ is merely locally Lipschitz in the sense of Assumption 1, and the inequality

$$\frac{\left(\frac{\partial}{\partial t} + \Delta + \mathrm{tr}_g h\right)\Psi}{\Psi} \geqq 0$$

holds in the distributional sense, and $\varphi \geqq 0$, *then*

$$\frac{P_{\varphi,\Psi}(r_1)}{r_1^n} - \frac{P_{\varphi,\Psi}(r_0)}{r_0^n} \leqq -\int_{r_0}^{r_1} \frac{n}{r^{n+1}} \int_{E_r} (\psi + n\log r)\left(\frac{\partial \varphi}{\partial t} - \Delta\varphi\right) d\mu\, dt\, dr. \tag{2.6}$$

Remark 8. If φ solves the heat equation and Ψ solves the conjugate heat equation, then (2.5) implies that $P_{\varphi,\Psi}(r)/r^n$ is independent of r. See Example 5 (above) and Corollary 23 (below).

Proof of Theorem 7. We begin by assuming that Ψ is smooth. In the proof, we write $P(\cdot) \equiv P_{\varphi,\Psi}(\cdot)$. For most of the proof, we will work with a modified function, namely

$$P(r,s) := \int_{E_r \cap (\mathscr{M}^n \times [a,s])} [|\nabla\psi|^2 - \psi_{(r)}(\mathrm{tr}_g\, h)]\varphi\, d\mu\, dt, \tag{2.7}$$

arising from restriction to the time interval $[a,s]$, for some $s \in (a,b)$. As a result, we will only be working on domains on which Ψ and its derivatives are bounded, and the convergence of integrals will not be in doubt. A limit $s \nearrow b$ will be taken at the end.

Let $\zeta : \mathbb{R} \to [0,1]$ be a smooth function with the properties that $\zeta(y) = 0$ for $y \leqq 0$ and $\zeta'(y) \geqq 0$. Let $Z : \mathbb{R} \to [0,\infty)$ denote the primitive of ζ defined by $Z(y) = \int_{-\infty}^{y} \zeta(x)\, dx$. One should keep in mind that ζ can be made very close to the Heaviside function, in which case $Z(y)$ will lie a little below $[y]_+$.

For $r \in (0, \bar{r}]$ and $s \in (a,b)$, we define

$$Q(r,s) := \int_{\mathscr{M}^n \times [a,s]} [|\nabla\psi|^2 \zeta(\psi_{(r)}) - Z(\psi_{(r)})(\mathrm{tr}_g\, h)]\varphi\, d\mu\, dt, \tag{2.8}$$

which should be regarded as a perturbation of $P(r,s)$, and will relieve us of some technical problems arising from the fact that we have no control on the regularity of E_r. Note that $\zeta(\psi_{(r)})$ and $Z(\psi_{(r)})$ have support in E_r. Therefore, the convergence of the integrals is guaranteed.

In the following computations, we suppress the dependence of Q on s and assume that each integral is over the space-time region $\mathscr{M}^n \times [a,s]$ unless otherwise stated. One has

$$\begin{aligned} \frac{r^{n+1}}{n}\frac{d}{dr}\left[\frac{Q(r)}{r^n}\right] &= \frac{r}{n}Q'(r) - Q(r) \\ &= \int [|\nabla\psi|^2 \zeta'(\psi_{(r)}) - Z'(\psi_{(r)})(\mathrm{tr}_g\, h)]\varphi\, d\mu\, dt - Q(r) \\ &= \int [|\nabla\psi|^2 \zeta'(\psi_{(r)})]\varphi\, d\mu\, dt - \int [\zeta(\psi_{(r)})(\mathrm{tr}_g\, h)]\varphi\, d\mu\, dt \\ &\quad - \int [|\nabla\psi|^2 \zeta(\psi_{(r)})]\varphi\, d\mu\, dt + \int [Z(\psi_{(r)})(\mathrm{tr}_g\, h)]\varphi\, d\mu\, dt. \end{aligned} \tag{2.9}$$

The first integral and the last integral in the last equality on the right-hand side require further attention.

For the first of these, we keep in mind that $\nabla\psi = \nabla\psi_{(r)}$ and compute

$$
\begin{aligned}
(2.10)\qquad &\int[|\nabla\psi|^2\zeta'(\psi_{(r)})]\varphi\,d\mu\,dt \\
&= \int\langle\nabla\psi,\nabla\big(\zeta(\psi_{(r)})\big)\rangle\varphi\,d\mu\,dt \\
&= -\int[(\Delta\psi)\zeta(\psi_{(r)})\varphi+\langle\nabla\psi_{(r)},\nabla\varphi\rangle\zeta(\psi_{(r)})]\,d\mu\,dt \\
&= \int[-(\Delta\psi)\zeta(\psi_{(r)})\varphi+(\Delta\varphi)\psi_{(r)}\zeta(\psi_{(r)})+\langle\nabla\psi,\nabla\varphi\rangle\psi_{(r)}\zeta'(\psi_{(r)})]\,d\mu\,dt,
\end{aligned}
$$

the calculation being valid on each time slice.

For the fourth integral, we compute that at each time $t\in(a,b)$, one has

$$
\frac{d}{dt}\int\limits_{\mathscr{M}^n\times\{t\}}Z(\psi_{(r)})\varphi\,d\mu=\int\limits_{\mathscr{M}^n\times\{t\}}\left[Z'(\psi_{(r)})\frac{\partial\psi}{\partial t}\varphi+Z(\psi_{(r)})\frac{\partial\varphi}{\partial t}+Z(\psi_{(r)})\varphi(\mathrm{tr}_g\,h)\right]d\mu,
$$

the final term coming from differentiation of the volume form. Integrating over the time interval $[a,s]$ and using the facts that $Z'=\zeta$ and that $Z(\psi_{(r)})=0$ at $t=a$ (which holds because $\psi_{(r)}\leqq 0$ at $t=a$ by Remark 4), we find that

$$
\begin{aligned}
(2.11)\qquad \int Z(\psi_{(r)})(\mathrm{tr}_g\,h)\varphi\,d\mu\,dt &= -\int\left[\zeta(\psi_{(r)})\frac{\partial\psi}{\partial t}\varphi+Z(\psi_{(r)})\frac{\partial\varphi}{\partial t}\right]d\mu\,dt \\
&\quad+\int\limits_{\mathscr{M}^n\times\{s\}}Z(\psi_{(r)})\varphi\,d\mu,
\end{aligned}
$$

where the integrals are still over $\mathscr{M}^n\times[a,s]$ unless otherwise indicated.

We now combine (2.9) with (2.10) and (2.11) to obtain

$$
\begin{aligned}
(2.12)\qquad \frac{r^{n+1}}{n}\frac{d}{dr}\left[\frac{Q(r)}{r^n}\right] &= -\int\left(\frac{\partial\psi}{\partial t}+\Delta\psi+|\nabla\psi|^2+\mathrm{tr}_g\,h\right)\zeta(\psi_{(r)})\varphi\,d\mu\,dt \\
&\quad+\int(\Delta\varphi)\psi_{(r)}\zeta(\psi_{(r)})\,d\mu\,dt-\int\frac{\partial\varphi}{\partial t}Z(\psi_{(r)})\,d\mu\,dt \\
&\quad+\int\langle\nabla\psi,\nabla\varphi\rangle\psi_{(r)}\zeta'(\psi_{(r)})\,d\mu\,dt+\int\limits_{\mathscr{M}^n\times\{s\}}Z(\psi_{(r)})\varphi\,d\mu.
\end{aligned}
$$

The entire identity may now be multiplied by n/r^{n+1} and integrated with respect to r between r_0 and r_1, where $0<r_0<r_1\leqq\bar{r}$, to get an identity for the quantity $Q(r_1,s)/r_1^n-Q(r_0,s)/r_0^n$.

We may simplify the resulting expression by picking an appropriate sequence of valid functions ζ and passing to the limit. Precisely, we pick a smooth $\zeta_1:\mathbb{R}\to[0,1]$ with the properties that $\zeta_1(y)=0$ for $y\leqq 1/2$, $\zeta_1(y)=1$ for $y\geqq 1$, and $\zeta_1'(y)\geqq 0$. Then we define a sequence $\zeta_k:\mathbb{R}\to[0,1]$ by $\zeta_k(y)=\zeta_1(2^{k-1}y)$. As k increases, this sequence increases pointwise to the characteristic function of $(0,\infty)$. The corresponding Z_k converge uniformly to the function $y\mapsto[y]_+$. Crucially, we also can make use of the facts that $\zeta_k(\psi_{(r)})$ converges to the characteristic function of E_r in $L^1(\mathscr{M}^n\times[a,b])$ and that $\psi_{(r)}\zeta_k'(\psi_{(r)})$ is a bounded sequence of functions on $\mathscr{M}^n\times[a,b)$ with disjoint supports for each k. Indeed, the support of ζ_k' lies within the interval $(2^{-k},2^{1-k})$.

For each $r\in(0,\bar{r}]$, we have $Q(r,s)\to P(r,s)$ as $k\to\infty$. Using the dominated convergence theorem, our expression becomes

$$
\begin{aligned}
(2.13)\qquad \frac{P(r_1,s)}{r_1^n}-\frac{P(r_0,s)}{r_0^n}
&= -\int_{r_0}^{r_1}\frac{n}{r^{n+1}}\int_{E_r\cap(\mathscr{M}^n\times[a,s])}\left(\frac{\partial\psi}{\partial t}+\Delta\psi+|\nabla\psi|^2+\operatorname{tr}_g h\right)\varphi\,d\mu\,dt\,dr\\
&\quad-\int_{r_0}^{r_1}\frac{n}{r^{n+1}}\int_{E_r\cap(\mathscr{M}^n\times[a,s])}\psi_{(r)}\left(\frac{\partial\varphi}{\partial t}-\Delta\varphi\right)d\mu\,dt\,dr\\
&\quad+\int_{r_0}^{r_1}\frac{n}{r^{n+1}}\int_{\mathscr{M}^n\times\{s\}}[\psi_{(r)}]_+\varphi\,d\mu\,dr.
\end{aligned}
$$

Now we may take the limit as $s\nearrow b$. By Assumption 3, the final term converges to zero, and we end up with (2.5) as desired.

Next we turn to the case that ψ is merely Lipschitz, in the sense of Assumption 1. Given $s\in(a,b)$ and functions ζ and Z as above, there exists a sequence of smooth functions ψ_j on $\mathscr{M}^n\times[a,b]$ such that $\psi_j\to\psi$ in both $W^{1,2}$ and C^0 on the set $E_{\bar r}\cap(\mathscr{M}^n\times[a,s])$. By hypothesis on our Lipschitz ψ, we have

$$
I:=\int_{\mathscr{M}^n\times[a,s]}-\left(\frac{\partial\psi}{\partial t}+\Delta\psi+|\nabla\psi|^2+\operatorname{tr}_g h\right)\zeta(\psi_{(r)})\varphi\,d\mu\,dt\leqq 0,
$$

where we make sense of the Laplacian term via integration by parts, namely

$$
\int-(\Delta\psi)\zeta(\psi_{(r)})\varphi\,d\mu\,dt:=\int[\langle\nabla\psi,\nabla\varphi\rangle\zeta(\psi_{(r)})+|\nabla\psi|^2\zeta'(\psi_{(r)})\varphi]\,d\mu\,dt.
$$

By definition of ψ_j, we have

$$
\lim_{j\to\infty}\int_{\mathscr{M}^n\times[a,s]}-\left(\frac{\partial\psi_j}{\partial t}+\Delta\psi_j+|\nabla\psi_j|^2+\operatorname{tr}_g h\right)\zeta\big((\psi_j)_{(r)}\big)\varphi\,d\mu\,dt=I\leqq 0,
$$

uniformly in $r\in(0,\bar r]$. Consequently, we may carry out the same calculations that we did in the first part of the proof to obtain an inequality for the quantity $Q(r_1,s)/r_1^n-Q(r_0,s)/r_0^n$, with ψ_j in place of ψ. We then pass to the limit as $j\to\infty$ to obtain the inequality

$$
\begin{aligned}
(2.14)\qquad \frac{Q(r_1,s)}{r_1^n}-\frac{Q(r_0,s)}{r_0^n}
&\leqq \int_{r_0}^{r_1}\frac{n}{r^{n+1}}\int_{\mathscr{M}^n\times[a,s]}(\Delta\varphi)\psi_{(r)}\zeta(\psi_{(r)})\,d\mu\,dt\,dr\\
&\quad-\int_{r_0}^{r_1}\frac{n}{r^{n+1}}\int_{\mathscr{M}^n\times[a,s]}\frac{\partial\varphi}{\partial t}Z(\psi_{(r)})\,d\mu\,dt\,dr\\
&\quad+\int_{r_0}^{r_1}\frac{n}{r^{n+1}}\int_{\mathscr{M}^n\times[a,s]}\langle\nabla\psi,\nabla\varphi\rangle\psi_{(r)}\zeta'(\psi_{(r)})\,d\mu\,dt\,dr\\
&\quad+\int_{r_0}^{r_1}\frac{n}{r^{n+1}}\int_{\mathscr{M}^n\times\{s\}}Z(\psi_{(r)})\varphi\,d\mu\,dr
\end{aligned}
$$

for our Lipschitz ψ. Finally, we replace ζ with the same sequence of cut-off functions ζ_k that we used before (thus approximating the Heaviside function), take the limit as $k \to \infty$, and then take the limit as $s \nearrow b$. This gives the inequality (2.6). $\square$

The argument above may be compared to proofs of earlier results, especially the proof [5] of the local monotonicity formula for mean curvature flow.

There is an alternative formula for (2.4) that we find useful in the sequel:

Lemma 9. *Suppose that* $\big(\mathscr{M}^n, g(t)\big)$ *is a smooth one-parameter family of complete Riemannian manifolds evolving by* (1.1) *for* $t \in [a,b]$, *that* $\Psi : \mathscr{M}^n \times [a,b) \to (0,\infty)$ *satisfies Assumptions* 1–3, *that* $\bar{r} > 0$ *is determined by Assumption* 3 *and Remark* 4, *and that* $0 < r_0 < r_1 \leqq \bar{r}$.

If $\varphi \equiv 1$ *and* $\dfrac{\partial \psi}{\partial t} + |\nabla \psi|^2 \in L^1(E_{\bar{r}})$, *then for all* $r \in (0, \bar{r}]$, *one has*

$$P_{\varphi,\Psi}(r) = \int\limits_{E_r} \left(\frac{\partial \psi}{\partial t} + |\nabla \psi|^2 \right) d\mu\, dt.$$

Proof. In the case that $\varphi \equiv 1$, substituting formula (2.11) into formula (2.8) yields

$$Q(r,s) = \int\limits_{\mathscr{M}^n \times [a,s]} \left(\frac{\partial \psi}{\partial t} + |\nabla \psi|^2 \right) \zeta(\psi_{(r)})\, d\mu\, dt - \int\limits_{\mathscr{M}^n \times \{s\}} Z(\psi_{(r)})\, d\mu.$$

Although (2.11) was derived assuming smoothness of ψ, one can verify that it holds for locally Lipschitz Ψ satisfying Assumption 1 by approximating ψ by a sequence of smooth ψ_j (as in the proof of Theorem 7) and then passing to the limit as $j \to \infty$. Then if $\dfrac{\partial \psi}{\partial t} + |\nabla \psi|^2 \in L^1(E_{\bar{r}})$, one may (again as in the proof of Theorem 7) choose a sequence ζ_k along which $Q(r,s) \to P(r,s)$ as $k \to \infty$ and then let $s \nearrow b$ to obtain the stated formula. $\square$

3. A local gradient estimate

In order to apply Theorem 7 to a fundamental solution of the heat equation of an evolving manifold in Section 5, we need a local gradient estimate. One approach would be to adapt existing theory of local heat kernel asymptotics. Instead, we prove a more general result which may be of independent interest. Compare [17], [22], [24], the recent [33], [34], and [36], [37].

