Turbulence of near-inertial waves in the continuously stratified fluid

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1 Motivation

Quasi-inertial oscillations are ubiquitous in the atmosphere and the oceans. The measured frequency spectra of atmospheric and oceanic perturbations always exhibit a pronounced peak at the (local) inertial frequency, f. The physical reason of the persistence of these oscillations is simple: the minimal frequency of inertia-gravity waves in the atmosphere, or ocean, is f, and the dispersion surface has a minimum at this value. Hence, the group velocity of near-inertial waves (NIW) is close to zero and they need long times to be evacuated.

Motivated by the persistence of NIW, we consider below their kinetics, assuming a large ensemble of weakly-nonlinear NIW with random phases. We apply the wave-turbulence (WT) technique to obtain the stationary energy spectra from the kinetic equation (KE) for the wave-density.

2 WT approach to IGW: a brief review

WT approach to *internal* inertia-gravity waves (IGW) has a long history which is intertwinned with strong turbulence approach to rotating stratified flows (Herring and collaborators, Lesieur and collaborators, Carnevale and collaborators, etc etc)

Muller, Olbers & collaborators (Hasselmann's school, e.g. Muller and Olbers, 1975, Olbers, 1976), and McComas (McComas & Muller, 1981) established kinetic equation by using Hamiltonian structure in Lagrangian variables (correct incorporation of the pressure constraint was problematic), and studied its asymptotics/approximate/numeric solutions. Critisized by Holloway (1980,1984).

Pelinovsky & Raevsky (1977) obtained the anisotropic IGW power-law spectra from dimensional reasons using Clebsch variables. Voronovich (1979) also used Clebsch variables to obtain the KE.

Clebsch variables are also subject to constraints.

Daubner & Zeitlin (1996), and Caillol & Zeitlin (2000) used direct averaging of the eqns of motion to derive KE and used Kuznetsov method to obtain the anisotropic power-law spectra for IGW. They also showed decoupling of wave and vortex components of motion in low orders, and thus possibility to apply WT (lot of work on the decoupling: Riley and coll., Majda and coll., Mahalov and coll.)

Y. Lvov & Tabak, 2001, 2004 rederived KE and spectra by Kuznetsov method in isopycnal coordinates by using the Hamiltonian form by Holm & Long (1984) and wave-projection by Falkovich & Medvedev (1992). They were apparently unaware of previous literature.

For technical reasons most of the earlier results were obtained in the (horizontally) short wave limit, when rotation effects may be neglected (opposite to NIW limit).

3 PE in density/enthropy coordinates, and their Hamiltonian structure

3.1 Primitive equations

Continuously stratified non-dissipative hydrostatic primitive PE (oceanic case) :

$$\frac{dv_\perp}{dt} + f\hat{z} \times v_\perp + \frac{1}{\rho_0}\nabla_\perp p = 0, \qquad (1)$$

$$\frac{1}{\rho}p_z + g = 0, \qquad (2)$$

$$\nabla_\perp v_\perp + w_z = 0, \qquad (3)$$

$$\frac{d\rho}{dt} = 0, \qquad \frac{d}{dt} = \partial_t + (v_\perp, \nabla_\perp) + w\partial_z \qquad (4)$$

Here ρ_0 is a large constant background fluid density, and ρ is its small variation: $\rho_0 \gg (\rho - \rho_0)$ (the Boussinesq approximation), p is the hydrostatic pressure, $v_\perp$ and w are horizontal and vertical components of the fluid velocity.

