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Dynamics of the Bose–Einstein condensation

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Abstract

Bose–Einstein condensation is a very general physical phenomenon which takes place not only in the systems of bosonic atoms, but also in optical wave systems. Many important features of the condensation in such diverse systems can be captured by the nonlinear Schroedinger model. Within this model we develop a statistical description in which the condensate is nonlinearly coupled to wave turbulence described by a kinetic equation. Our focus will be on the strong-condensate regime in which the three-wave interaction replaces the four-wave process operating on the preceding stages of an explosive condensate formation and its initial growth. In the strong-condensate regime, the condensate growth accelerates and becomes quadratic in time. This regime will proceed until the wave dispersion drops below a critical value and the state of dispersionless acoustic turbulence forms.

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1. Introduction

Bose–Einstein condensation [1,2], a phenomenon which is observed at ultra-low temperatures, has been a subject of renewed theoretical interest thanks to recent experimental discoveries in this area [3–5] (a good review of this subject can be found in [6]). A lot of theoretical conclusions about condensate dynamics are based on the assumption that the condensate band is near thermodynamical equilibrium with some temperature T and chemical potential μ. Often, however, the condensation is so rapid that the gas is in a very non-equilibrium state which has to be described using a kinetic rather than thermodynamic theory [9–11]. Such an approach, using the quantum kinetic equation with various phenomenological assumptions about the scattering amplitudes, was developed by Gardiner et al. [9,10].

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For dilute gases with a large number of atoms the condensation can be described by the nonlinear Schroedinger (NLS) equation (also known as Gross–Pitaevsky equation) [7,8]:

$$\mathrm{i}\Psi_t + \Delta\Psi - |\Psi|^2\Psi = 0, \tag{1}$$

where Ψ is the order parameter. The NLS model works when the energy occupation numbers are large so that the effect of the quantum fluctuations is negligible. This and the other underlying assumptions of the NLS model are sometimes violated in real experiments and more realistic models are needed. However, NLS is a good “minimal” model, which allows to achieve a qualitative understanding of the basic condensation mechanisms in more realistic systems as well as in the optical turbulence. NLS model was used to derive a kinetic equation of waves in [12]. This kinetic equation describes “wave turbulence”—statistical fields of weakly nonlinear dispersive waves which are more relevant to the non-equilibrium condensation kinetics than the thermodynamic descriptions. This kinetic equation for waves was used in [11,13] to describe the initial stages of the condensation as a self-similar process. In the kinetic equation approach, the cooling process is naturally described as a Kolmogorov-type down-scale energy cascade whereas the condensation is explained by an up-scale (inverse) cascade of another invariant, the number of particles. Numerically, the condensate formation in 2D was studied by Dyachenko and Falkovich [14]. The process of condensation passes in time through four different regimes.

1.1. Explosive formation of a condensate component

Let us assume that the system is excited by a weak forcing (see the next section for the conditions on the forcing amplitude) so that the kinetic equation approach can be used. Let this forcing be concentrated near a wavenumber $k = k_0$ and let there be neither forcing nor dissipation away from this scale except, perhaps, for a very high-k dissipation. Kagan et al. [11] have shown that under these conditions the wave excitations will engulf all the large-scale range in a finite time t_c after which a coherent condensate state starts forming. Semikoz and Tkachev [13] considered an early after t_c using the kinetic equation in which they put a finite number of particles at $k = 0$, an approach valid when the low wavenumber states are not yet coherent. However, buildup of correlations and breakdown of the kinetic equation description (and correspondingly its self-similar “explosive” solution) happens slightly before $t = t_c$ at a scale $k = k_{NL} > 0$ where the nonlinearity becomes strong [16]. The initial stage of the condensation have also been studied numerically by Dyachenko and Falkovich [14] and more recently by Josserand and Pomeau [15].