Let $\big(\mathscr{M}^n, g(t)\big)$ be a smooth one-parameter family of complete Riemannian manifolds evolving by (1.1) for $t \in [0, \bar{t}]$. We shall abuse notation by writing $g(\tau)$ to mean $g\big(\tau(t)\big)$, where

$$\tau(t) := \bar{t} - t.$$

In the remainder of this section, we state our results solely in terms of τ. In particular, $g(\tau)$ satisfies $\frac{\partial}{\partial\tau}g = -2h$ on $\mathscr{M}^n \times [0, \bar{t}]$.

Given $\bar{x} \in \mathscr{M}^n$ and $\rho > 0$, define

$$\Omega(\rho) := \bigcup_{0 \leqq \tau \leqq \bar{t}} \left(B_{g(\tau)}(\bar{x}, \rho) \times \{\tau\}\right) \subseteqq \mathscr{M}^n \times [0, \bar{t}]. \tag{3.1}$$

We now prove a local *a priori* estimate for bounded positive solutions of the conjugate heat equation

$$\left(\frac{\partial}{\partial\tau} - \Delta - \mathrm{tr}_g\, h\right) v = 0. \tag{3.2}$$

We will apply this in Section 5.

Theorem 10. *Let $\left(\mathscr{M}^n, g(\tau)\right)$ be a smooth one-parameter family of complete Riemannian manifolds evolving by $\frac{\partial}{\partial\tau}g = -2h$ for $0 \leqq \tau \leqq \bar{t}$. Assume there exist $k_1, k_2, k_3 \geqq 0$ such that*

$$h \leqq k_1 g, \quad \mathrm{Rc} \geqq -k_2 g, \quad \textit{and} \quad |\nabla(\mathrm{tr}_g\, h)| \leqq k_3$$

in the space-time region $\Omega(2\rho)$ given by (3.1). *Assume further that $v(\tau)$ solves* (3.2) *and satisfies $0 < v \leqq A$ in $\Omega(2\rho)$.*

Then there exist a constant C_1 depending only on n and an absolute constant C_2 such that at all $(x, \tau) \in \Omega(\rho)$, one has

$$\frac{|\nabla v|^2}{v^2} \leqq \left(1 + \log\frac{A}{v}\right)^2 \left[\frac{1}{\tau} + C_1 k_1 + 2k_2 + k_3 + \sqrt{k_3} + \frac{C_1\sqrt{k_2}\rho\coth(\sqrt{k_2}\rho) + C_2}{\rho^2}\right].$$

Proof. By scaling, we may assume that $A = 1$. We define[6)]

$$f := \log v \quad \text{and} \quad w := |\nabla \log(1 - f)|^2,$$

computing that

$$\left(\frac{\partial}{\partial\tau} - \Delta\right) f = |\nabla f|^2 + (\mathrm{tr}_g\, h).$$

Then using Bochner-Weitzenböck, we calculate that

$$\left(\frac{\partial}{\partial\tau} - \Delta\right)|\nabla f|^2 = 2h(\nabla f, \nabla f) - 2\,\mathrm{Rc}(\nabla f, \nabla f) - 2|\nabla\nabla f|^2 + 2\langle\nabla(\mathrm{tr}_g\, h + |\nabla f|^2), \nabla f\rangle$$

[6)] Note that w is used by Souplet-Zhang [33], Theorem 1.1, in generalizing Hamilton's result [7]. A similar function is employed by Yau [36]. Also see related work of the third author [26].

and

$$\left(\frac{\partial}{\partial\tau}-\Delta\right)w=\frac{1}{(1-f)^2}[2h(\nabla f,\nabla f)-2\,\mathrm{Rc}(\nabla f,\nabla f)+2\langle\nabla(\mathrm{tr}_g h),\nabla f\rangle]$$
$$-\frac{2}{(1-f)^2}\left[|\nabla\nabla f|^2+\frac{\langle\nabla|\nabla f|^2,\nabla f\rangle}{1-f}+\frac{|\nabla f|^4}{(1-f)^2}\right]$$
$$-4\frac{|\nabla f|^4}{(1-f)^4}+2\frac{(\mathrm{tr}_g h)|\nabla f|^2+|\nabla f|^4}{(1-f)^3}-2f\frac{\langle\nabla|\nabla f|^2,\nabla f\rangle}{(1-f)^3}.$$

By rewriting the last term above as

$$-2f\frac{\langle\nabla|\nabla f|^2,\nabla f\rangle}{(1-f)^3}=-2\frac{f}{1-f}\langle\nabla w,\nabla f\rangle+4\frac{|\nabla f|^4}{(1-f)^4}-4\frac{|\nabla f|^4}{(1-f)^3}$$

and cancelling terms, we obtain

$$\left(\frac{\partial}{\partial\tau}-\Delta\right)w=\frac{1}{(1-f)^2}[2h(\nabla f,\nabla f)-2\,\mathrm{Rc}(\nabla f,\nabla f)+2\langle\nabla(\mathrm{tr}_g h),\nabla f\rangle]$$
$$-\frac{2}{(1-f)^2}\left[|\nabla\nabla f|^2+\frac{\langle\nabla|\nabla f|^2,\nabla f\rangle}{1-f}+\frac{|\nabla f|^4}{(1-f)^2}\right]$$
$$+2\frac{(\mathrm{tr}_g h)|\nabla f|^2-|\nabla f|^4}{(1-f)^3}+2\frac{-f}{1-f}\langle\nabla w,\nabla f\rangle.$$

Now let $\eta(s)$ be a smooth nonnegative cutoff function such that $\eta(s)=1$ when $s\leqq 1$ and $\eta(s)=0$ when $s\geqq 2$, with $\eta'\leqq 0$, $|\eta'|\leqq C_2$, $(\eta')^2\leqq C_2\eta$, and $\eta''\geqq -C_2$. Define

$$u(x,\tau):=\eta\left(\frac{d_{g(\tau)}(\bar{x},x)}{\rho}\right).$$

Observe that at each fixed τ, u is smooth in space off of the $g(\tau)$ cut locus of $\bar{x}$. However, for our purposes of applying the maximum principle, Calabi's trick allows us to proceed as though u were smooth everywhere. Thus, we calculate that

$$\frac{|\nabla u|^2}{u}\leqq\frac{C_2}{\rho^2}$$

and

$$\frac{\partial u}{\partial\tau}\leqq C_2k_1$$

and

$$-\Delta u\leqq\frac{C_1\sqrt{k_2}\rho\coth(\sqrt{k_2}\rho)+C_2}{\rho^2}.$$

Now let $G := uw$ and compute that

$$\left(\frac{\partial}{\partial\tau}-\Delta\right)(\tau G) = G + \tau u\left[\left(\frac{\partial}{\partial\tau}-\Delta\right)w\right] + \tau w\left[\left(\frac{\partial}{\partial\tau}-\Delta\right)u\right] - 2\tau\langle\nabla u,\nabla w\rangle.$$

For any $\tau_1 > 0$, consider τG on $\mathscr{M}^n \times [0,\tau_1]$. At any point (x_0,τ_0) where τG attains its maximum on $\mathscr{M}^n \times [0,\tau_1]$, we have $0 \leqq \left(\frac{\partial}{\partial\tau}-\Delta\right)(\tau G)$ and

$$\begin{aligned}\left(\frac{\partial}{\partial\tau}-\Delta\right)(\tau G) \leqq\ & G - 2\tau\langle\nabla u,\nabla w\rangle \\ & + 2\tau u\left[\frac{h(\nabla f,\nabla f) - \mathrm{Rc}(\nabla f,\nabla f) + \langle\nabla(\mathrm{tr}_g h),\nabla f\rangle}{(1-f)^2} + \frac{(\mathrm{tr}_g h)|\nabla f|^2}{(1-f)^3}\right] \\ & + 2\tau u\left[\frac{-f}{1-f}\langle\nabla w,\nabla f\rangle - \frac{|\nabla f|^4}{(1-f)^3}\right] \\ & + \tau w\left[C_2k_1 + \frac{C_1\sqrt{k_2}\rho\coth(\sqrt{k_2}\rho)+C_2}{\rho^2}\right].\end{aligned}$$

Using the fact that $\nabla G(x_0,t_0) = 0$, we can replace $u\nabla w$ by $-w\nabla u$ above. Then multiplying both sides of the inequality by $u \in [0,1]$ and using $1/(1-f) \leqq 1$, we obtain

$$\begin{aligned}0 \leqq\ & G + 2\tau\{[(n+1)k_1+k_2]G + k_3\sqrt{G}\} \\ & + 2\tau G|\nabla u|\,|\nabla f|\left(\frac{-f}{1-f}\right) - 2\tau(1-f)G^2 \\ & + \tau G\left\{C_2k_1 + \frac{C_1\sqrt{k_2}\rho\coth(\sqrt{k_2}\rho)+C_2}{\rho^2}\right\}.\end{aligned}$$

Noticing that $2k_3\sqrt{G} \leqq k_3(G+1)$ and that

$$\begin{aligned}2\tau G|\nabla u|\,|\nabla f|\left(\frac{-f}{1-f}\right) &\leqq \tau G\left(\frac{|\nabla f|^2}{1-f}u + \frac{|\nabla u|^2}{u}\frac{f^2}{1-f}\right) \\ &\leqq \tau(1-f)G^2 + \tau G\frac{C_2}{\rho^2}\frac{f^2}{1-f},\end{aligned}$$

we estimate at (x_0,t_0) that

$$\begin{aligned}0 \leqq\ & \tau k_3 + G\left\{1 + \tau\left[C_1k_1 + 2k_2 + k_3 + \frac{C_1\sqrt{k_2}\rho\coth(\sqrt{k_2}\rho)+C_2}{\rho^2}\right]\right\} \\ & + \tau G\frac{C_2}{\rho^2}\frac{f^2}{1-f} - \tau(1-f)G^2.\end{aligned}$$

Dividing both sides by $\tau(1-f)$ while noting that $1/(1-f) \leqq 1$ and $-f/(1-f) \leqq 1$, we get

$$0 \leqq k_3 + G\left[\frac{1}{\tau} + C_1 k_1 + 2k_2 + k_3 + \frac{C_1\sqrt{k_2}\rho\coth(\sqrt{k_2}\rho) + C_2}{\rho^2}\right] - G^2,$$

from which we can conclude that

$$G \leqq \frac{1}{\tau} + C_1 k_1 + 2k_2 + k_3 + \sqrt{k_3} + \frac{C_1\sqrt{k_2}\rho\coth(\sqrt{k_2}\rho) + C_2}{\rho^2}$$

at (x_0, τ_0). Hence $W(\tau_1) := \tau_1 \sup_{x \in B_{g(\tau)}(\bar{x}, \rho)} w(x, \tau_1)$ may be estimated by

$$\begin{aligned} W(\tau_1) &\leqq \tau_0 G(x_0, \tau_0) \\ &\leqq 1 + \tau_0\left[C_1 k_1 + 2k_2 + k_3 + \sqrt{k_3} + \frac{C_1\sqrt{k_2}\rho\coth(\sqrt{k_2}\rho) + C_2}{\rho^2}\right] \\ &\leqq 1 + \tau_1\left[C_1 k_1 + 2k_2 + k_3 + \sqrt{k_3} + \frac{C_1\sqrt{k_2}\rho\coth(\sqrt{k_2}\rho) + C_2}{\rho^2}\right]. \end{aligned}$$

Since $\tau_1 > 0$ was arbitrary, the result follows. □

Remark 11. In the special case that $h \equiv 0$, we have

$$\frac{|\nabla v|^2}{v^2} \leqq \left(1 + \log\frac{A}{v}\right)^2\left[\frac{1}{\tau} + 2k_2 + \frac{C_1\sqrt{k_2}\rho\coth(\sqrt{k_2}\rho) + C_2}{\rho^2}\right]$$

at (x, τ), for all times $\tau \in [0, \bar{t}]$ and points $x \in B_{g(\tau)}(\bar{x}, \rho)$, which slightly improves a result of [33].

4. Reduced volume for Ricci flow

Our first application of Theorem 7 is to Ricci flow. Let $\big(\mathscr{M}^n, g(t)\big)$ be a complete solution of Ricci flow that remains smooth for $0 \leqq t \leqq \bar{t}$. This corresponds to $h = -\mathrm{Rc}$ and $\mathrm{tr}_g h = -R$ in (1.1).

4.1. Localizing Perelman's reduced volume. Perelman [31], §7, has discovered a remarkable quantity that may be regarded as a kind of parabolic distance for Ricci flow. Define $\tau(t) := \bar{t} - t$, noting that $g\big(\tau(t)\big)$ then satisfies $\frac{\partial}{\partial\tau} g = 2\,\mathrm{Rc}$ for $0 \leqq \tau \leqq \bar{t}$. Fix $\bar{x} \in \mathscr{M}^n$ and regard $(\bar{x}, 0)$ (in (x, τ) coordinates) as a space-time origin. The space-time action of a smooth path γ with $\gamma(0) = (\bar{x}, 0)$ and $\gamma(\tau) = (x, \tau)$ is

$$\mathscr{L}(\gamma) := \int_0^\tau \sqrt{\sigma}\left(\left|\frac{d\gamma}{d\sigma}\right|^2 + R\right) d\sigma \tag{4.1a}$$

$$= \int_0^{\sqrt{\tau}} \left(\frac{1}{2}\left|\frac{d\gamma}{ds}\right|^2 + 2s^2 R\right) ds \quad (s = \sqrt{\sigma}). \tag{4.1b}$$

Taking the infimum over all such paths, Perelman defines the reduced distance from $(\bar{x}, 0)$ to (x, τ) as

$$\ell(x,\tau) = \ell_{(\bar{x},0)}(x,\tau) := \frac{1}{2\sqrt{\tau}} \inf_{\gamma} \mathscr{L}(\gamma), \tag{4.2}$$

and observes that

$$v(x,\tau) = v_{(\bar{x},0)}(x,\tau) := \frac{1}{(4\pi\tau)^{n/2}} e^{-\ell(x,\tau)} \tag{4.3}$$

is a subsolution of the conjugate heat equation $u_\tau = \Delta u - Ru$ in the barrier sense [28], hence in the distributional sense.[7] It follows that the reduced volume (essentially a Gaussian weighted volume)

$$\tilde{V}(t) = \tilde{V}_{(\bar{x},0)}(t) := \int_{\mathscr{M}^n} v(x,\tau)\, d\mu \tag{4.4}$$

is a monotonically increasing function of t which is constant precisely on shrinking gradient solitons. (Compare with Example 4 above.)

The interpretations of ℓ as parabolic distance and $\tilde{V}$ as Gaussian weighted volume are elucidated by the following examples.