3.2 PE in isopycnal/isentropic coordinates

ρ – independent vertical coordinate, z – new dependent variable. $\rightarrow$

$$\psi_\rho = gz, \quad \psi = p + g\rho z, \tag{5}$$

where ψ is the Montgomery stream-function. Fluid velocity:

$$(u, v, \tilde{w}) = \frac{d}{dt}(x, y, \rho). \tag{6}$$

$\Rightarrow \tilde{w} = 0$, a well-known advantage of the isopycnal (isentropic) coordinates. The "old" vertical velocity:

$$w = \frac{dz}{dt} = z_t + uz_x + vz_y. \tag{7}$$

Horizontal motion, $\frac{D}{Dt} = \partial_t + (v_\perp, \nabla_\perp)$:

$$\frac{Dv_\perp}{Dt} + f\hat{z} \times v_\perp + \frac{1}{\rho_0}\nabla_\perp \psi = 0. \tag{8}$$

Mass conservation:

$$\rho_0 dxdydz = \rho_0 z_\rho dxdyd\rho, \tag{9}$$

which implies:

$$(\rho_0 z_\rho)_t + (u\rho_0 z_\rho)_x + (v\rho_0 z_\rho)_y = 0. \tag{10}$$

Therefore

$$u_t + uu_x + vu_y - fv + (\psi/\rho_0)_x = 0, \tag{11}$$

$$v_t + uv_x + vv_y + fu + (\psi/\rho_0)_y = 0, \tag{12}$$

$$(\rho_0 z_\rho)_t + (u\rho_0 z_\rho)_x + (v\rho_0 z_\rho)_y = 0 \quad \psi_\rho = z. \tag{13}$$

By introducing the potential vorticity (PV):

$$q = \frac{f + v_x - u_y}{z_\rho}, \tag{14}$$

it is easy to see that an arbitrary function function of density and PV $G(q, \rho)$ is a Lagrangian invariant:

$$G_t + uG_x + vG_y = 0, \tag{15}$$

The local energy conservation is expressed as:

$$\left(\frac{1}{2}\left(\rho_0 z_\rho (u^2 + v^2) - gz^2\right)\right)_t + \left(\rho_0 z_\rho u \left(\frac{u^2 + v^2}{2} + \psi\right)\right)_x + \left(\rho_0 z_\rho v \left(\frac{u^2 + v^2}{2} + \psi\right)\right)_y + (\psi z_t)_\rho = 0 \tag{16}$$

The Hamiltonian is the energy of the system in the isopycnal coordinates:

$$H = \frac{1}{2} \int \left(h(u^2 + v^2) + g z^2 \right) dx dy d\rho, \tag{17}$$

where the pseudo-height h is:

$$h = \rho_0 z_\rho, \tag{18}$$

and the Poisson bracket is (Holm & Long, 1989):

$$\begin{pmatrix} u \\ v \\ h \end{pmatrix}_t + \hat{J} \begin{pmatrix} \delta H/\delta u = hu \\ \delta H/\delta v = hv \\ \delta H/\delta h = B \end{pmatrix} = 0. \tag{19}$$

The Poisson bracket operator $\hat{J}$:

$$\hat{J} = \begin{pmatrix} 0 & -q & \partial_x \\ q & 0 & \partial_y \\ \partial_x & \partial_y & 0 \end{pmatrix}, \tag{20}$$

and the variables resulting from functional differentiations of the Hamiltonian are the pseudo-height, the PV, and the Bernoulli function:

$$B = \frac{u^2 + v^2}{2} + \frac{\psi}{\rho_0}. \tag{21}$$

Remarkably, the Poisson bracket (20) is identical to that of the RSW model (Falkovich & Medvedev, 1992) and the Hamiltonian (17) has a similar form.

3.3 Wave-vortex decoupling

Following Falkovich & Medvedev, 1992, we define new dependent variables: $\mathcal{W} = (h, \varphi, q)$, where φ is the velocity potential, instead of $\mathcal{U} = (u, v, h)$. Poisson bracket becomes:

$$\hat{J} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & \partial'_x q \partial'_y - \partial'_y q \partial'_x & \left[\nabla' q \nabla - q\right] \frac{1}{h} \\ 0 & \frac{1}{h}\left[q - \nabla q \nabla'\right] & \frac{1}{h}\partial_x q \partial_y \frac{1}{h} - \frac{1}{h}\partial_y q \partial_x \frac{1}{h} \end{pmatrix}. \qquad (22)$$