1.2. Initial condensate growth

The condensate at this stage is far from its pure spatially uniform state with an infinite correlation length. Numerical experiments [17] show a large number of vortices present at this stage which are involved in a complicated chaotic motion in which vortex annihilation dominates the process of the new vortex creations. The phase correlation length is roughly equal to the mean distance between vortices and is, therefore, increasing. This is nothing but the well-known Kibble–Zurek phase transition [21,22] which has been intensely discussed in the statistical physics of superfluids and cosmology within many models including NLS [23]. However, the dynamics of this transition to the coherent phase via vortices, being a strongly nonlinear process, remains poorly understood theoretically and requires further study.

1.3. Strong condensate

Very soon, there will be only very few isolated vortices left, whereas most of the space will be occupied by a quasi-uniform coherent condensate. It will significantly affect the wave dynamics (i.e. the higher energy states). This regime was studied by Dyachenko et al. [19] based on a renormalized kinetic equation obtained using the

Bogolubov transformation [24]. They showed that presence of the condensate changes the dominant nonlinear process from being four-wave to a three-wave one which, in particular, leads to an exponent change in the power-law energy spectrum. However, the condensate amplitude was considered in [19] to be an external independent of time parameter. In the present paper, we will generalize this approach by considered a coupled system of the wave turbulence and the condensate which allows us to find the rate at which the condensate amplitude grows in time.

1.4. Late stages of the condensate growth

The growth of the condensate in the third regime is accompanied by a weakening of the wave dispersion as will be shown later in this paper. Therefore, at some point the system will reach a state in which the dispersion is less than the forcing growth rate. Effectively, this will be a dispersionless acoustic wave field possibly dominated by weak shocks interacting with each other. Applicability of the kinetic equation description is still poorly understood but some effort in this direction was undertaken in [20]. In this paper we do not study this regime, although we derive the time at which it occurs and we discuss its possible consequences in the last section.

In the present paper, we concentrate on the strong-condensate regime only leaving out the study of the early and the late stages of the condensate growth. We will also ignore the spatial inhomogeneity present in real experiments due to the magnetic trapping. Study of the inhomogeneity effects on the condensate growth rate can be done in future using a WKB technique developed recently for this case in [25].

2. The model

To describe the Bose gas with repulsion we will use the nonlinear Schroedinger (NLS) equation which we will write in the following form:

$$i\Psi_t + \Delta\Psi - |\Psi|^2\Psi + \lambda(t)\Psi = i\hat{\gamma}\Psi, \tag{2}$$

where the term $i\hat{\gamma}\Psi$ describes the forcing and dissipation ($\hat{\gamma}$ is a convolution-type operator which becomes a multiplication by a function $\gamma(\mathbf{k})$ in the Fourier space). Term $\lambda(t)\Psi$ is introduced for convenience: it corresponds to a change of variables $\Psi \to \Psi\ \exp(\int \lambda\, dt)$ and it will be fixed later in a way that makes the condensate variable real.

Presence of the condensate means that Ψ can be written as

$$\Psi = \Psi_0 + \Psi_1, \tag{3}$$

where Ψ_0 is independent of the coordinate part describing the condensate amplitude and Ψ_1 is a zero-mean part:

$$\langle\Psi_1\rangle = \lim_{V\to 0} \frac{1}{V}\int \Psi_1\, d\mathbf{x} \tag{4}$$

(the integration is over the volume V). Below, we will choose $\lambda(t)$ in such a way that Ψ_0 is purely real for all t.