Example 9. Let $(\mathscr{M}^n, g)$ be a Riemannian manifold of nonnegative Ricci curvature, and let q be any smooth superharmonic function ($\Delta q \leqq 0$). In their seminal paper [22], Li and Yau define

$$\rho(x,\tau) = \inf\left\{ \frac{1}{4\tau}\int_0^1 \left|\frac{d\gamma}{d\sigma}\right|^2 d\sigma + \tau \int_0^1 q\big(\gamma(\sigma)\big)\, d\sigma \right\},$$

where the infimum is taken over all smooth paths from an origin $(\bar{x}, 0)$. As a special case of their more general results [22], Theorem 4.3, they observe that $(4\pi\tau)^{-n/2} e^{-\rho(x,\tau)}$ is a distributional subsolution of the linear parabolic equation $\left(\frac{\partial}{\partial\tau} - \Delta + q\right) u = 0$.

Example 10. Let $(\mathbb{R}^n, g)$ denote Euclidean space with its standard flat metric. Given $\lambda \in \mathbb{R}$, define $X = \operatorname{grad}\left(\frac{\lambda}{4}|x|^2\right)$. Then one has $0 = \mathrm{Rc} = \lambda g - \mathscr{L}_X g$. Hence there is a Ricci soliton structure (i.e. an infinitesimal Ricci soliton) on Euclidean space, called the Gaussian soliton. It is nontrivial whenever $\lambda \neq 0$.

Take $\lambda = 1$ to give $(\mathbb{R}^n, g)$ the structure of a gradient shrinking soliton. Then $\gamma(\sigma) = \sqrt{\sigma/\tau}\, x$ is an $\mathscr{L}$-geodesic from $(0,0)$ to (x,τ). Thus the reduced distance is $\ell_{(0,0)}(x,\tau) = |x|^2/4\tau$ and the reduced volume integrand is exactly the heat kernel $v_{(0,0)}(x,\tau) = (4\pi\tau)^{-n/2} e^{-|x|^2/4\tau}$. Hence $\tilde{V}_{(0,0)}(t) \equiv 1$. (Compare [21], §15.)

[7] See [38] for a direct proof of the distributional inequality.

Example 11. Let $S^n_{r(\tau)}$ denote the round sphere of radius $r(\tau) = \sqrt{2(n-1)\tau}$. This is a positive Einstein manifold, hence a homothetically shrinking (in t) solution of Ricci flow. Along any sequence (x_k, τ_k) of smooth origins approaching the singularity $\mathcal{O}$ at $\tau = 0$, one gets a smooth function $\ell_{\mathcal{O}}(x,\tau) := \lim_{k\to\infty} \ell_{(x_k,\tau_k)}(x,\tau) \equiv n/2$ measuring the reduced distance from $\mathcal{O}$. Hence $\tilde{V}_{\mathcal{O}}(t) \equiv [(n-1)/(2\pi e)]^{n/2}\,\mathrm{Vol}(S^n_1)$ for all $t < 0$. (See [4], §7.1.)

Our first application of Theorem 7 is where Ψ is Perelman's reduced-volume density v (4.3). Let ℓ denote the reduced distance (4.2) from a smooth origin $(\bar{x}, \bar{t})$ and assume there exists $k \in (0,\infty)$ such that $\mathrm{Rc} \geqq -kg$ on $\mathscr{M}^n \times [0, \bar{t}]$. In what follows, we will freely use results from the Appendix (Section 7, below).

Lemma 39 guarantees that ℓ is locally Lipschitz, hence that Assumption 1 is satisfied. (Also see [38] or [4].) The estimate in Part (1) of Lemma 28 ensures that Assumption 2 is satisfied. Assumption 3 follows from combining that estimate, Corollary 32, and Lemma 40. Here we may take any $\bar{r} > 0$ satisfying $\bar{r}^2 < \min\{\bar{t}/c, 4\pi\}$, where $c = e^{4k\bar{t}/3}/(4\pi)$. So for $r \in (0, \bar{r}]$, consider

$$P_{\varphi,v}(r) := \int_{E_r} \left[|\nabla \ell|^2 + R\left(n \log \frac{r}{\sqrt{4\pi\tau}} - \ell \right) \right] \varphi \, d\mu \, dt.$$

Notice that $|\nabla \ell|^2$ replaces the term $\dfrac{|x-\bar{x}|^2}{4\tau^2}$ in the heatball formulas for Euclidean space and solutions of mean curvature flow. See Examples 5 and 7, respectively.

Remark 12. For $r \in (0, \bar{r}]$, one may write $P_{1,v}(r)$ in either alternative form

$$P_{1,v}(r) = \int_{E_r} \left(\frac{n}{2\tau} + \ell_\tau + |\nabla \ell|^2 \right) d\mu \, dt \tag{4.5a}$$

$$= \int_{E_r} \left(\frac{n}{2\tau} - \frac{1}{2}\tau^{-3/2}\mathscr{K} \right) d\mu \, dt. \tag{4.5b}$$

Here $\mathscr{K}(x,\tau) = \int_0^\tau \sigma^{3/2} \mathscr{H}(d\gamma/d\sigma)\, d\sigma$ is computed along a minimizing $\mathscr{L}$-geodesic γ, where $\mathscr{H}(X) = 2\,\mathrm{Rc}(X,X) - (R_\tau + 2\langle \nabla R, X\rangle + R/\tau)$ is Hamilton's traced differential Harnack expression.

If $R \geqq 0$ and $\varphi \geqq 0$ on $E_{\bar{r}}$, then for all $r \in (0, \bar{r}]$, one has

$$P_{\varphi,v}(r) = \int_{E_r} [|\nabla \ell|^2 + R\psi_{(r)}] \varphi \, d\mu \, dt \geqq \int_{E_r} |\nabla \ell|^2 \varphi \, d\mu \, dt \geqq 0. \tag{4.6}$$

If $\big(\mathscr{M}^n, g(0)\big)$ has nonnegative curvature operator and $\bar{r}^2 < 4\pi\bar{t}(1 - 1/C)$ for some $C > 1$, then for all $r \in (0, \bar{r}]$,

$$P_{1,v}(r) \leqq \int_{E_r} \frac{n/2 + C\ell}{\tau} \, d\mu \, dt. \tag{4.7}$$

Proof. By Part (2) of Lemma 39, the arguments of Lemma 40 apply to show that $\psi_t + |\nabla\psi|^2 = \dfrac{n}{2\tau} + \ell_\tau + |\nabla\ell|^2 \in L^1(E_{\bar r})$. Hence Lemma 9 and identities (7.5) and (7.6) of [28] imply formulae (4.5).

Since $\psi_{(r)} > 0$ in E_r, the inequalities in (4.6) are clear.

If $\big(\mathscr{M}^n, g(0)\big)$ has nonnegative curvature operator, Hamilton's traced differential Harnack inequality [18] implies that

$$\mathscr{H}\left(\frac{d\gamma}{d\sigma}\right) \geqq -R\left(\frac{1}{\sigma} + \frac{1}{\bar t - \sigma}\right) = -\frac{\bar t}{\bar t - \sigma}\,\frac{R}{\sigma}$$

along a minimizing $\mathscr{L}$-geodesic γ. Hence

$$-\frac{1}{2}\tau^{-3/2}\mathscr{K} \leqq \frac{\bar t}{\bar t - \tau}\,\frac{\tau^{-3/2}}{2}\int_0^\tau \sqrt{\sigma}\left(R + \left|\frac{d\gamma}{d\sigma}\right|^2\right) d\sigma = \frac{\bar t}{\bar t - \tau}\,\frac{\ell}{\tau}.$$

By Lemma 31, one has $\tau < r^2/4\pi$, which gives estimate (4.7). □

Our main result in this section is as follows. Recall that $\psi_{(r)} := n\log\left(\dfrac{r}{\sqrt{4\pi\tau}}\right) - \ell$.

Corollary 13. *Let $\big(\mathscr{M}^n, g(t)\big)$ be a complete solution of Ricci flow that remains smooth for $0 \leqq t \leqq \bar t$ and satisfies $\mathrm{Rc} \geqq -kg$. Let φ be any smooth nonnegative function of (x,t) and let $c = e^{4k\bar t/3}/(4\pi)$. Then whenever $0 < r_0 < r_1 < \min\{\sqrt{\bar t/c}, 2\sqrt{\pi}\}$, one has*

$$\frac{P_{\varphi,v}(r_1)}{r_1^n} - \frac{P_{\varphi,v}(r_0)}{r_0^n} \leqq -\int_{r_0}^{r_1}\frac{n}{r^{n+1}}\int_{E_r}\psi_{(r)}\left(\frac{\partial\varphi}{\partial t} - \Delta\varphi\right) d\mu\, dt\, dr. \tag{4.8}$$

Furthermore,

$$\varphi(\bar x, \bar t) = \lim_{r\searrow 0}\frac{P_{\varphi,v}(r)}{r^n}. \tag{4.9}$$

In particular,

$$\varphi(\bar x, \bar t) \geqq \frac{P_{\varphi,v}(r_1)}{r_1^n} + \int_0^{r_1}\frac{n}{r^{n+1}}\int_{E_r}\psi_{(r)}\left[\left(\frac{\partial}{\partial t} - \Delta\right)\varphi\right] d\mu\, dt\, dr. \tag{4.10}$$

Proof. The quantity $\Psi = v$ satisfies $\dfrac{\partial v}{\partial t} + \Delta v - Rv \geqq 0$ as a distribution. (This is implied by Perelman's barrier inequality [28], (7.13); see [38], Lemma 1.12, for a direct proof.) Hence we may apply Theorem 7 in the form (2.6) to obtain (4.8).

Formula (7.6) of Perelman [28] implies that

$$P_{\varphi,v}(r) = \int\limits_{E_r} \left[\frac{\ell}{\tau} + R\psi_{(r)} - R - \tau^{-3/2}\mathscr{K} \right] \varphi \, d\mu \, dt.$$

By Corollary 32, there is a precompact neighborhood $\mathscr{U}$ of $\bar{x}$ with $E_r \subseteqq \mathscr{U} \times [0, cr^2]$ for all $r > 0$ under consideration. By Lemma 35, there exists a precompact set $\mathscr{V}$ such that the images of all minimizing $\mathscr{L}$-geodesics from $(\bar{x}, 0)$ to points in $\mathscr{U} \times [0, cr^2]$ are contained in the set $\mathscr{V} \times [0, cr^2]$, in which one has uniform bounds on all curvatures and their derivatives. So by Lemma 28, one has $\dfrac{\ell}{\tau} = \dfrac{d_0^2(\bar{x}, x)}{4\tau^2} + O\left(\dfrac{1}{\tau}\right)$ and $R\psi_{(r)} = R\left(n \log \dfrac{r}{\sqrt{4\pi\tau}} - \ell\right) = O\left(\dfrac{1}{\tau}\right)$ as $\tau \searrow 0$. By Corollary 37, $\tau^{-3/2}\mathscr{K}$ is also $O\left(\dfrac{1}{\tau}\right)$ as $\tau \searrow 0$. Adapting the arguments in the proof of Lemma 40, one concludes that

$$\lim_{r \searrow 0} \frac{P_{\varphi,v}(r)}{r^n} = \lim_{r \searrow 0} \left\{ \frac{1}{r^n} \int\limits_{E_r} \frac{d_0^2(\bar{x}, x)}{4\tau^2} \varphi \, d\mu \, dt \right\} = \varphi(\bar{x}, \bar{t}),$$

exactly as in the calculation for Euclidean space. (Also see Corollary 23, below.) □

An example of how this result may be applied is the following local Harnack inequality, which follows directly from (4.10).

Remark 14. Assume the hypotheses of Corollary 13 hold. If $R \geqq 0$ on E_{r_1}, then

$$R(\bar{x}, \bar{t}) \geqq \frac{1}{r_1^n} \int\limits_{E_{r_1}} [|\nabla \ell|^2 + R\psi_{(r)}] R \, d\mu \, dt + \int\limits_0^{r_1} \frac{2n}{r^{n+1}} \int\limits_{E_r} \psi_{(r)} |\mathrm{Rc}|^2 \, d\mu \, dt \, dr.$$

The inequality (4.8) is sharp in the following sense.

Corollary 15. *Let $\left(\mathscr{M}^n, g(t)\right)$ be a complete solution of Ricci flow that is smooth for $0 \leqq t \leqq \bar{t}$, with $\mathrm{Rc} \geqq -kg$. If equality holds in* (4.8) *for $\varphi \equiv 1$, then $\left(E_r, g(t)\right)$ is isometric to a shrinking gradient soliton for all $r < \min\{\sqrt{\bar{t}/c}, 2\sqrt{\pi}\}$.*

Proof. From the proof of Theorem 7, it is easy to see that

$$\frac{d}{dr}\left(\frac{P_{1,v}(r)}{r^n}\right) = -\frac{n}{r^{n+1}} \int\limits_{E_r} \frac{\left(\dfrac{\partial}{\partial t} + \Delta + \mathrm{tr}_g h\right) v}{v} d\mu \, dt$$

for almost all $r < \min\{\sqrt{\bar{t}/c}, 2\sqrt{\pi}\}$. Therefore, equality in (4.8) implies that v is a distributional solution of the parabolic equation

$$\left(\frac{\partial}{\partial \tau} - \Delta + R\right) v = 0$$

in E_r for almost all small r. By parabolic regularity, v is actually smooth. This implies that one has equality in the chain of inequalities

$$\Delta\ell - |\nabla\ell|^2 + R - \frac{n}{2\tau} \leqq \ell_\tau = -\left(-R + \frac{n}{2\tau} - \frac{1}{2}\tau^{-3/2}\mathscr{K}\right) + \frac{n-2\ell}{2\tau} \leqq -\Delta\ell + \frac{n-2\ell}{2\tau}$$

that follow from equations (7.13), (7.5), and (7.10) of [28]. Hence one has

$$u := \tau(2\Delta\ell - |\nabla\ell|^2 + R) + \ell - n = 0.$$

By [28], equation (9.1) (where the roles of u and v are reversed), this implies that

$$0 = \left(\frac{\partial}{\partial\tau} - \Delta + R\right)(uv) = -2\tau\left|\mathrm{Rc} + \nabla\nabla\ell - \frac{1}{2\tau}g\right|^2 v.$$

This is possible only if $\big(E_r, g(t)\big)$ has the structure of a shrinking gradient soliton with potential function ℓ. □

Remark 16. For applications of Corollary 15 to regularity theorems for Ricci flow, see [27] by the third author.

4.2. Comparing global and local quantities. Corollaries 13 and 15 suggest a natural question: how does the purely local monotone quantity $P_{1,v}(r)/r^n$ compare to Perelman's global monotone quantity $\tilde{V}(t) = \int_{\mathscr{M}^n} v\, d\mu$? A path to a partial answer begins with an observation that generalizes Example 11 above.

Cao, Hamilton, and Ilmanen [1] prove that any complete gradient shrinking soliton $\big(\mathscr{M}^n, g(t)\big)$ that exists up to a maximal time $T < \infty$ and satisfies certain noncollapsing and curvature decay hypotheses converges as $t \nearrow T$ to an incomplete (possibly empty) metric cone $(\mathscr{C}, d)$, which is smooth except at the parabolic vertex $\mathscr{O}$. The convergence is smooth except on a compact set (possibly all of $\mathscr{M}^n$) that vanishes into the vertex.[8] Furthermore, they prove that along a sequence (x_k, τ_k) approaching $\mathscr{O}$, a limit $\ell_{\mathscr{O}}(x,\tau) := \lim \ell_{(x_k,\tau_k)}(x,\tau)$ exists for all $x \in \mathscr{M}^n$ and $\tau(t) > 0$. They show that the central density function

$$\Theta^{\mathrm{RF}}_{\mathscr{O}}(t) := \tilde{V}_{\mathscr{O}}(t) = \lim_{k\to\infty} \tilde{V}_{(x_k,\tau_k)}(t)$$

of the parabolic vertex $\mathscr{O}$ is independent of time and satisfies $\Theta^{\mathrm{RF}}_{\mathscr{O}}(t) \equiv e^{\nu}$, where ν is the constant entropy of the soliton $\big(\mathscr{M}^n, g(\tau)\big)$.