Here $\partial'_x = \partial_x \nabla^{-2}$, $\partial'_y = \partial_y \nabla^{-2}$, $\nabla' = (\partial'_x, \partial'_y)^T$, $\nabla = (\nabla_\perp, \partial_z)$ and the superscript T means matrix transposition. To get (22) from (20) the following rules are applied. First, the change of variables from the old $\mathcal{U}$ to new $\mathcal{W}$ functional variables is made. Then the expressions of $\mathcal{W}$ in terms of $\mathcal{U}$ are diffrentiated by t and the transition matrix $\mathcal{M}$ is obtained: $\mathcal{W}_t = \mathcal{M}\,\mathcal{U}_t$. The new Poisson bracket operator $\hat{J}_{new} = \mathcal{M}\,\hat{J}_{old}\,\mathcal{M}^\dagger$ follows. $\hat{J}_{new}$ is then expressed in terms of the new variables.

From (22) it follows that if the PV is a function of density alone $q = q_0(\rho)$, the Poisson bracket has constant coefficients:

$$\hat{J} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{23}$$

Hence, on the constrained manifold $q = q_0(\rho)$ we get canonical Hamiltonian variables h and φ:

$$\frac{\partial h}{\partial t} = \frac{\delta H}{\delta \varphi}, \qquad \frac{\partial \varphi}{\partial t} = -\frac{\delta H}{\delta h}. \tag{24}$$

The velocity field is determined from

$$u = \frac{\partial \varphi}{\partial x} - \frac{\partial}{\partial y}\Delta^{-1}(hq_0 - f), \qquad v = \frac{\partial \varphi}{\partial y} + \frac{\partial}{\partial x}\Delta^{-1}(hq_0 - f)\ . \tag{25}$$

4 Internal inertia-gravity waves

4.1 The state of rest and small perturbations

The rest state of the system is

$$u_0 = 0, \quad v_0 = 0, \quad h_0(\rho) = -\left(\frac{1}{\rho_0}\frac{d\bar{\rho}}{dz}\right)^{-1} = \frac{g}{N^2}, \quad q_0 = f/h_0, \quad (26)$$

where $\bar{\rho} = \bar{\rho}(z)$ is the density at rest expressed as a function of z, $N(\rho)$ is the (variable) Brunt-Vaisala frequency

$$N^2 = -\frac{g}{\rho_0}\frac{d\bar{\rho}}{dz}.$$

Consider a small perturbation of the rest state $\eta = h - h_0$. Then

$$u = \frac{\partial \varphi}{\partial x} - f \frac{\partial}{\partial y} \Delta^{-1} \frac{\eta}{h_0}, \qquad v = \frac{\partial \varphi}{\partial y} + \frac{\partial}{\partial x} \Delta^{-1} \frac{\eta}{h_0} \tag{27}$$

and the kinetic energy has a power expansion starting at the second-order term. The expansion of the potential energy around the state of rest contains the first-order terms:

$$z^2 = \left(\partial_\rho^{-1} \frac{h_0 + \eta}{\rho_0} \right)^2 = \left(\partial_\rho^{-1} \frac{h_0}{\rho_0} \right)^2 + 2 \left(\partial_\rho^{-1} \frac{h_0}{\rho_0} \right) \left(\partial_\rho^{-1} \frac{\eta}{\rho_0} \right) + \left(\partial_\rho^{-1} \frac{\eta}{\rho_0} \right)^2 . \tag{28}$$

Due to the fact that the system conserves the generalized mass

$$M = \int L(\rho) h dx dy d\rho,$$

where $L(\rho)$ is an arbitrary functional depending on ρ, the linear terms may be removed by introducing the available potential energy. The wave Hamiltonian takes the form $H = H_2 + H_3$, where

$$\begin{aligned} H_2 &= \frac{1}{2}\int \left[h_0 \left(\frac{\partial \varphi}{\partial x} - f \frac{\partial}{\partial y} \Delta^{-1} \frac{\eta}{h_0} \right)^2 + h_0 \left(\frac{\partial \varphi}{\partial y} + \frac{\partial}{\partial x} \Delta^{-1} \frac{\eta}{h_0} \right)^2 \right. \\ &+ \left. g \left(\partial_\rho^{-1} \frac{\eta}{\rho_0} \right)^2 \right] dx dy d\rho, \end{aligned} \tag{29}$$

$$H_3 = \frac{1}{2}\int \left[\eta \left(\frac{\partial \varphi}{\partial x} - f \frac{\partial}{\partial y} \Delta^{-1} \frac{\eta}{h_0} \right)^2 + \eta \left(\frac{\partial \varphi}{\partial y} + \frac{\partial}{\partial x} \Delta^{-1} \frac{\eta}{h_0} \right)^2 \right] dx dy d\rho, \tag{30}$$