We assume that the growth-rate function $\gamma(\mathbf{k})$ is independent of the direction of $\mathbf{k}$ and is positive is a narrow range around the scale $|\mathbf{k}| = k_0$ followed by a large inertial range with $\gamma(\mathbf{k}) = 0$ which ends at some $|\mathbf{k}| = k_d \gg k_0$ beyond which the wave turbulence is dissipated, e.g. $\gamma(\mathbf{k}) < 0$ for $|\mathbf{k}| > k_d$. This is a standard assumption which allows deal with the fundamental turbulence mechanisms, such as the energy cascade, in their purity. However, the theory can be easily generalized for anisotropic $\gamma(\mathbf{k})$ and situations where there is no dissipation range present. Let us also assume that

$$\epsilon = \frac{\gamma}{k_0^2} \ll 1, \tag{5}$$

which means that the forcing rate is much less than the linear frequency. This will also mean that the nonlinear interaction rate (which is balanced by the forcing rate in the steady state) is less than the linear frequency. In other words, one can view turbulence in this case as a large set of nearly monochromatic weakly interacting waves and can be described by the weak turbulence theory. In this paper, we will additionally assume that

$$\epsilon \ll \frac{k_0}{\Psi_0} \ll 1, \tag{6}$$

which means that the main nonlinear process in this case is a three-wave resonant interaction which corresponds to the third regime described in Section 1.

3. Hamiltonian formalism

Let us write the NLS equation in a Hamiltonian form:

$$\mathrm{i}\Psi_t = \frac{\delta H}{\delta \Psi^*} + \mathrm{i}\hat{\gamma}\Psi \tag{7}$$

with Hamiltonian

$$H = \int \left(|\nabla \Psi|^2 + \frac{1}{2}|\Psi|^4 - \lambda(t)|\Psi|^2 \right) \mathrm{d}\mathbf{x}. \tag{8}$$

The conservative part of (7) obeys a variational principle

$$\delta S = 0 \tag{9}$$

with the action

$$S = \int L \, \mathrm{d}t \tag{10}$$

and the Lagrangian

$$L = \frac{\mathrm{i}}{2} \int (\Psi_t \Psi^* - \Psi_t^* \Psi) \, \mathrm{d}\mathbf{x} - H. \tag{11}$$

Let us consider the amplitude-phase representation of the order parameter Ψ:

$$\Psi = A \, \mathrm{e}^{\mathrm{i}\phi}. \tag{12}$$

Then

$$L = -\int A^2 \phi_t \, \mathrm{d}\mathbf{x} - H \tag{13}$$

and, taking into account that the Lagrangian is only defined up to an arbitrary time derivative, we can finally choose

$$L = 2 \int A_t A \phi \, \mathrm{d}\mathbf{x} - H. \tag{14}$$

This form suggest that the variable

$$p = 2A\phi \tag{15}$$

is a Hamiltonian momentum which is canonically conjugate to the coordinate A. In new variables A and p we have

$$L = \int A_t p \, \mathrm{d}\mathbf{x} - H \tag{16}$$

and Eq. (7) can be re-written as

$$A_t = \frac{\delta H}{\delta p} + \mathrm{i}\hat{\gamma} A, \tag{17}$$

$$p_t = -\frac{\delta H}{\delta A}, \tag{18}$$

where

$$H = \int \left\{ (\nabla A)^2 + \frac{1}{2} A^4 - \lambda(t) A^2 + \frac{1}{4} \left(\nabla p - \frac{p \nabla A}{A} \right)^2 \right\} \mathrm{d}\mathbf{x}. \tag{19}$$

In the case when no forcing or dissipation are present ($\gamma = 0$), there is a conservation of the total number of particles which can be written in terms of the new variables as

$$N = \int A^2 \, \mathrm{d}\mathbf{x}. \tag{20}$$

In the case $\gamma \neq 0$, the balance equation for N is

$$\dot{N} = 2 \int \hat{\gamma} A^2 \, \mathrm{d}\mathbf{x}. \tag{21}$$

Note that the energy $E = H$ is not generally conserved in the new variables (even for $\gamma = 0$) because of an explicit time dependence of H via $\lambda(t)$.