On a compact soliton, there is a pointwise version of the Cao-Hamilton-Ilmanen result, due to Bennett Chow and the third author:

Lemma 17. *If $\big(\mathscr{M}^n, g(\tau)\big)$ is a compact shrinking (necessarily gradient) soliton, then the limit $\ell_{\mathscr{O}}(x,\tau)$ exists for all $x \in \mathscr{M}^n$ and $\tau(t) > 0$. This limit agrees up to a constant with the soliton potential function $f(x,\tau)$.*

See [4] for a proof.

[8] See [11] for examples where $(\mathscr{C}, d) = \lim_{\tau\searrow 0}\big(\mathscr{M}^n, g(\tau)\big)$ is nonempty.

Recall that the entropy of a compact Riemannian manifold $(\mathscr{M}^n, g)$ is

$$\nu(\mathscr{M}^n, g) := \inf\left\{\mathscr{W}(g, f, \tau) : f \in C_0^\infty, \tau > 0, \int_{\mathscr{M}^n} (4\pi\tau)^{-n/2} e^{-f}\, d\mu = 1\right\},$$

where

$$\mathscr{W}(g, f, \tau) := \int_{\mathscr{M}^n} [\tau(|\nabla f|^2 + R) + f - n](4\pi\tau)^{-n/2} e^{-f}\, d\mu. \tag{4.11}$$

(Compare with Example 3.) Under the coupled system

$$\frac{\partial}{\partial t} g = -2\,\mathrm{Rc}, \tag{4.12a}$$

$$\left(\frac{\partial}{\partial t} + \Delta\right) f = |\nabla f|^2 - R + \frac{n}{2\tau}, \tag{4.12b}$$

$$\frac{d\tau}{dt} = -1, \tag{4.12c}$$

the functional $\mathscr{W}\big(g(t), f(t), \tau(t)\big)$ is monotone increasing in time and is constant precisely on a compact shrinking gradient soliton with potential function f, where (after possible normalization) one has

$$\mathrm{Rc} + \nabla\nabla f - \frac{1}{2\tau} g \overset{\circ}{=} 0. \tag{4.13}$$

Here and in the remainder of this section, the symbol $\overset{\circ}{=}$ denotes an identity that holds on a shrinking gradient soliton.

We are now ready to answer the question we posed above regarding the relationship between $P_{1,v}(r)/r^n$ and $\tilde{V}(t)$. (Compare with Example 7.)

Corollary 18. *Let $\big(\mathscr{M}^n, g(t)\big)$ be a compact shrinking Ricci soliton that vanishes into a parabolic vertex $\mathscr{O}$ at time T. Then for all $t < T$ and $r > 0$, one has*

$$\Theta_{\mathscr{O}}^{\mathrm{RF}}(t) := \tilde{V}_{\mathscr{O}}(t) = \frac{P_{1,v}(r)}{r^n},$$

where $P_{1,v}(r) = \int_{E_r} \left[|\nabla \ell|^2 + R\left(n \log \frac{r}{\sqrt{4\pi\tau}} - \ell\right)\right] d\mu\, dt$ *is computed with* $\ell = \ell_{\mathscr{O}}$.

Proof. It will be easiest to regard everything as a function of $\tau(t) := T - t > 0$. Because $\big(\mathscr{M}^n, g(\tau)\big)$ is a compact shrinking soliton, there exist a time-independent metric $\bar{g}$ and function $\bar{f}$ on $\mathscr{M}^n$ such that $\mathrm{Rc}(\bar{g}) + \bar{\nabla}\bar{\nabla}\bar{f} - \frac{1}{2}\bar{g} = 0$. The solution of Ricci flow is then $g(\tau) = \tau\xi_\tau^*(\bar{g})$, where $\{\xi_\tau\}_{\tau>0}$ is a one-parameter family of diffeomorphisms such that $\xi_1 = \mathrm{id}$ and $\frac{\partial}{\partial\tau}\xi_\tau(x) = -\tau^{-1}\,\mathrm{grad}_{\bar{g}}\,\bar{f}(x)$. The soliton potential function satisfies $f(x, \tau) = \xi_\tau^*\bar{f}(x)$ and $f_\tau = -\|\nabla f\|^2$. (Notice that (4.13) implies that system (4.12) holds.)

Let $\Psi = (4\pi\tau)^{-n/2} e^{-\ell(x,\tau)}$, where ℓ is the reduced distance from the parabolic vertex $\mathcal{O}$. By Lemma 17, $\ell = f + C$. So Assumptions 1 and 2 are clearly satisfied. Because

$$\int_{\mathcal{M}^n \times \{\tau\}} |\psi| \, d\mu = O[\tau^{n/2} \log(\tau^{-n/2})] \quad \text{and} \quad \int_{\mathcal{M}^n \times \{\tau\}} |\nabla \psi|^2 \, d\mu = O(\tau^{n/2-1})$$

as $\tau \searrow 0$, Assumption 3 is satisfied as well. Because $\frac{\partial}{\partial \tau} \psi = |\nabla f|^2 - \frac{n}{2\tau}$, Lemma 9 implies that

$$P_{1,v}(r) = \int_{E_r} \left(\frac{n}{2\tau} + \ell_\tau + |\nabla \ell|^2 \right) d\mu \, dt \overset{\circ}{=} \int_{E_r} \frac{n}{2\tau} \, d\mu \, dt.$$

(Compare Remark 12.) Computing $\tilde{V}(\tau) = \tilde{V}_{\mathcal{O}}(\tau)$, one finds that

$$\begin{aligned}
\tilde{V}(1) &= \int_{\mathcal{M}^n} (4\pi)^{-n/2} e^{-\ell(x,1)} \, d\mu\big(g(1)\big) \\
&= \int_0^\infty \mathrm{Vol}_{g(1)}\{x : (4\pi)^{-n/2} e^{-\ell(x,1)} \geqq z\} \, dz \\
&\overset{\circ}{=} \int_0^\infty \frac{n}{2\tau} \, \mathrm{Vol}_{g(\tau)}[\xi_\tau^{-1}\{x : (4\pi\tau)^{-n/2} e^{-\ell(x,1)} \geqq 1\}] \, d\tau \quad (z = \tau^{n/2}) \\
&= \int_0^\infty \frac{n}{2\tau} \, \mathrm{Vol}_{g(\tau)} \left\{ y : \ell(y,\tau) < n \log \frac{1}{\sqrt{4\pi\tau}} \right\} d\tau \\
&= \int_{E_1} \frac{n}{2\tau} \, d\mu \, dt \\
&\overset{\circ}{=} P_{1,v}(1).
\end{aligned}$$

But on a shrinking gradient soliton, $P_{1,v}(r)/r^n$ is independent of $r > 0$, while $\tilde{V}(\tau)$ is independent of $\tau > 0$. Since they agree at $r = 1$ and $\tau = 1$, they agree everywhere. □

Since the reduced distance and reduced volume are invariant under parabolic rescaling, similar considerations apply to solutions whose rescaled limits are shrinking gradient solitons.

4.3. Localizing forward reduced volume. In [12], Feldman, Ilmanen, and the third author introduce a *forward reduced distance*

$$\ell_+(x,t) := \inf_\gamma \frac{1}{2\sqrt{t}} \int_0^t \sqrt{s} \left(\left| \frac{d\gamma}{ds} \right|^2 + R \right) ds.$$

Here the infimum is taken over smooth paths γ from an origin $(\bar{x}, 0)$ to (x, t). Define

$$u(x,t) = (4\pi t)^{-n/2} e^{-\ell_+(x,t)}$$

and $\psi = \log u$. In [25], it is proved that $\left(\frac{\partial}{\partial t} - \Delta - R \right) u \leqq 0$ holds in the distributional sense if $\big(\mathcal{M}^n, g(t)\big)$ is a complete solution of Ricci flow with bounded nonnegative curvature

operator for $0 \leqq t \leqq T$. Following the same arguments as in the proof of Corollary 13 then leads to the following result for

$$P_{\varphi,u}(r) = \int\limits_{E_r} \left[|\nabla \ell_+|^2 - R\left(n \log \frac{r}{\sqrt{4\pi t}} - \ell_+ \right) \right] \varphi \, d\mu \, dt.$$

Corollary 19. *Let $\left(\mathscr{M}^n, g(t)\right)$ be a complete solution of Ricci flow with bounded nonnegative curvature operator for $0 \leqq t \leqq T$. Let φ be any smooth nonnegative function. Then whenever $0 < r_0 < r_1 < \sqrt{4\pi T}$, one has*

$$\frac{P_{\varphi,u}(r_1)}{r_1^n} - \frac{P_{\varphi,u}(r_0)}{r_0^n} \leqq \int\limits_{r_0}^{r_1} \frac{n}{r^{n+1}} \int\limits_{E_r} (\psi + n \log r) \left(\frac{\partial \varphi}{\partial t} + \Delta \varphi \right) d\mu \, dt \, dr. \tag{4.14}$$

In direct analogy with Corollary 15, one also has the following.

Corollary 20. *Let $\left(\mathscr{M}^n, g(t)\right)$ be a complete solution of Ricci flow with bounded nonnegative curvature operator for $0 \leqq t \leqq T$. If equality holds in (4.14) with $\varphi \equiv 1$, then $\left(E_r, g(t)\right)$ is isometric to an expanding gradient soliton for all $r < \sqrt{4\pi T}$.*

5. Mean-value theorems for heat kernels

In this section, we apply Theorem 7 to heat kernels of evolving Riemannian manifolds, especially those evolving by Ricci flow, with stationary (i.e. time-independent) manifolds appearing as an interesting special case.

Let $\left(\mathscr{M}^n, g(t)\right)$ be a smooth family of Riemannian manifolds evolving by (1.1) for $t \in [0, \bar{t}]$. We will again abuse notation by regarding certain evolving quantities, where convenient, as functions of $x \in \mathscr{M}^n$ and $\tau(t) := \bar{t} - t$.

A smooth function $\Psi : (\mathscr{M}^n \times [0, \bar{t}]) \backslash (\bar{x}, 0) \to \mathbb{R}_+$ is called a *fundamental solution* of the conjugate heat equation

$$\left(\frac{\partial}{\partial \tau} - \Delta - \mathrm{tr}_g h \right) \Psi = 0 \tag{5.1}$$

with singularity at $(\bar{x}, 0)$ if Ψ satisfies (5.1) at all $(x, \tau) \in \mathscr{M}^n \times (0, \bar{t}]$, with $\lim\limits_{\tau \searrow 0} \Psi(\cdot, \tau) = \delta_{\bar{x}}$ in the sense of distributions. We call a minimal fundamental solution of (5.1) a *heat kernel*.

For any smooth family $\left(\mathscr{M}^n, g(t)\right)$ of complete Riemannian manifolds, it is well known that a heat kernel Ψ always exists and is unique. Moreover, Ψ is bounded outside any compact space-time set containing $(\bar{x}, 0)$ in its interior.[9] If Ψ is the conjugate heat kernel for $\left(\mathscr{M}^n, g(t)\right)$, then (2.4) takes the form

[9] There are several standard constructions, all of which utilize local properties that the manifold inherits from $\mathbb{R}^n$. See the fine survey [16] and references therein.

$$P_{\varphi,\Psi}(r) = \int_{E_r} [|\nabla \log \Psi|^2 - \log(r^n \Psi)(\mathrm{tr}_g h)]\varphi \, d\mu \, dt.$$

It is clear that Assumptions 1 and 2 are always satisfied. In particular, $E_{\bar{r}}$ is compact for $\bar{r} > 0$ sufficiently small. We shall prove that Assumption 3 is also valid for such $\bar{r}$. For this, we need a purely local observation about Ψ near $(\bar{x}, 0)$.

Lemma 21. *For $t \in [0, \bar{t}]$, let $\big(\mathcal{M}^n, g(t)\big)$ be a smooth family of (possibly incomplete) Riemannian manifolds. Suppose that Ψ is any fundamental solution of (5.1) with singularity at $(\bar{x}, 0)$. For any $\varepsilon > 0$, there exist a precompact neighborhood Ξ of $\bar{x}$, a time $\bar{\tau} \in (0, \bar{t}]$, and a smooth function $\Phi : \Xi \times [0, \bar{\tau}] \to \mathbb{R}_+$ with $\Phi(\bar{x}, 0) = 1$ such that for all $(x, \tau) \in (\Xi \times [0, \bar{\tau}]) \backslash (\bar{x}, 0)$, one has*

$$\left| \Psi(x,\tau) - \Phi(x,\tau) \cdot \frac{1}{(4\pi\tau)^{n/2}} \exp\left(-\frac{d^2_{g(\tau)}(\bar{x}, x)}{4\tau} \right) \right| \leqq \varepsilon. \tag{5.2}$$

Proof. One begins with Garofalo and Lanconelli's asymptotics [15], Theorem 2.1, for a fundamental solution with respect to a Riemannian metric on $\mathbb{R}^n$ which is Euclidean outside of an arbitrarily large compact neighborhood of the origin. The first step is a straightforward adaptation of their proof to the case $h \neq 0$. The second step is to glue a large ball centered at $\bar{x} \in \mathcal{M}^n$ into Euclidean space, obtaining a manifold $\big(\mathbb{R}^n, \tilde{g}(t)\big)$ which is identical to $\big(\mathcal{M}^n, g(t)\big)$ on a large neighborhood of $\bar{x}$ and to which the refined asymptotics apply. The difference of the fundamental solutions Ψ and $\tilde{\Psi}$ for $\big(\mathcal{M}^n, g(t)\big)$ and $\big(\mathbb{R}^n, \tilde{g}(t)\big)$, respectively, starts at zero as a distribution. By the comparison principle, it stays uniformly small for a short time. □

We now consider Assumption 3. Let $\bar{r} > 0$ be given. Apply Lemma 21 with $\varepsilon = \bar{r}^{-n}/2$. By shrinking Ξ and $\bar{\tau}$ if necessary, we may assume without loss of generality that $1/2 \leqq \Phi \leqq 2$ in $\Xi \times [0, \bar{\tau}]$. Because $\Psi(\cdot, \tau) \to \delta_{\bar{x}}$ as $\tau \searrow 0$, we may also assume $\bar{\tau} > 0$ is small enough that $E_{\bar{r}}(\tau) \subseteq \Xi$ for all $\tau \in (0, \bar{\tau}]$, where $E_{\bar{r}}(\tau) := E_{\bar{r}} \cap (\mathcal{M}^n \times \{\tau\})$. Then in $\bigcup_{\tau \in (0, \bar{\tau}]} E_{\bar{r}}(\tau)$, one has

$$\frac{1}{(4\pi\tau)^{n/2}} \exp\left(-\frac{d^2_{g(\tau)}(\bar{x}, x)}{4\tau} \right) \geqq \frac{\Psi(x,\tau) - \varepsilon}{\Phi(x,\tau)} \geqq \frac{1}{4\bar{r}^n}, \tag{5.3}$$

which implies that $d^2_{g(\tau)}(\bar{x}, \cdot) \leqq 4\tau \left[\frac{n}{2} \log \frac{1}{\tau} + \log 4 - \frac{n}{2} \log(4\pi) + \log \bar{r}^n \right]$ there. Reduce $\bar{\tau} > 0$ if necessary so that $\bar{\tau} \leqq 4^{(n-2)/n} \pi \bar{r}^{-2}$. Then one has

$$d^2_{g(\tau)}(\bar{x}, \cdot) \leqq 4n\tau \log \frac{1}{\tau} \tag{5.4}$$

in $E_{\bar{r}}(\tau)$ for all $\tau \in (0, \bar{\tau}]$. Since $\Psi > \bar{r}^{-n} = 2\varepsilon$ in $E_{\bar{r}}$, one also has

$$\frac{\Psi}{2} \leqq \Psi - \varepsilon \leqq \frac{\Phi(x,\tau)}{(4\pi\tau)^{n/2}} \leqq \frac{2}{(4\pi\tau)^{n/2}}. \tag{5.5}$$

If necessary, reduce $\bar{\tau} > 0$ further so $\bar{\tau} \leqq \bar{r}^{-1}$ and $\bar{\tau} \leqq 4^{(n-2)/n}\pi$. Then $\psi := \log \Psi$ satisfies

$$|\psi| \leqq n \log \frac{1}{\tau}$$

in $E_{\bar{r}}(\tau)$ for all $\tau \in (0, \bar{\tau}]$. By (5.4), this proves that $\lim_{\tau \searrow 0} \int_{E_{\bar{r}}(\tau)} |\psi| \, d\mu = 0$.