4.2 Small-amplitude waves on the background of constant stratification and their interaction

Consider the constant rest state $h_0 = g/N^2 = \text{const}$. The normal variables $b_{\mathbf{k}}$ are introduces in the standard way:

$$\varphi_{\mathbf{k}} = i\sqrt{\frac{\omega_{\mathbf{k}}}{2h_0 p^2}}\left(b_{\mathbf{k}} - b^*_{-\mathbf{k}}\right), \quad \eta_{\mathbf{k}} = \sqrt{\frac{h_0 p^2}{2\omega_{\mathbf{k}}}}\left(b_{\mathbf{k}} + b^*_{-\mathbf{k}}\right). \tag{31}$$

On the background of $N = const$, the linearization of (11) - (13) gives harmonic waves IGW with the frequency:

$$\omega_{\mathbf{k}} = \sqrt{f^2 + \frac{g h_0 p^2}{\rho_0^2 q^2}}, \tag{32}$$

where the wave vector $\mathbf{k}$ is split into horizontal and vertical parts as:

$$\mathbf{k} = (k_x, k_y, k_z) = (\mathbf{p}, q), \quad k = \sqrt{k_x^2 + k_y^2 + k_z^2}, \quad p = \sqrt{k_x^2 + k_y^2}. \tag{33}$$

The quadratic Hamiltonian takes the standard form

$$H_2 = \int \omega_{\mathbf{k}} |b_{\mathbf{k}}|^2 d\mathbf{k}. \tag{34}$$

In terms of the normal variables $b_{\mathbf{k}}$ the Fourier transforms of velocity and η are:

$$u_{\mathbf{k}} = A_{\mathbf{k}} b_{\mathbf{k}} + A^*_{-\mathbf{k}} b^*_{-\mathbf{k}}, \quad v_{\mathbf{k}} = B_{\mathbf{k}} b_{\mathbf{k}} + B^*_{-\mathbf{k}} b^*_{-\mathbf{k}}, \quad \eta_{\mathbf{k}} = C_{\mathbf{k}} b_{\mathbf{k}} + C^*_{-\mathbf{k}} b^*_{-\mathbf{k}}, \tag{35}$$

$$A_{\mathbf{k}} = \frac{-k_x \omega_{\mathbf{k}} + i k_y f}{p\sqrt{2h_0 \omega_{\mathbf{k}}}}, \quad B_{\mathbf{k}} = \frac{-k_y \omega_{\mathbf{k}} - i k_x f}{p\sqrt{2h_0 \omega_{\mathbf{k}}}}, \quad C_{\mathbf{k}} = \frac{k^x h_0}{p\sqrt{2h_0 \omega_{\mathbf{k}}}}. \tag{36}$$

The interaction Hamiltonian (30)

$$H_3 = \frac{1}{2} \int \eta(u^2 + v^2) dx dy d\rho, \tag{37}$$

thus, takes the standard form:

$$H_3 = \int d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3) V_{123} b_1 b_2 b_3^* + \text{c.c.} + \tag{38}$$

$$\int d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) U_{123} b_1 b_2 b_3 + \text{c.c.}. \tag{39}$$

The matrix elements have the following symmetries with respect to permutations: $V_{123} = V_{213}$ and $U_{123} = U_{213} = U_{321} = U_{132}$.