4. Normal variables in the condensate presence

Let us consider weak perturbations on background of a strong condensate:

$$A = A_0 + A_1, \qquad p = p_0 + p_1, \qquad \int A_1 \, \mathrm{d}\mathbf{x} = 0, \quad A_1 \ll A_0. \tag{22}$$

We now choose $\lambda(t) = A_0^2$ which gives $p_0 = 0$. This corresponds to a choice of coordinates in which the condensate order parameter is purely real. Substituting (22) into (19) we have

$$H = H_0 + H_2 + H_3 \tag{23}$$

where the subscript denoted the order of the term with respect to the perturbation amplitudes. Only the quadratic and the cubic Hamiltonians are important for the dynamic of the perturbations:

$$H_2 = \int \left\{ \frac{1}{4} (\nabla p)^2 + A_0^2 A_1^2 \right\} \mathrm{d}\mathbf{x}, \tag{24}$$

$$H_3 = \int \left\{ 2 A_0 A_1^3 - \frac{p}{2A_0} (\nabla p \cdot \nabla A) \right\} \mathrm{d}\mathbf{x}. \tag{25}$$

Now we can diagonalize the quadratic Hamiltonian by introducing normal variables $a_{\mathbf{k}}$ via the following formulae:

$$A_{\mathbf{k}} = \sqrt{\frac{\omega_{\mathbf{k}}}{2A_0^2 + k^2}} (a_{\mathbf{k}} + a^*_{-\mathbf{k}}), \tag{26}$$

$$p_{\mathbf{k}} = \mathrm{i}\sqrt{\frac{\omega_{\mathbf{k}}}{k^2}}(a_{\mathbf{k}} - a^*_{-\mathbf{k}}), \tag{27}$$

where the $A_{\mathbf{k}}$ and $p_{\mathbf{k}}$ are the Fourier transforms of A_1 and p_1, respectively (e.g. $A_{\mathbf{k}} = (2\pi)^{-3/2}\int A_1(\mathbf{x})\,\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{x}}\,\mathrm{d}\mathbf{x}$) and

$$\omega_{\mathbf{k}} = k\sqrt{2A_0^2 + k^2} \tag{28}$$

is the wave frequency re-normalized by the condensate. Substituting the new variables from (26) and (27) into the Hamiltonian leads to a diagonal form of this operator with

$$H_2 = \int \omega_{\mathbf{k}} a_{\mathbf{k}} a^*_{\mathbf{k}}\,\mathrm{d}\mathbf{k}. \tag{29}$$

The number of particles in terms of the new variables has the form

$$N = \int \frac{2\omega_{\mathbf{k}}}{2A_0^2 + k^2}|a_{\mathbf{k}}|^2\,\mathrm{d}\mathbf{k} + A_0^2. \tag{30}$$

Among the terms in the cubic Hamiltonian, the following are important for the statistical theory,[1]

$$H_3 = \frac{1}{2}\int V_{\mathbf{k},\mathbf{k}_1,\mathbf{k}_2}(a^*_{\mathbf{k}} a_{\mathbf{k}_1} a_{\mathbf{k}_2} + a_{\mathbf{k}} a^*_{\mathbf{k}_1} a^*_{\mathbf{k}_2})\delta(\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2)\,\mathrm{d}\mathbf{k}\,\mathrm{d}\mathbf{k}_1\,\mathrm{d}\mathbf{k}_2, \tag{31}$$

where the interaction coefficient $V_{\mathbf{k},\mathbf{k}_1,\mathbf{k}_2}$ has the following form

$$V_{\mathbf{k},\mathbf{k}_1,\mathbf{k}_2} = \frac{\sqrt{\omega_{\mathbf{k}}\omega_{\mathbf{k}_1}\omega_{\mathbf{k}_2}}}{(2\pi)^{3/2}}\left\{\frac{6A_0}{(\alpha_{\mathbf{k}}\alpha_{\mathbf{k}_1}\alpha_{\mathbf{k}_2})^{1/2}} + \frac{1}{2}A_0\left[\frac{\mathbf{k}\cdot\mathbf{k_1}}{kk_1\alpha_{\mathbf{k}_2}} + \frac{\mathbf{k}\cdot\mathbf{k}_2}{kk_2\alpha_{\mathbf{k}_1}} + \frac{\mathbf{k}_1\cdot\mathbf{k}_2}{k_1k_2\alpha_{\mathbf{k}}}\right]\right\}, \tag{32}$$