If $\bar{\tau} \leqq \bar{r}^2$, then $\Psi \leqq \bar{r}^{-n} \leqq \bar{\tau}^{-n/2}$ outside $E_{\bar{r}}$. So by (5.5), there exists $c = c(n)$ such that $\Psi(\cdot, \tau) \leqq e^c \tau^{-n/2}$ for all $\tau \in (0, \bar{\tau}]$. By (5.4), $E_{\bar{r}}(\tau) \subseteqq B_{g(\tau)}(\bar{x}, \rho)$ for $\rho := \sqrt{5n/e}$. Since $\Psi > \bar{r}^{-n}$ in $E_{\bar{r}}$, Theorem 10 yields C independent of x and τ in $\bigcup_{0<\tau\leqq\bar{\tau}} B_{g(\tau)}(\bar{x}, 2\rho)$ such that for any $\tau \in (0, \bar{\tau}]$, one has

$$|\nabla\psi|^2 \leqq \left(\frac{1}{\tau} + C\right)\left(1 + c + n \log \bar{r} - \frac{n}{2} \log \frac{\tau}{2}\right)^2 \tag{5.6}$$

in $E_{\bar{r}} \cap (\mathscr{M}^n \times [\tau/2, \tau])$. If $\bar{r} > 0$ is small enough that $E_{\bar{r}}$ is compact, this estimate and (5.4) prove that $\int_{E_{\bar{r}}} |\nabla\psi|^2 \, d\mu \, dt < \infty$, which establishes Assumption 3.

Remark 22. Assumption 3 is valid for *all* $\bar{r} > 0$ in any manifold $\big(\mathscr{M}^n, g(t)\big)_{t<\bar{t}}$ for which the kernel Ψ vanishes at infinity in space-time, i.e. if for every $\varepsilon > 0$, there exists a compact set $K \subset \mathscr{M}^n \times (-\infty, \bar{t}]$ such that $\Psi \leqq \varepsilon$ outside K.

Our main result in this section is the following consequence of Theorem 7. The reader is invited to compare it with Corollary 13 (above) for Perelman's reduced volume density. Recall that $\psi_{(r)} := \log(r^n \Psi)$.

Corollary 23. *Suppose that* $\big(\mathscr{M}^n, g(t)\big)$ *is a smooth family of complete Riemannian manifolds evolving by* (1.1) *for* $t \in [0, \bar{t}]$. *Let* $\Psi : (\mathscr{M}^n \times [0, \bar{t}]) \backslash (\bar{x}, 0) \to \mathbb{R}_+$ *be the kernel of the conjugate heat equation* (5.1) *with singularity at* $(x, \tau) = (\bar{x}, 0)$. *Let* φ *be any smooth function of* (x, t). *Then there is* $\bar{r} > 0$ *such that if* $0 < r_0 < r_1 < \bar{r}$, *then*

$$\frac{P_{\varphi,\Psi}(r_1)}{r_1^n} - \frac{P_{\varphi,\Psi}(r_0)}{r_0^n} = -\int_{r_0}^{r_1} \frac{n}{r^{n+1}} \int_{E_r} \psi_{(r)} \left(\frac{\partial\varphi}{\partial t} - \Delta\varphi\right) d\mu \, dt \, dr.$$

Furthermore, one has

$$\varphi(\bar{x}, \bar{t}) = \lim_{r \searrow 0} \frac{P_{\varphi,\Psi}(r)}{r^n},$$

and thus

$$\varphi(\bar{x}, \bar{t}) = \frac{P_{\varphi,\Psi}(r_1)}{r_1^n} + \int_0^{r_1} \frac{n}{r^{n+1}} \int_{E_r} \psi_{(r)} \left(\frac{\partial\varphi}{\partial t} - \Delta\varphi\right) d\mu \, dt \, dr.$$

Proof. Now that we have verified Assumptions 1–3, everything follows directly from Theorem 7 except for the representation formula $\varphi(\bar{x}, \bar{t}) = \lim_{r \searrow 0} [P_{\varphi,\Psi}(r)/r^n]$, which we will prove by a blow-up argument. Without loss of generality, we may assume that $\varphi(\bar{x}, \bar{t}) = 1$. Here is the set-up. Identify $\mathbb{R}^n$ with $T_{\bar{x}}\mathscr{M}^n$, and let $y \in \mathscr{M}^n$ denote the

image of $\vec{y} \in \mathbb{R}^n$ under the exponential map $\exp_{\bar{x}}(\cdot)$ for g at $\tau = 0$. For $r > 0$, define $\varphi^r(\vec{y}, \tau) := \varphi(ry, r^2\tau)$, $\Psi^r(\vec{y}, \tau) := r^n \Psi(ry, r^2\tau)$, and $\Psi^0(\vec{y}, \tau) := (4\pi\tau)^{-n/2} e^{-|\vec{y}|^2/4\tau}$. Let $d\mu^r(\cdot, \tau)$ denote the pullback of $r^{-n}\, d\mu(\cdot, r^2\tau)$ under the map $\vec{y} \mapsto \exp_{\bar{x}}(r\vec{y})$. For $\delta \geqq 0$, consider the 'truncations' defined by

$$E_r^\delta := E_r \cap (\mathscr{M}^n \times (\delta r^2, \bar{t}]),$$

$$\hat{E}_r^\delta := \{(\vec{y}, \tau) : \tau > \delta \text{ and } \Psi^r(\vec{y}, \tau) > 1\},$$

$$\hat{E}_0^\delta := \{(\vec{y}, \tau) : \tau > \delta \text{ and } \Psi^0(\vec{y}, \tau) > 1\},$$

$$P_r^\delta := \int\limits_{E_r^\delta} [|\nabla \log \Psi|^2 - (\mathrm{tr}_g\, h) \log(r^n \Psi)] \varphi \, d\mu \, d\tau,$$

$$P_0^\delta := \int\limits_{\hat{E}_0^\delta} |\nabla \log \Psi^0|^2 \, d\vec{y} \, d\tau.$$

The proof consists of two claims, which together imply the result.

The first claim is that if $0 < \delta \ll 1$, then $\lim\limits_{r \searrow 0} [P_r^\delta / r^n] = P_0^\delta$. Pulling back, one computes $P_r^\delta = r^n \int\limits_{\hat{E}_r^\delta} [|\nabla \log \Psi^r|^2 - r^2 (\mathrm{tr}_g\, h) \log \Psi^r] \varphi^r \, d\mu^r \, d\tau$. By Lemma 21, $\Psi^r \to \Psi^0$ as $r \searrow 0$ uniformly on any $\Omega \subset\subset \mathbb{R}^n \times [\delta, \bar{t}]$. By parabolic regularity, $\chi(\hat{E}_r^\delta) \to \chi(\hat{E}_0^\delta)$ in $L^1(\mathbb{R}^n)$ as $r \searrow 0$. Since $d\mu^r \to d\vec{y}$ and $\varphi(\bar{x}, \bar{t}) = 1$, the claim follows.

The second claim is that for any $\eta > 0$, there exists some $\delta \in (0, 1/100)$ such that $0 \leqq [P_{\varphi, \Psi}(r) - P_r^\delta] / r^n < \eta$ for all small $r > 0$. By Lemma 21, if $r \leqq 1$ is so small that $1/2 \leqq \Phi \leqq 2$ in $E_r \cap (\Xi \times [0, \bar{\tau}])$, then $(4\pi\tau)^{-n/2} \exp\big(-d^2_{g(\tau)}(\bar{x}, x)/4\tau\big) \geqq \dfrac{1}{4r^n}$ there. (Compare (5.3).) Furthermore, $d^2_{g(\tau)}(\bar{x}, \cdot) \leqq 4\tau \left(\dfrac{n}{2} \log \dfrac{r^2}{\tau} + \log 4 \right) \leqq 4\tau n \log \dfrac{r^2}{\tau}$ in $E_r \backslash E_r^\delta$, since $\dfrac{r^2}{\tau} \geqq 4$. Because $\tau \leqq 1$ in E_r for all small $r > 0$, Theorem 10 gives C such that $|\nabla \log \Psi|^2 \leqq \dfrac{C}{\tau} \left(\log \dfrac{r^2}{\tau} \right)^2$ in $E_r \backslash E_r^\delta$. (Here we used $r^{-n} \leqq \Psi \leqq e^c \tau^{-n/2}$; compare (5.6).) Therefore,

$$\int\limits_{E_r \backslash E_r^\delta} |\nabla \log \Psi|^2 \, d\mu \leqq C' \int\limits_0^{\delta r^2} \tau^{\frac{n-2}{2}} \left(\log \frac{r^2}{\tau} \right)^{\frac{n+4}{2}} d\tau \leqq C'' r^n \delta^{n/2} \left(\log \frac{1}{\delta} \right)^{\frac{n+4}{2}}.$$

The second claim, hence the theorem, follows readily. □

Remark 24. In the special case that $L(\cdot, t)$ is a divergence-form, uniformly elliptic operator on Euclidean space $\mathbb{R}^n$ and Ψ is the kernel of its adjoint L^*, the results of Corollary 23 appear in [10], Theorems 1 and 2, for φ solving $\left(\dfrac{\partial}{\partial t} - L \right) \varphi = 0$, and in [14], Theorem 1.5, for arbitrary smooth φ.

We conclude this section with two results for the special case of the conjugate heat kernel Ψ of a *fixed* Riemannian manifold $(\mathscr{M}^n, g)$.

Our first observation is that one can adapt the argument of [10] to obtain a mean-value representation theorem in terms of an integral on 'heat spheres'. This approach is naturally related to the interpretation of equation (1.4) as a space-time Green's formula. To give the argument, we introduce some additional notation. Consider the space-time manifold $\tilde{M}^{n+1} = \mathscr{M}^n \times \mathbb{R}$ equipped with the metric $\tilde{g}(x,t) = g(x) + dt^2$, where t is the global $\mathbb{R}$-coordinate. Applying Green's formula to a bounded space-time domain D in $\tilde{M}^{n+1}$ with the vector field $\varphi\Psi\dfrac{\partial}{\partial t} - \Psi\nabla\varphi + \varphi\nabla\Psi$, we get

$$\begin{aligned}(5.7)\qquad \int_D \left(\frac{\partial\varphi}{\partial t} - \Delta\varphi\right)\Psi\, d\mu\, dt &= \int_D \left[\left(\frac{\partial\varphi}{\partial t} - \Delta\varphi\right)\Psi + \varphi\left(\frac{\partial\Psi}{\partial t} + \Delta\Psi\right)\right] d\mu\, dt \\ &= \int_D \operatorname{div}_{\tilde{g}}\left(\varphi\Psi\frac{\partial}{\partial t} - \Psi\nabla\varphi + \varphi\nabla\Psi\right) d\mu\, dt \\ &= \int_{\partial D}\left\langle \varphi\Psi\frac{\partial}{\partial t} - \Psi\nabla\varphi + \varphi\nabla\Psi, \tilde{\nu}\right\rangle_{\tilde{g}} d\tilde{A},\end{aligned}$$

where $\tilde{\nu}$ is the unit outward normal and $d\tilde{A}$ the area element of ∂D, both taken with respect to $\tilde{g}$. For $s \geqq 0$, we follow [10] in defining

$$D_r^s = \{(x,\tau) \in E_r : \tau > s\}$$

and two portions of its space-time boundary,

$$P_1^s = \{(x,\tau) : \Psi = r^{-n}, \tau > s\} \quad \text{and} \quad P_2^s = \{(x,\tau) \in \overline{D}_r^s : \tau = s\}.$$

Applying (5.7) to D_r^s yields

$$\begin{aligned} 0 &= \int_{D_r^s}\left(\frac{\partial\varphi}{\partial t} - \Delta\varphi\right)\Psi\, d\mu\, dt \\ &= \int_{P_2^s}\varphi\Psi\, d\mu + \int_{P_1^s}\left\langle \varphi\Psi\frac{\partial}{\partial t} - \Psi\nabla\varphi + \varphi\nabla\Psi, \tilde{\nu}\right\rangle_{\tilde{g}} d\tilde{A} \\ &= \int_{P_2^s}\varphi\Psi\, d\mu + \frac{1}{r^n}\int_{P_1^s}\left\langle \varphi\frac{\partial}{\partial t} - \nabla\varphi, \tilde{\nu}\right\rangle_{\tilde{g}} d\tilde{A} + \int_{P_1^s}\varphi\langle\nabla\Psi, \tilde{\nu}\rangle_{\tilde{g}}\, d\tilde{A}.\end{aligned}$$

Letting $s \searrow 0$, we obtain

$$\begin{aligned}\varphi(\bar{x},0) &= \lim_{s\searrow 0}\int_{P_2^s}\varphi\Psi\, d\mu \\ &= -\frac{1}{r^n}\int_{P_1^0}\left\langle \varphi\frac{\partial}{\partial t} - \nabla\varphi, \tilde{\nu}\right\rangle_{\tilde{g}} d\tilde{A} - \int_{P_1^0}\varphi\langle\nabla\Psi, \tilde{\nu}\rangle_{\tilde{g}}\, d\tilde{A} \\ &= -\frac{1}{r^n}\int_{D_r^0}\left(\frac{\partial}{\partial t} - \Delta\right)\varphi\, d\mu\, dt + \int_{P_1^0}\varphi\frac{|\nabla\Psi|^2}{\sqrt{|\Psi_t|^2 + |\nabla\Psi|^2}} d\tilde{A}.\end{aligned}$$

Summing together and noticing that $P_1^0 = \partial E_r$, we get the following mean-value theorem, which is naturally related to Corollary 23 by the coarea formula.