In terms of the coefficients $A_{\mathbf{k}}$, $B_{\mathbf{k}}$, and $C_{\mathbf{k}}$ the matrix elements are:

$$V_{123} = \frac{1}{2}\left(C_1(A_2A_3^* + B_2B_3^*) + C_2(A_1A_3^* + B_1B_3^*) + C_3^*(A_1A_2 + B_1B_2)\right), \tag{40}$$

$$U_{123} = \frac{1}{6}\left(C_1(A_2A_3 + B_2B_3) + C_2(A_1A_3 + B_1B_3) + C_3(A_1A_2 + B_1B_2)\right) \tag{41}$$

With the help of unit vectors $\mathbf{e}_i = \mathbf{p}_i/p_i$ we get :

$$\begin{aligned}2V_{123}\sqrt{8h_0\omega_1\omega_2\omega_3} = &\quad p_1((\omega_2\omega_3 + f^2)(e_2, e_3) + if(\omega_2 + \omega_3)[e_2, e_3]) \\ + &\quad p_2((\omega_1\omega_3 + f^2)(e_1, e_3) + if(\omega_1 + \omega_3)[e_1, e_3]) \\ + &\quad p_3((\omega_1\omega_2 - f^2)(e_1, e_2) - if(\omega_1 - \omega_2)[e_1, e_2]),\end{aligned}$$

$$\begin{aligned}6U_{123}\sqrt{8h_0\omega_1\omega_2\omega_3} = &\quad p_1((\omega_2\omega_3 - f^2)(e_2, e_3) - if(\omega_2 - \omega_3)[e_2, e_3]) \\ + &\quad p_2((\omega_1\omega_3 - f^2)(e_1, e_3) - if(\omega_1 - \omega_3)[e_1, e_3]) \\ + &\quad p_3((\omega_1\omega_2 - f^2)(e_1, e_2) - if(\omega_1 - \omega_2)[e_1, e_2]).\end{aligned}$$

4.3 The case of near-inertial waves

Consider waves with $\omega_{\mathbf{k}} \approx f$. They are necessarily long in the horizontal direction, $p \ll q$, propagating almost vertically, and their dispersion in the first order is:

$$\omega_{\mathbf{k}} \approx f \left(1 + \frac{g h_0}{2 f^2 \rho_0^2} \frac{p^2}{q^2}\right). \quad (42)$$

This dispersion law is of the non-decay type – an example of anisotropic waves with a non-decay scale-invariant dispersion.

Hence, three-wave interactions are non-resonant, and can be removed by a canonical transformation (Zakharov, Lvov & Falkovich, 1992). In the leading order the three-wave matrix elements have the form

$$V_{123} = \sqrt{\frac{f}{8h_0}} \left(p_1((e_2, e_3) + i[e_2, e_3]) + p_2((e_1, e_3) + i[e_1, e_3])\right), \; U_{123} = 0. \quad (43)$$

The effective four-wave matrix element resulting after elimination of the non-resonant three-wave interactions is:

$$T_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3} = 2\frac{V_{\mathbf{k}+\mathbf{k}_1,\mathbf{k},\mathbf{k}_1} V^*_{\mathbf{k}_2+\mathbf{k}_3,\mathbf{k}_2,\mathbf{k}_3}}{f} - 2\frac{V_{\mathbf{k},\mathbf{k}_2,\mathbf{k}-\mathbf{k}_2} V^*_{\mathbf{k}_3,\mathbf{k}_1,\mathbf{k}_3-\mathbf{k}_1}}{f}$$
$$-2\frac{V_{\mathbf{k},\mathbf{k}_3,\mathbf{k}-\mathbf{k}_3} V^*_{\mathbf{k}_2,\mathbf{k}_1,\mathbf{k}_2-\mathbf{k}_1}}{f} - 2\frac{V_{\mathbf{k}_1,\mathbf{k}_3,\mathbf{k}_1-\mathbf{k}_3} V^*_{\mathbf{k}_2,\mathbf{k},\mathbf{k}_2-\mathbf{k}}}{f}$$
$$-2\frac{V_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_1-\mathbf{k}_2} V^*_{\mathbf{k}_3,\mathbf{k},\mathbf{k}_3-\mathbf{k}}}{f}. \qquad (44)$$

5 Wave-turbulence of waves with scale-invariant anisotropic dispersion

5.1 General formalism

The dispersion relation (42) consists of a constant plus some power-law anisotropic scale-invariant correction. Due to its non-decay character, only the correction enters in the collision integral for the ensemble of random-phase NIW .