where

$$\alpha_{\mathbf{k}} = 2A_0^2 + k^2. \tag{33}$$

In the limit of strong condensate, $A_0 \gg k^2$, we have

$$\omega_{\mathbf{k}} = \sqrt{2}A_0 k\left(1 + \frac{k^2}{4A_0^2} + \cdots\right) \tag{34}$$

and

$$V_{\mathbf{k},\mathbf{k}_1,\mathbf{k}_2} = \frac{(kk_1k_2)^{1/2}}{A_0^{1/2}}. \tag{35}$$

5. Statistical description

Let us introduce the particle density n_k which for homogeneous turbulence satisfies the relation

$$\langle a_{\mathbf{k}} a^*_{\mathbf{k}'}\rangle = n_k\delta(\mathbf{k} - \mathbf{k}'). \tag{36}$$

[1] For brievity, we omitted some terms which do not contribute to the resonant interaction and which, therefore, are sub-dominant in the statistical evolution process. More details on this can be found in [19].

For the energy and the number of particles integrals we have

$$E = \int \omega_{\mathbf{k}} n_k \, \mathrm{d}\mathbf{k}, \tag{37}$$

$$N = \int \frac{2\omega_{\mathbf{k}}}{2\rho + k^2} n_k \, \mathrm{d}\mathbf{k} + \rho, \tag{38}$$

where $\rho = A_0^2$ is the condensate density. In the strong-condensate limit, $\rho \gg k^2$, we have

$$N = \frac{E}{\rho} + \rho. \tag{39}$$

This is quite remarkable formula. As we will show later, $E/\rho^2 \to 0$ so that asymptotically $N \to \rho$, i.e. all the particles are concentrated in the condensate.

The waves with frequency (34) can satisfy three-wave resonances and, taking into account that the cubic Hamiltonian is finite, we can conclude that the statistical description is dominated by a three-wave process and write a standard kinetic equation corresponding to this case

$$\begin{aligned} \dot{n} = \pi \int |V_{kk_1k_2}|^2 f_{k12} \delta_{\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2} \delta_{\omega_{\mathbf{k}}-\omega_{\mathbf{k}_1}-\omega_{\mathbf{k}_2}} \, \mathrm{d}\mathbf{k}_1 \, \mathrm{d}\mathbf{k}_2 \\ - 2\pi \int |V_{k_1kk_2}|^2 f_{1k_2} \delta_{\mathbf{k}_1-\mathbf{k}-\mathbf{k}_2} \delta_{\omega_{\mathbf{k}_1}-\omega_{\mathbf{k}}-\omega_{\mathbf{k}_2}} \, \mathrm{d}\mathbf{k}_1 \, \mathrm{d}\mathbf{k}_2 + 2\gamma_k n_k, \end{aligned} \tag{40}$$

where $f_{k_{12}} = n_{\mathbf{k}_1} n_{\mathbf{k}_2} - n_{\mathbf{k}}(n_{\mathbf{k}_1} + n_{\mathbf{k}_2})$. Note that in absence of a condensate, the dispersion relation is $\omega = k^2$ and it also allows three-wave resonances. However, the amplitude of the three-wave process is zero because there is no cubic Hamiltonian present in this case.