Theorem 25. *Let $(\mathscr{M}^n, g)$ be a complete fixed manifold. Let Ψ denote the conjugate heat kernel with singularity at $(x,\tau) = (\bar{x}, 0)$. If a smooth function φ of (x,t) solves the heat equation, then*

$$\varphi(\bar{x},\bar{t}) = \int\limits_{\partial E_r} \frac{|\nabla\Psi|^2}{\sqrt{|\Psi_t|^2 + |\nabla\Psi|^2}} \varphi \, d\tilde{A}.$$

For the ε-regularity theorems for Ricci flow derived by the third author [27], we need a mean-value inequality for nonnegative supersolutions. For this purpose, assume that the Ricci curvature of $(\mathscr{M}^n, g)$ satisfies $\mathrm{Rc} \geqq (n-1)kg$ for some $k \in \{-1, 0, 1\}$. Let $(\mathscr{M}_k^n, \tilde{g})$ denote the simply connected space form of constant sectional curvature k, and let Ψ_k denote its conjugate heat kernel centered at $\tilde{x} \in \mathscr{M}_k^n$. Then there exists $\tilde{\Psi}_k : [0,\infty) \times (0,\infty) \to (0,\infty)$ such that $\Psi_k(x,\tau) = \tilde{\Psi}_k\big(d_k(\tilde{x},x),\tau\big)$, where d_k denotes the distance function of $(\mathscr{M}_k^n, \tilde{g})$.

Fix an origin $(\bar{x},\bar{\tau}) \in \mathscr{M}^n \times \mathbb{R}$. Again let $\tau := \bar{t} - t$, and let $\tilde{\Psi}$ denote the transplant of Ψ_k to $(\mathscr{M}^n, g)$, i.e.

$$\tilde{\Psi}(x,\tau) := \tilde{\Psi}_k\big(d_g(\bar{x},x),\tau\big). \tag{5.8}$$

As above, let $\tilde{\psi}_{(r)} = \log(r^n\tilde{\Psi})$ and $\tilde{E}_r = \{(x,\tau) \in \mathscr{M}^n \times \mathbb{R} : \tilde{\psi}_{(r)}(x,\tau) > 0\}$. Define

$$\tilde{I}^{(\bar{x},0)}(r) := \frac{1}{r^n} \int\limits_{\tilde{E}_r} |\nabla \log\tilde{\Psi}|^2 \, d\mu \, dt. \tag{5.9}$$

Then the following mean-value inequality follows from Theorem 7.

Corollary 26. *Let $(\mathscr{M}^n, g)$ be a complete Riemannian manifold such that $\mathrm{Rc} \geqq (n-1)kg$ for some $k \in \{-1,0,1\}$. Let $\tilde{\Psi}$ be defined by (5.8), and let $\varphi \geqq 0$ be any smooth supersolution of the heat equation, i.e. $\left(\frac{\partial}{\partial t} - \Delta\right)\varphi \geqq 0$. Then*

$$\varphi(\bar{x},\bar{t}) \geqq \frac{1}{r^n} \int\limits_{\tilde{E}_r} |\nabla \log\tilde{\Psi}|^2 \varphi \, d\mu \, dt.$$

In particular, $\tilde{I}^{(\bar{x},0)}(r) \leqq 1$ holds for all $r > 0$, and $\frac{d}{dr}\tilde{I}^{(\bar{x},0)}(r) \leqq 0$ holds in the sense of distributions.

If equality holds for $\varphi \equiv 1$, then the largest metric ball in $\tilde{E}_r$ is isometric to the corresponding ball in the simply-connected space form of constant sectional curvature k.

Proof. The inequalities follow from Theorem 7 by the results of Cheeger-Yau [3] that $\left(\frac{\partial}{\partial\tau} - \Delta\right)\tilde{\Psi}(x,\tau) \leqq 0$ and $\tilde{\Psi}(x,\tau) \geqq \Psi(x,\tau)$, where Ψ is the conjugate heat kernel of $(\mathscr{M}^n, g)$. The implication of equality is a consequence of the rigidity derived from equality in the Bishop volume comparison theorem. (See [2].) $\square$

6. Average energy for Ricci flow

Again assume $\big(\mathscr{M}^n, g(t)\big)$ is a smooth complete solution of Ricci flow for $t \in [0, \bar{t}]$. Let Ψ denote a fundamental solution to the conjugate heat equation

$$\left(\frac{\partial}{\partial t} + \Delta - R\right)\Psi = 0 \tag{6.1}$$

centered at $(\bar{x}, \bar{t})$. The traditional notation in this case is $\Psi = e^{-f}$, i.e. $f := -\psi$.

Perelman [28] has discovered that the average energy

$$\mathscr{F}(t) = \int\limits_{\mathscr{M}^n \times \{t\}} (\Delta f + R) e^{-f}\, d\mu = \int\limits_{\mathscr{M}^n \times \{t\}} (|\nabla f|^2 + R) e^{-f}\, d\mu$$

is a monotonically (weakly) increasing function of t. Our result in this situation gives a quantity which is not just monotonic, but constant in its parameter.

Corollary 27. *Suppose that $\big(\mathscr{M}^n, g(t)\big)$ is a smooth, compact solution of Ricci flow for $t \in [0, \bar{t}]$, with $\bar{t} < \infty$. Suppose further that $\Psi : \mathscr{M}^n \times [0, \bar{t}] \to (0, \infty)$ is a fundamental solution of* (6.1) *with singularity at $(\bar{x}, \bar{t})$. Define $f := -\log \Psi$.*

Then for all $\bar{f} \in \mathbb{R}$ below some threshold value, we have

$$\int\limits_{\{f < \bar{f}\}} (\Delta f + R) e^{-\bar{f}}\, d\mu\, dt = 1,$$

where

$$\{f < \bar{f}\} := \{(x, t) \in \mathscr{M}^n \times [a, b) : f(x, t) < \bar{f}\}.$$

Proof. The arguments in Section 5 (above) verify that the hypotheses of Lemma 9 are satisfied. Since

$$\frac{\partial \psi}{\partial t} + |\nabla \psi|^2 = -\Delta \psi + R,$$

one then has

$$P_{1,\Psi}(r) = \int\limits_{E_r} (-\Delta \psi + R)\, d\mu\, dt = \int\limits_{E_r} (\Delta f + R)\, d\mu\, dt.$$

At this point, we change variables from r to $\bar{f} := n \log r$. We then get

$$\frac{P_{1,\Psi}(r)}{r^n} = \int\limits_{\{f < \bar{f}\}} (\Delta f + R) e^{-\bar{f}}\, d\mu\, dt,$$

whence the conclusion follows from Corollary 23 in Section 5. □

7. Appendix. Simple estimates for reduced geometry

For the convenience of the reader, we provide certain elementary estimates involving reduced geometry in a form adapted to this paper. The reader should note that most of the estimates solely for reduced distance are essentially contained in Ye's notes [38], though not always in the form stated here. (Also see [4].)

Notation. Assume that $\big(\mathscr{M}^n, g(\tau)\big)$ is a smooth one-parameter family of complete (possibly noncompact) manifolds satisfying $\frac{\partial}{\partial\tau} g = 2\,\mathrm{Rc}$ for $0 \leqq \tau \leqq \bar{\tau}$. Unless otherwise noted, all Riemannian quantities are measured with respect to $g(\tau)$. All quantities in reduced geometry are calculated with respect to a fixed origin $\mathscr{O} = (\bar{x}, 0)$. We denote the metric distance from x to y with respect to $g(\tau)$ by $d_\tau(x, y)$ and write $d_\tau(x) = d_\tau(\bar{x}, x)$. We define $B_\tau(x, r) = \{y \in \mathscr{M}^n : d_\tau(x, y) < r\}$ and write $B_\tau(r) = B_\tau(\bar{x}, r)$. Perelman's *space-time action* $\mathscr{L}$, *reduced distance* ℓ, and *reduced volume density* v are defined above in (4.1), (4.2), and (4.3), respectively. We will also use the *space-time distance* $L(x, \tau) := \inf\{\mathscr{L}(\gamma) : \gamma(0) = (\bar{x}, 0), \gamma(\tau) = (x, \tau)\}$.

7.1. Bounds for reduced distance. Given $k \geqq 0$ and $K \geqq 0$, define

$$\underline{\ell}(x, \tau) = e^{-2k\tau}\frac{d_0^2(x)}{4\tau} - \frac{nk}{3}\tau \tag{7.1}$$

and

$$\bar{\ell}(x, \tau) = e^{2K\tau}\frac{d_0^2(x)}{4\tau} + \frac{nK}{3}\tau. \tag{7.2}$$

Our first observation directly follows Ye [38].

Lemma 28. *The reduced distance $\ell(x, \tau)$ has the following properties.*

(1) *If there is $k \geqq 0$ such that* $\mathrm{Rc} \geqq -kg$ *on* $\mathscr{M}^n \times [0, \bar{\tau}]$, *then* $\ell(x, \tau) \geqq \underline{\ell}(x, \tau)$.

(2) *If there is $K \geqq 0$ such that* $\mathrm{Rc} \leqq Kg$ *on* $\mathscr{M}^n \times [0, \bar{\tau}]$, *then* $\ell(x, \tau) \leqq \bar{\ell}(x, \tau)$.

Proof. (1) Observe that $g(\tau) \geqq e^{-2k\tau} g(0)$. By (4.1), the $\mathscr{L}$-action of an arbitrary path γ from $(\bar{x}, 0)$ to (x, τ) is

$$\begin{aligned}
\mathscr{L}(\gamma) &= \int_0^{\sqrt{\tau}} \left(\frac{1}{2}\left|\frac{d\gamma}{ds}\right|^2 + 2s^2 R\right) ds \\
&\geqq \frac{1}{2} e^{-2k\tau} \int_0^{\sqrt{\tau}} \left|\frac{d\gamma}{ds}\right|_0^2 ds - 2nk \int_0^{\sqrt{\tau}} s^2\, ds \\
&\geqq e^{-2k\tau}\frac{d_0^2(x)}{2\sqrt{\tau}} - \frac{2nk}{3}\tau^{3/2}.
\end{aligned}$$

Since γ was arbitrary, one has $\ell(x, \tau) = \frac{1}{2\sqrt{\tau}} \inf_\gamma \mathscr{L}(\gamma) \geqq \underline{\ell}(x, \tau)$.

(2) Observe that $g(\tau) \leqq e^{2K\tau} g(0)$. Let β be a path from $(\bar{x}, 0)$ to (x, τ) that is minimal and of constant speed with respect to $g(0)$. Then as above,

$$\mathscr{L}(\beta) \leqq e^{2K\tau} \frac{d_0^2(x)}{2\sqrt{\tau}} + \frac{2nK}{3} \tau^{3/2}.$$

Hence $\ell(x, \tau) \leqq \dfrac{1}{2\sqrt{\tau}} \mathscr{L}(\beta) \leqq \bar{\ell}(x, \tau)$. $\square$

Remark 29. If $\mathrm{Rc} \geqq -kg$ on $\mathscr{M}^n \times [0, \bar{\tau}]$, it follows from Part (1) of Lemma 28 (by standard arguments) that minimizing $\mathscr{L}$-geodesics exist and are smooth.

7.2. Bounds for reduced-volume heatballs. Recall that the reduced-volume density is $v(x, \tau) = (4\pi\tau)^{-n/2} e^{-\ell(x,\tau)}$. For $r > 0$, define the reduced-volume heatball

$$E_r = \{(x, \tau) \in \mathscr{M}^n \times (0, \bar{\tau}] : v(x, \tau) > r^{-n}\} \tag{7.3}$$

$$= \left\{(x, \tau) \in \mathscr{M}^n \times (0, \bar{\tau}] : \ell(x, \tau) < n \log \frac{r}{\sqrt{4\pi\tau}}\right\} \tag{7.4}$$

and define $c(k, \bar{\tau})$ by

$$c = \frac{e^{4k\bar{\tau}/3}}{4\pi}. \tag{7.5}$$

Given $r > 0$, $k \geqq 0$, $\tau > 0$, define

$$\rho(r, k, \tau) = e^{k\tau} \sqrt{\left(2n\tau \log \frac{r^2}{4\pi\tau} + \frac{4}{3} nk\tau^2\right)_+}. \tag{7.6}$$

Note that $\rho(r, 0, \tau)$ agrees with $R_r(\tau)$ in [9]. It is easy to see that for each $r > 0$ and $k \geqq 0$, one has $\rho(r, k, \tau) > 0$ for all sufficiently small $\tau > 0$.

Remark 30. If $\mathrm{Rc} \geqq -kg$ on $\mathscr{M}^n \times [0, \bar{\tau}]$, then Part (1) of Lemma 28 implies that $(x, \tau) \in E_r$ only if $x \in B_0\big(\rho(r, k, \tau)\big)$.

Lemma 31. *Assume* $0 < r^2 \leqq \min\{\bar{\tau}/c, 4\pi\}$. *If* $cr^2 \leqq \tau \leqq \bar{\tau}$, *then* $\rho(r, k, \tau) = 0$.

Proof. When $\tau = cr^2$, one has

$$\frac{k}{3}\tau + \frac{1}{2} \log \frac{r^2}{4\pi\tau} \leqq \frac{k\bar{\tau}}{3} + \frac{1}{2} \log \frac{1}{4\pi c} = -\frac{k\bar{\tau}}{3} \leqq 0,$$

while for $cr^2 \leqq \tau \leqq \bar{\tau}$, one has

$$\begin{aligned} \frac{\partial}{\partial\tau}\left(\frac{k}{3}\tau^2 + \frac{1}{2}\tau \log \frac{r^2}{4\pi\tau}\right) &= \frac{2k}{3}\tau + \frac{1}{2} \log \frac{r^2}{4\pi\tau} - \frac{1}{2} \\ &\leqq \frac{2k}{3}\bar{\tau} + \frac{1}{2} \log \frac{1}{4\pi c} - \frac{1}{2} \leqq -\frac{1}{2}. \quad \square \end{aligned}$$

Corollary 32. *Assume that* $\mathrm{Rc} \geqq -kg$ *on* $\mathscr{M}^n \times [0, \bar{\tau}]$ *for some* $k \geqq 0$ *and that* $0 < r^2 \leqq \min\{\bar{\tau}/c, 4\pi\}$. *Then*

$$E_r \subseteqq \bigcup_{0<\tau<cr^2} B_0\big(\rho(r,k,\tau)\big) \times \{\tau\}.$$

7.3. Gradient estimates for reduced distance. Local gradient estimates for curvatures evolving by Ricci flow originated in [32], §7. Recall the following version.

Proposition 33 (Hamilton [19], §13). *Suppose* $g(\tau)$ *solves backward Ricci flow for* $\tau_0 \leqq \tau \leqq \tau_1$ *on an open set* $\mathscr{U}$ *of* $\mathscr{M}^n$ *with* $\bar{B}_{\tau_1}(x, 2\lambda) \subset \mathscr{U}$. *There exists* C_n *depending only on n such that if* $|\mathrm{Rm}| \leqq M$ *on* $\mathscr{U} \times [\tau_0, \tau_1]$, *then*

$$|\nabla \mathrm{Rm}| \leqq C_n M \sqrt{\frac{1}{\lambda^2} + \frac{1}{\tau_1 - \tau} + M}$$

on $B_{\tau_1}(x, \lambda) \times [\tau_0, \tau_1)$.

If there is a global bound on curvature, the situation is quite simple:

Remark 34. If $|\mathrm{Rm}| \leqq M$ on $\mathscr{M}^n \times [0, \bar{\tau}]$, then for every $\tau^* < \bar{\tau}$ there exists $A = A(n, M, \tau^*)$ such that $|\nabla R| \leqq A$ on $\mathscr{M}^n \times [0, \tau^*]$.

More generally, the following 'localization lemma' often provides adequate local bounds.

Lemma 35. *Assume* $\mathrm{Rc} \geqq -kg$ *on* $\mathscr{M}^n \times [0, \bar{\tau}]$. *Then for every* $\lambda > 0$ *and* $\tau^* \in (0, \bar{\tau})$, *there exists* λ^* *such that the image of any minimizing* $\mathscr{L}$*-geodesic from* $(\bar{x}, 0)$ *to any* $(x, \tau) \in B_0(\lambda) \times (0, \tau^*]$ *is contained in* $B_0(\lambda^*)$. *In particular, there exist constants* C, C' *depending only on* $|\mathrm{Rm}|$ *in a space-time cylinder* $\Omega(\lambda, \tau^*, \bar{\tau})$ *such that* $\mathrm{Rc} < Cg$ *and* $|\nabla R| \leqq C'$ *on* $B_0(\lambda^*) \times [0, \tau^*]$.

Proof. By smoothness, there exists K such that $\mathrm{Rc} \leqq Kg$ on $B_0(\lambda) \times [0, \tau^*]$. Applying Part (2) of Lemma 28 along radial geodesics from $\bar{x}$ shows that

$$\sup_{(x,\tau) \in B_0(\lambda) \times [0, \tau^*]} [\tau \ell(x, \tau)] \leqq e^{2K\tau^*} \frac{\lambda^2}{4} + \frac{nK}{3} (\tau^*)^2.$$

Define

$$\lambda^* = 2e^{k\tau^*} \sqrt{e^{2K\tau^*} \frac{\lambda^2}{4} + \frac{n}{3}(k + K)(\tau^*)^2}.$$

Let $(x, \tau) \in B_0(\lambda) \times (0, \tau^*]$ be arbitrary and let γ be any minimizing $\mathscr{L}$-geodesic from $(\bar{x}, 0)$ to (x, τ). Then for every $\sigma \in [0, \tau]$, one obtains

$$d_0\big(\gamma(\sigma)\big) \leqq 2e^{k\tau} \sqrt{\tau \ell(x, \tau) + \frac{nk}{3} \tau^2} < \lambda^*$$

by following the proof of Part (1) of Lemma 28. This proves that the image of γ is contained in $B_0(\lambda^*)$.

Now define $\tau' = \tau^* + \frac{1}{2}(\bar{\tau} - \tau^*)$ and choose λ' large enough that $B_0(\lambda^*) \subseteqq B_{\tau'}(\lambda')$. By smoothness, there exists M such that $|\mathrm{Rm}| \leqq M$ on $B_{\tau'}(3\lambda') \times [0, \tau']$. So by Proposition 33, there exists C' such that $|\nabla R| \leqq C'$ on $B_{\tau'}(\lambda') \times [0, \tau^*]$. Clearly, $\mathrm{Rc} \leqq Cg$ on $B_{\tau'}(\lambda') \times [0, \tau^*]$ as well. □

Lemma 36. *Assume that there exists an open set $\mathscr{U} \subseteqq \mathscr{M}^n$ and $0 \leqq \tau_0 \leqq \tau_1 \leqq \bar{\tau}$ such that $|\nabla R| \leqq A$ on $\mathscr{U} \times [\tau_0, \tau_1]$. Let $\gamma : [0, \tau_1] \to \mathscr{U}$ be an $\mathscr{L}$-geodesic and let $\Gamma(\tau_0) = \lim_{\tau \searrow \tau_0} \left(\sqrt{\tau} \left| \frac{d\gamma}{d\tau} \right| \right)$, which is well defined for all $\tau_0 \geqq 0$.*

(1) *If $\mathrm{Rc} \geqq -kg$ on $\mathscr{U} \times [\tau_0, \tau_1]$, then for all $\tau \in [\tau_0, \tau_1]$, one has*

$$\left| \frac{d\gamma}{d\tau} \right| \leqq \frac{1}{2\sqrt{\tau}} \left[\left(2\Gamma(\tau_0) + \frac{A}{k}\sqrt{\tau_1} \right) e^{k(\tau - \tau_0)} - \frac{A}{k}\sqrt{\tau_1} \right] \quad (k > 0)$$

$$\leqq \frac{1}{2\sqrt{\tau}} [2\Gamma(\tau_0) + A\sqrt{\tau_1}(\tau - \tau_0)] \quad (k = 0).$$

(2) *If $\mathrm{Rc} \leqq Kg$ on $\mathscr{U} \times [\tau_0, \tau_1]$, then for all $\tau \in [\tau_0, \tau_1]$, one has*

$$\left| \frac{d\gamma}{d\tau} \right| \geqq \frac{1}{2\sqrt{\tau}} \left[\left(2\Gamma(\tau_0) + \frac{A}{K}\sqrt{\tau_1} \right) e^{K(\tau_0 - \tau)} - \frac{A}{K}\sqrt{\tau_1} \right] \quad (K > 0)$$

$$\geqq \frac{1}{2\sqrt{\tau}} [2\Gamma(\tau_0) + A\sqrt{\tau_1}(\tau_0 - \tau)] \quad (K = 0).$$

Proof. It will be more convenient to regard γ as a function of $s = \sqrt{\tau}$. Let $\dot{\gamma} = \frac{d\gamma}{d\tau}$ and $\gamma' = \frac{d\gamma}{ds} = 2s\dot{\gamma}$. The Euler-Lagrange equation satisfied by γ is

$$\nabla_{\dot{\gamma}}\dot{\gamma} = \frac{1}{2}\nabla R - 2\,\mathrm{Rc}(\dot{\gamma}) - \frac{1}{2\tau}\dot{\gamma}.$$

In terms of s, this becomes

$$\nabla_{\gamma'}\gamma' = 2s^2\nabla R - 4s\,\mathrm{Rc}(\gamma'),$$

which is nonsingular at $s = 0$. The computation

$$\frac{d}{ds}|\gamma'|^2 = \frac{d\tau}{ds}\frac{\partial}{\partial\tau} g(\gamma', \gamma') + 2g(\nabla_{\gamma'}\gamma', \gamma')$$

$$= 4s^2\langle \nabla R, \gamma' \rangle - 4s\,\mathrm{Rc}(\gamma', \gamma')$$

shows that $|\gamma'|$ satisfies the differential inequalities

$$\frac{d}{ds}|\gamma'| \leqq 2ks|\gamma'| + 2As^2 \tag{7.7}$$

and

$$\frac{d}{ds}|\gamma'| \geqq -2Ks|\gamma'| - 2As^2. \tag{7.8}$$

Let $s_0 = \sqrt{\tau_0}$ and $s_1 = \sqrt{\tau_1}$. Define

$$\bar{\psi}(s) = \left(|\gamma'(s_0)| + \frac{As_1}{k}\right)e^{k(s^2 - s_0^2)} - \frac{As_1}{k}$$

and

$$\underline{\psi}(s) = \left(|\gamma'(s_0)| + \frac{As_1}{K}\right)e^{K(s_0^2 - s^2)} - \frac{As_1}{K},$$

replacing these by their limits if either k or K is zero. Note that $\underline{\psi}(s_0) = |\gamma'(s_0)| = \bar{\psi}(s_0)$. It is readily verified that $\bar{\psi}$ is a supersolution of (7.7) and that $\underline{\psi}$ is a subsolution of (7.8). So one has $\underline{\psi}(s) \leqq \left|\frac{d\gamma}{ds}\right| \leqq \bar{\psi}(s)$ for $s_0 \leqq s \leqq s_1$, as claimed. □

Corollary 37. *Assume that* $\mathrm{Rc} \geqq -kg$ *on* $\mathscr{M}^n \times [0, \bar{\tau}]$. *Then for any* $\lambda > 0$ *and* $\tau^* \in (0, \bar{\tau})$, *there exist positive constants* η *and* C *such that for any minimizing* $\mathscr{L}$*-geodesic* γ *from* $(\bar{x}, 0)$ *to* $(x, \tau) \in B_0(\lambda) \times (0, \tau^*]$, *one has*

$$\min_{[0,\tau]}\left(\sqrt{\sigma}\left|\frac{d\gamma}{d\sigma}\right|\right) \geqq \eta \max_{[0,\tau]}\left(\sqrt{\sigma}\left|\frac{d\gamma}{d\sigma}\right|\right) - C.$$

Furthermore, for all $\sigma \in (0, \tau]$, *one has*

$$\left|\frac{d\gamma}{d\sigma}\right|^2 \leqq \frac{2}{\eta^2}\left[\frac{\ell(\gamma(\tau), \tau) + C^2}{\sigma} + \frac{nk}{3}\right].$$

Proof. By Lemma 35, there exists a neighborhood $\mathscr{U}$ containing the image of γ such that $\mathrm{Rc} \leqq Kg$ and $|\nabla R| \leqq A$ in $\mathscr{U} \times [0, \tau^*]$. Using this, the first statement is easy to verify.

To prove the second statement, let $x = \gamma(\tau)$, so that $L(x, \tau) = \mathscr{L}(\gamma)$. Then as in Lemma 28, one has

$$L(x, \tau) + \frac{2nk}{3}\tau^{3/2} \geqq \int_0^{\hat{\tau}} \sqrt{\sigma}\left|\frac{d\gamma}{d\sigma}\right|^2 d\sigma$$

for any $\hat{\tau} \in (0, \tau]$. Let $\psi = \min_{[0,\tau]}\left(\sqrt{\sigma}\left|\frac{d\gamma}{d\sigma}\right|\right)$ and $\Psi = \max_{[0,\tau]}\left(\sqrt{\sigma}\left|\frac{d\gamma}{d\sigma}\right|\right)$. Then for any $\delta \in (0, \hat{\tau})$ one has

$$L(x,\tau)+\frac{2nk}{3}\tau^{3/2} \geqq 2\sqrt{\hat{\tau}}\psi^2$$
$$\geqq 2\sqrt{\hat{\tau}}\left(\frac{\eta^2}{2}\Psi^2 - C^2\right)$$
$$\geqq \frac{\eta^2}{2}\frac{\sqrt{\hat{\tau}}}{\sqrt{\hat{\tau}}-\sqrt{\hat{\tau}-\delta}}\int_{\hat{\tau}-\delta}^{\hat{\tau}}\sqrt{\sigma}\left|\frac{d\gamma}{d\sigma}\right|^2 d\sigma - 2C^2\sqrt{\hat{\tau}}.$$

Consequently, one obtains

$$\int_{\hat{\tau}-\delta}^{\hat{\tau}}\sqrt{\sigma}\left|\frac{d\gamma}{d\sigma}\right|^2 d\sigma \leqq \frac{\delta}{\eta^2\hat{\tau}}\left[L(x,\tau)+2C^2\sqrt{\hat{\tau}}+\frac{2nk}{3}\tau^{3/2}\right], \tag{7.9}$$

whence the second statement follows. □

Lemma 38. *Assume* $\mathrm{Rc} \geqq -kg$ *on* $\mathscr{M}^n \times [0,\bar{\tau}]$. *Let* $\lambda > 0$ *and* $\tau^* < \bar{\tau}$ *be given.*

(1) *There exists* C *such that for all* $x \in B_0(\lambda)$ *and* $\tau \in (0,\tau^*]$, *one has*

$$|L(x,\tau\pm\delta) - L(x,\tau)| \leqq C\left(\frac{1}{\sqrt{\tau}}+\sqrt{\tau}\right)\delta$$

whenever $\delta \in (0,\tau/3)$ *and* $\tau \pm \delta \in [0,\tau^*]$.

(2) *There exists* C *such that for all* $x \in B_0(\lambda)$ *and* $\tau \in (0,\tau^*]$, *one has*

$$|L(x,\tau\pm\delta) - L(x,\tau)| \leqq C\left(\frac{L}{\tau}+\frac{1}{\sqrt{\tau}}+\sqrt{\tau}\right)\delta$$

whenever $\delta \in (0,\tau/3)$ *and* $\tau \pm \delta \in [0,\tau^*]$.

Proof. Let α be a minimizing $\mathscr{L}$-geodesic from $(0,\bar{x})$ to (x,τ). By Lemma 35, we may assume that $\mathrm{Rc} < Kg$ and $|\nabla R| \leqq A$ in $\mathscr{U} \times [0,\tau^*]$, where $\mathscr{U}$ is a neighborhood of the image of α.

To bound L at a later time in terms of L at an earlier time, let β denote the constant path $\beta(\sigma) = x$ for $\tau \leqq \sigma \leqq \tau+\delta$. Because α is minimizing and $\mathscr{L}$ is additive, one has $L(x,\tau) = \mathscr{L}(\alpha)$ and $L(x,\tau+\delta) \leqq \mathscr{L}(\alpha)+\mathscr{L}(\beta)$. Hence there exists C_n depending only on n such that

$$L(x,\tau+\delta) - L(x,\tau) \leqq \mathscr{L}(\beta) = \int_{\tau}^{\tau+\delta}\sqrt{\sigma}R\,d\sigma \leqq [C_n(k+K)\sqrt{\tau}]\delta.$$

To bound L at an earlier time in terms of L at a later time, define a path γ from $(\bar{x},0)$ to $(x,\tau-\delta)$ by

$$\gamma(\sigma) = \alpha(\sigma), \qquad 0 \leqq \sigma \leqq \tau - 2\delta,$$
$$\gamma(\sigma) = \alpha\big(2\sigma - (\tau - 2\delta)\big), \quad \tau - 2\delta < \sigma \leqq \tau - \delta.$$

Observe that the image of γ lies in $\mathscr{U}$ and that

$$\mathscr{L}(\gamma) \leqq \mathscr{L}(\alpha) - \int_{\tau-2\delta}^{\tau} \sqrt{\sigma} R\big(\alpha(\sigma)\big)\, d\sigma + 4\int_{\tau-2\delta}^{\tau-\delta} \sqrt{\sigma}\left|\frac{d\alpha}{d\sigma}\right|^2 d\sigma + \int_{\tau-2\delta}^{\tau-\delta} \sqrt{\sigma} R\big(\gamma(\sigma)\big)\, d\sigma.$$

By Part (1) of Lemma 36, there exists C' such that $\left|\frac{d\alpha}{d\sigma}\right|^2 \leqq C'/\tau$ for $\sigma \geqq \tau - 2\delta \geqq \tau/3$. Since $\mathscr{L}(\alpha) = L(x,\tau)$, it follows that

$$L(x,\tau-\delta) - L(x,\tau) \leqq \mathscr{L}(\gamma) - \mathscr{L}(\alpha) \leqq C_n\left[\frac{C'}{\sqrt{\tau}} + (k+K)\sqrt{\tau}\right]\delta.$$

This proves the first statement.

To prove the second statement, use (7.9) to estimate $\int_{\tau-2\delta}^{\tau-\delta} \sqrt{\sigma}\left|\frac{d\alpha}{d\sigma}\right|^2 d\sigma$. □

Lemma 39. *If* $\mathrm{Rc} \geqq -kg$ *on* $\mathscr{M}^n \times [0,\bar{\tau}]$, *then* $\ell : \mathscr{M}^n \times (0,\bar{\tau})$ *is locally Lipschitz.*

(1) *For any* $\lambda > 0$ *and* $\tau^* < \bar{\tau}$, *there exists* C *such that*

$$\left|\ell_\tau + \frac{\ell}{2\tau}\right| \leqq C\left(\frac{1}{\tau} + 1\right)$$

everywhere in $B_0(\lambda)$ *and almost everywhere in* $(0,\tau^*]$, *and such that*

$$|\nabla \ell| \leqq C\left(\frac{1}{\tau} + 1\right)$$

everywhere in $(0,\tau^*]$ *and almost everywhere in* $B_0(\lambda)$.