6. Asymptotic solution

For the collision integral on the RHS of (40) one can write and estimate:

$$S_{\mathrm{nl}} \sim \frac{|V_k|^2}{\omega_k} k^3 n_k^2 \sim \frac{k^5 n_k^2}{\rho}. \tag{41}$$

It is interesting that ρ appears in the denominator which is caused by two reasons. First, $|V_k|^2 \sim 1/A_0$, and secondly $\omega \approx A_0 k$. We will exploit this property and seek solution to the stationary version of Eq. (40) in the form $n = \rho\tilde{n}$ where $\tilde{n}$ is independent of ρ. At the scales much smaller than the forcing scale but greater than the dissipative scale (i.e. in the inertial range) the spectrum corresponds to an energy cascade and has a Kolmogorov-type power-law scaling [18,19]:

$$\tilde{n} = Ck^{-9/2}, \tag{42}$$

where C is an order one constant. Above the forcing scale the wave turbulence is in thermodynamic equilibrium:

$$\tilde{n} = \frac{T}{k}, \tag{43}$$

where T is a constant (temperature).

The energy integral can be written in terms of $\tilde{n}$ as follows:

$$\int \omega_{\mathbf{k}} n_k \, \mathrm{d}\mathbf{k} \approx \rho^{3/2} \int k\tilde{n}_k \, \mathrm{d}\mathbf{k}. \tag{44}$$

Thus:

$$\frac{E}{\rho^2} \sim \rho^{-1/2} \to 0 \quad \text{for } \rho \to \infty, \tag{45}$$

which, together with (39), means that most of the particles at large time are in the condensate, $N \approx \rho$. This allows us to rewrite (21) as

$$\dot{\rho} = \frac{2}{\rho} \int \gamma_k \omega_k n_k \, d\mathbf{k} \tag{46}$$

or, in terms of the condensate amplitude $A_0 = \sqrt{\rho}$:

$$\dot{A}_0 = 2 \int \gamma_k k \tilde{n}_k \, d\mathbf{k}. \tag{47}$$

Integrating this equation we have

$$A_0 = \left(2 \int \gamma_k k \tilde{n}_k \, d\mathbf{k}\right) t \tag{48}$$

that means that the condensate intensity growth quadratically with time:

$$\rho \sim t^2. \tag{49}$$

7. Discussion

In this paper, we considered a coupled system of wave turbulence and condensate governed by the NLS equation. We described a regime when the condensate is strong and the waves evolve via a three-wave interaction process. We established that asymptotically both the condensate and the wave turbulence amplitudes grow linearly in time $A_0 \sim \sqrt{\langle A_1^2 \rangle} \sim t$. The growth of the condensate density in this regime is quadratic, $\rho \sim t^2$, and is, therefore, accelerated with respect to the linear growth associated with the straightforward steady inverse cascade. This result is different from the linear growth observed numerically in [14]. The most likely reason for this discrepancy is that the forcing computed in [14] was not of the “instability” type as in the present paper. Also, they considered the 2D NLS known to be a very special from the point of view of the nonlinear dynamics.

The weak turbulence description used in this paper is valid when the nonlinear frequency correction is much less than the frequency dispersion. The nonlinear frequency correction is of the same order as the nonlinear damping. In turn, the nonlinear damping is equal to the forcing γ in the steady state which is realized for $\tilde{n}$, i.e. when the condensate dependence is scaled out of the equation. Putting this dependence back means that the spectrum n grows proportionally to ρ as so does the nonlinear frequency correction, $\omega_{nl} \sim \rho\gamma$. On the other hand, (34) gives us the linear dispersion $\omega_{disp} \sim k^2/\sqrt{\rho}$ which is decreasing with time as $1/t$. Thus, being valid initially for small γ's, the weak turbulence description breaks down at large time $t^* \sim \gamma^{-1/3}$. Note that A_1 grows at the same rate as A_0 so that their ratio remains constant. Thus, at t^* the nonlinearity is still small, $A_1/A_0 \ll 1$, which means that the dispersionless regime at $t > t^*$ remains weakly nonlinear. Such a regime is still poorly understood and requires further study. It is possible that in this regime is qualitatively similar to the dispersive weak turbulence described in this paper or, alternatively, it will be dominated by weak shocks leading to a different power-law spectrum (the answer may also depend on the number of dimensions).

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