(2) *There exists* C *such that*

$$|\ell_\tau| \leqq C\left(\frac{\ell+1}{\tau} + 1\right)$$

everywhere in $\mathscr{M}^n$ *and almost everywhere in* $(0,\tau^*]$, *and such that*

$$|\nabla \ell|^2 \leqq C\left(\frac{\ell+1}{\tau} + 1\right)$$

everywhere in $(0,\tau^*]$ *and almost everywhere in* $\mathscr{M}^n$.

Proof. We again apply Lemma 35 to get bounds $\mathrm{Rc} \leqq Kg$ and $|\nabla R| \leqq A$ on $B_0(2\lambda^*) \times [0, \tau^*]$, where $B_0(\lambda^*)$ is a neighborhood of any minimizing geodesic from $(\bar{x}, 0)$ to a point $(x, \tau) \in B_0(\lambda) \times (0, \tau^*]$.

Wherever it is smooth, ℓ satisfies $\ell_\tau + \dfrac{\ell}{2\tau} = \dfrac{1}{2\sqrt{\tau}} L_\tau$. Thus local Lipschitz continuity in time and the estimates for ℓ_τ follow directly from Lemma 38 and Rademacher's Theorem.

To show local Lipschitz continuity in space, let $x, y \in B_0(\lambda)$ and $\tau \in (0, \tau^*]$ be given. We may assume that $d_\tau(x, y) \in (0, \tau/3)$. Let α be a minimizing $\mathscr{L}$-geodesic from $(\bar{x}, 0)$ to (x, τ) and let β be a unit-speed $g(\tau)$-geodesic from x to y. Let $\delta = d_\tau(x, y)$ and define a path γ from $(\bar{x}, 0)$ to (y, τ) by

$$\begin{aligned}
\gamma(\sigma) &= \alpha(\sigma), && 0 \leqq \sigma \leqq \tau - 2\delta, \\
\gamma(\sigma) &= \alpha\big(2\sigma - (\tau - 2\delta)\big), && \tau - 2\delta < \sigma \leqq \tau - \delta, \\
\gamma(\sigma) &= \beta\big(\sigma - (\tau - \delta)\big), && \tau - \delta < \sigma \leqq \tau.
\end{aligned}$$

Observe that the image of γ belongs to $B_0(2\lambda^*)$. Exactly as in the proof of Lemma 38, one finds there exist C_n and C' such that

$$\begin{aligned}
\mathscr{L}(\gamma) &\leqq \mathscr{L}(\alpha) - \int_{\tau-2\delta}^{\tau} \sqrt{\sigma} R\big(\alpha(\sigma)\big)\, d\sigma \\
&\quad + 4 \int_{\tau-2\delta}^{\tau-\delta} \sqrt{\sigma} \left|\frac{d\alpha}{d\sigma}\right|^2 d\sigma + \int_{\tau-\delta}^{\tau} \sqrt{\sigma} \left|\frac{d\beta}{d\sigma}\right|^2 d\sigma + \int_{\tau-2\delta}^{\tau} \sqrt{\sigma} R\big(\gamma(\sigma)\big)\, d\sigma \\
&\leqq \mathscr{L}(\alpha) + C_n \left[\frac{C'}{\sqrt{\tau}} + (k + K + e^{k\tau^*})\sqrt{\tau}\right]\delta.
\end{aligned}$$

Since α is minimizing, this implies that

$$L(y, \tau) - L(x, \tau) \leqq C\left(\frac{1}{\sqrt{\tau}} + \sqrt{\tau}\right) d_\tau(x, y).$$

Reversing the roles of x and y gives the same inequality for $L(x, \tau) - L(y, \tau)$. The first gradient estimate then follows by Rademacher's Theorem.

To prove the second gradient estimate, observe that local Lipschitz continuity of L implies that the $\mathscr{L}$-geodesic cut locus is a set of measure zero. If (x, τ) is not in the cut locus, then the first variation formula [28], (7.1) implies that $\nabla L(x, \tau) = 2\sqrt{\tau} \dfrac{d\alpha}{d\tau}$. The second gradient formula now follows from Corollary 37. □

7.4. Integration over reduced-volume heatballs. If v is the reduced-volume density and $\varphi : \mathscr{M}^n \times (0, \bar{\tau}) \to \mathbb{R}$ is a given function, then the function $P_{\varphi, v}(r)$ defined in (2.4) may be written as

$$P_{\varphi, v}(r) = \int_{E_r} F \varphi \, d\mu \, dt,$$

where

$$F = |\nabla \ell|^2 + R\left(n \log \frac{r}{\sqrt{4\pi\tau}} - \ell\right).$$

Lemma 40. *Assume that* $\mathrm{Rc} \geqq -kg$ *on* $\mathscr{M}^n \times [0, \bar{\tau}]$. *Then for any* $\tau^* \in (0, \bar{\tau})$, *there exists* C *independent of* φ *such that*

$$\frac{|P_{\varphi, v}(r)|}{r^n} \leqq C \sup_{\mathscr{M}^n \times (0, cr^2)} |\varphi|$$

whenever $0 < r^2 \leqq \min\{\tau^*/c, 4\pi\}$, *where* $c = e^{4k\bar{\tau}/3}/(4\pi)$.

Proof. For $0 < \tau \leqq \tau^*$, Part (2) of Lemma 39 implies that

$$|\nabla \ell|^2 \leqq \frac{C\ell + C'}{\tau}$$

almost everywhere in a precompact neighborhood $\mathscr{U}$ of $\bar{x}$. Here and in the rest of the proof, C, C', C'' denote positive constants that may change from line to line. By Corollary 32, we may assume that $\mathscr{U} \times [0, \tau^*]$ contains E_r for all $r > 0$ under consideration. Lemma 28 implies that

$$|\nabla \ell|^2 \leqq C \frac{d_0^2(x)}{\tau^2} + \frac{C'}{\tau}$$

almost everywhere in $\mathscr{U}$. Let $\lambda = e^{-k\tau}\rho(r, k, \tau)/(2\sqrt{n})$, where $\rho(r, k, \tau)$ is defined by (7.6). Together, Lemmata 28 and 31 show that

$$0 < n \log \frac{r}{\sqrt{4\pi\tau}} - \ell \leqq \frac{n}{\tau}\lambda^2 \leqq \frac{C}{\tau}(1 + \tau^2)$$

everywhere in E_r. Hence

$$|F| \leqq |\nabla \ell|^2 + n(k + K)\left(n \log \frac{r}{\sqrt{4\pi\tau}} - \ell\right) \leqq C \frac{d_0^2(x)}{\tau^2} + \frac{C'}{\tau}$$

almost everywhere in E_r. Since the volume forms $d\mu(\tau)$ are all comparable on $B_0(C\lambda) \times [0, \tau^*]$, it follows from the definition (7.6) of $\rho(r, k, \tau) = 2\sqrt{n}e^{k\tau}\lambda$ that

$$\begin{aligned} \int_{B_0(C\lambda)} |F| \, d\mu &\leqq C' \frac{\lambda^{n+2}}{\tau^2} + C'' \frac{\lambda^n}{\tau} \\ &\leqq C' \left[\tau^n + \tau^{\frac{n}{2}-1}\left(\log \frac{r^2}{4\pi\tau}\right)^{\frac{n}{2}+1}\right] + C'' \left[\tau^{n-1} + \tau^{\frac{n}{2}-1}\left(\log \frac{r^2}{4\pi\tau}\right)^{\frac{n}{2}}\right]. \end{aligned}$$

For $r > 0$ and $n \geqq 2$, the substitution $z = \tau/r^2$ shows that

$$\int_0^{cr^2} \tau^{\frac{n}{2}-1}\left(\log\frac{r^2}{4\pi\tau}\right)^{\frac{n+1\pm1}{2}} d\tau = r^n \int_0^c z^{\frac{n}{2}-1}\left(\log\frac{1}{4\pi z}\right)^{\frac{n+1\pm1}{2}} dz \leqq Cr^n.$$

Hence by Corollary 32, one has

$$\int_{E_r} |F|\, d\mu\, dt \leqq \int_0^{cr^2} \left(\int_{B_0(C\lambda)} |F|\, d\mu \right) d\tau \leqq C' r^n$$

whenever $0 < r^2 \leqq \min\{\bar{\tau}/c, 4\pi\}$. The result follows. □

References

[1] *Cao*, *Huai-Dong*, *Hamilton*, *Richard S.*, *Ilmanen*, *Tom*, Gaussian densities and stability for some Ricci solitons, arXiv: math.DG/0404165.

[2] *Cheeger*, *Jeff*, *Ebin*, *David G.*, Comparison theorems in Riemannian geometry, North-Holland Mathematical Library **9**, North-Holland Publishing Co., Amsterdam-Oxford; American Elsevier Publishing Co., Inc., New York 1975.

[3] *Cheeger*, *Jeff*, *Yau*, *Shing-Tung*, A lower bound for the heat kernel, Comm. Pure Appl. Math. **34** (1981), no. 4, 465–480.

[4] *Chow*, *Bennett*, *et al.*, The Ricci Flow: Techniques and Applications, Mathematical Surveys and Monographs, American Mathematical Society, Providence, RI, to appear.

[5] *Ecker*, *Klaus*, A local monotonicity formula for mean curvature flow, Ann. Math. **154** (2001), 503–525.

[6] *Ecker*, *Klaus*, Regularity theory for mean curvature flow, Progr. Nonlin. Diff. Equ. Appl. **57**, Birkhäuser Boston, Inc., Boston, MA, 2004.

[7] *Ecker*, *Klaus*, Local monotonicity formulas for some nonlinear diffusion equations, Calc. Var. Part. Diff. Equ. **23** (2005), no. 1, 67–81.

[8] *Ecker*, *Klaus*, *Huisken*, *Gerhard*, Mean curvature evolution of entire graphs, Ann. Math. (2) **130** (1989), no. 3, 453–471.

[9] *Evans*, *Lawrence C.*, *Gariepy*, *Ronald F.*, Wiener's criterion for the heat equation, Arch. Rat. Mech. Anal. **78** (1982), no. 4, 293–314.

[10] *Fabes*, *E. B.*, *Garofalo*, *N.*, Mean value properties of solutions to parabolic equations with variable coefficients, J. Math. Anal. Appl. **121** (1987), no. 2, 305–316.

[11] *Feldman*, *Mikhail*, *Ilmanen*, *Tom*, *Knopf*, *Dan*, Rotationally symmetric shrinking and expanding gradient Kähler-Ricci solitons, J. Diff. Geom. **65** (2003), no. 2, 169–209.

[12] *Feldman*, *Michael*, *Ilmanen*, *Tom*, *Ni*, *Lei*, Entropy and reduced distance for Ricci expanders, J. Geom. Anal. **15** (2005), no. 1, 49–62.

[13] *Fulks*, *Watson B.*, A mean value theorem for the heat equation, Proc. Amer. Math. Soc. **17** (1966), 6–11.

[14] *Garofalo*, *Nicola*, *Lanconelli*, *Ermanno*, Wiener's criterion for parabolic equations with variable coefficients and its consequences, Trans. Amer. Math. Soc. **308** (1988), no. 2, 811–836.

[15] *Garofalo*, *Nicola*, *Lanconelli*, *Ermanno*, Asymptotic behavior of fundamental solutions and potential theory of parabolic operators with variable coefficients, Math. Ann. **283** (1989), no. 2, 211–239.

[16] *Grigoŕyan*, *Alexander*, Estimates of heat kernels on Riemannian manifolds, Spectral theory and geometry (Edinburgh 1998), London Math. Soc. Lect. Note Ser. **273**, Cambridge Univ. Press, Cambridge 1999.

[17] *Hamilton*, *Richard S.*, A matrix Harnack estimate for the heat equation, Comm. Anal. Geom. **1** (1993), no. 1, 113–126.

[18] *Hamilton*, *Richard S.*, The Harnack estimate for the Ricci flow, J. Diff. Geom. **37** (1993), no. 1, 225–243.

[19] *Hamilton*, *Richard S.*, The formation of singularities in the Ricci flow, Surveys in differential geometry Vol. II (Cambridge, MA, 1993), Internat. Press, Cambridge, MA (1995), 7–136.

[20] *Huisken*, *Gerhard*, Asymptotic behavior for singularities of the mean curvature flow, J. Diff. Geom. **31** (1990), no. 1, 285–299.

[21] *Kleiner*, *Bruce*, *Lott*, *John*, Notes on Perelman's papers (25/05/2006 version), arXiv:math.DG/0605667.

[22] *Li*, *Peter*, *Yau*, *Shing-Tung*, On the parabolic kernel of the Schrödinger operator, Acta Math. **156** (1986), no. 3–4, 153–201.

[23] *Lott*, *John*, *Villani*, *Cedric*, Ricci curvature for metric-measure spaces via optimal transport, arXiv:math.DG/0412127.
[24] *Malliavin*, *Paul*, *Stroock*, *Daniel W.*, Short time behavior of the heat kernel and its logarithmic derivatives, J. Diff. Geom. **44** (1996), no. 3, 550–570.
[25] *Ni*, *Lei*, A matrix Li-Yau-Hamilton inequality for Kähler-Ricci flow, J. Diff. Geom. **75** (2007), 303–358.
[26] *Ni*, *Lei*, A note on Perelman's LYH type inequality, Comm. Anal. Geom. **14** (2006), 883–905.
[27] *Ni*, *Lei*, Mean value theorem on manifolds, Asian J. Math. **11** (2007), 277–304.
[28] *Perelman*, *Grisha*, The entropy formula for the Ricci flow and its geometric applications, arXiv:math.DG/0211159.
[29] *Pini*, *Bruno*, Sulle equazioni a derivate parziali, lineari del secondo ordine in due variabili, di tipo parabolico, Ann. Mat. Pura Appl. (4) **32** (1951), 179–204.
[30] *Pini*, *Bruno*, Maggioranti e minoranti delle soluzioni delle equazioni paraboliche, Ann. Mat. Pura Appl. (4) **37** (1954), 249–264.
[31] *Pini*, *Bruno*, Sulla soluzione generalizzata di Wiener per il primo problema di valori al contorno nel caso parabolico, Rend. Sem. Mat. Univ. Padova **23** (1954), 422–434.
[32] *Shi*, *Wan-Xiong*, Ricci deformation of the metric on complete noncompact Riemannian manifolds, J. Diff. Geom. **30** (1989), no. 2, 303–394.
[33] *Souplet*, *Philippe*, *Zhang*, *Qi S.*, Sharp gradient estimate and Yau's Liouville theorem for the heat equation on noncompact manifolds, arXiv:math.DG/0502079.
[34] *Stroock*, *Daniel W.*, *Turetsky*, *James*, Upper bounds on derivatives of the logarithm of the heat kernel, Comm. Anal. Geom. **6** (1998), no. 4, 669–685.
[35] *Watson*, *Neil A.*, A theory of subtemperatures in several variables, Proc. London Math. Soc. **26** (1973), 385–417.
[36] *Yau*, *Shing-Tung*, Harmonic functions on complete Riemannian manifolds, Comm. Pure Appl. Math. **28** (1975), 201–228.
[37] *Yau*, *Shing-Tung*, Harnack inequality for non-self-adjoint evolution equations, Math. Res. Lett. **2** (1995), no. 4, 387–399.
[38] *Ye*, *Rugang*, Notes on the reduced volume and asymptotic Ricci solitons of κ-solutions, http://www.math.ucsb.edu/~yer/solitonbirth.pdf.

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