The way of dealing with the turbulence of waves with scale-invariant dispersion laws of the *decay* type was found in the pioneering work of Kuznetsov, 1972. It was applied to the oposite limite of horizontally short IGW by Daubner & Zeitlin 1996, and Caillol & Zeitlin 2000. We are not aware of similar results for the non-decay case.

We assume a dispersion relation which is of non-decay type, power-law, scale-invariant and isotropic separately in the horizontal and vertical directions:

$$\omega(\mathbf{k}) = p^a q^b, \tag{45}$$

The four-wave matrix element is also assumed to be scale-invariant:

$$T(\lambda\mathbf{p}, \lambda\mathbf{p}_1, \lambda\mathbf{p}_2, \lambda\mathbf{p}_3, \mu\mathbf{q}, \mu\mathbf{q}_1, \mu\mathbf{q}_2, \mu\mathbf{q}_3) = \lambda^c \mu^d T(\mathbf{p}, \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{q}, \mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3). \tag{46}$$

The standard form of the collision integral of the four-wave kinetic equation is (Zakharov, L'vov & Falkovich, 1992):

$$I^{(4)}\left[N(\mathbf{k})\right] = \int d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 W_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3} f_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}, \tag{47}$$

where

$$W_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3} = \pi |T_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}|^2 \delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)\delta(\omega(\mathbf{k})+\omega(\mathbf{k}_1)-\omega(\mathbf{k}_2)-\omega(\mathbf{k}_3)), \tag{48}$$

$$\begin{aligned} f_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3} &= N(\mathbf{k}_1)N(\mathbf{k}_2)N(\mathbf{k}_3) + N(\mathbf{k})N(\mathbf{k}_2)N(\mathbf{k}_3) - N(\mathbf{k})N(\mathbf{k}_1)N(\mathbf{k}_2) \\ &- N(\mathbf{k})N(\mathbf{k}_1)N(\mathbf{k}_3). \end{aligned} \tag{49}$$

In order to factorize the collision integral $I^{(4)}$, which is sufficient for finding distributions $N(\mathbf{k})$ annihilating it and, thus, giving stationary solutions of the kinetic equation, we use the trick first proposed by Kats & Kontorovich, 1973.

The resonant wave *quadrangle* $(\mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$ defining the integration measure in the collision integral (47) may be transformed, with the help of the symmetry transformations $\hat{G}_i : \hat{G}_i \mathbf{k}_i = \mathbf{k} \; i = 1, 2, 3$ consisting of rotations and dilatations, into another resonant quadrangle $(\mathbf{k}, \mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3)$ in three different ways:

$$\mathbf{q}_1 = \hat{G}_1{}^2 \mathbf{k}_1, \; \mathbf{q}_2 = \hat{G}_1 \mathbf{k}_2, \; \mathbf{q}_3 = \hat{G}_1 \mathbf{k}_3; \tag{50}$$

$$\mathbf{q}_1 = \hat{G}_2 \mathbf{k}_3, \; \mathbf{q}_2 = \hat{G}_2{}^2 \mathbf{k}_2, \; \mathbf{q}_3 = \hat{G}_2 \mathbf{k}_1; \tag{51}$$

$$\mathbf{q}_1 = \hat{G}_3 \mathbf{k}_2, \; \mathbf{q}_2 = \hat{G}_3 \mathbf{k}_1, \; \mathbf{q}_3 = \hat{G}_3{}^2 \mathbf{k}_3. \tag{52}$$

For example, $\hat{G}_3$ acts on the horizontal and vertical wavenumbers, respectively, as follows:

$$\hat{G}_{3_H} = \lambda_3 \circ g_3, \quad \lambda_3 = \frac{p}{p_2}, \quad \hat{G}_{3_z} = \mu_3 = \frac{q}{q_2}. \tag{53}$$

Here g_3 and λ_3 are rotation and dilatation, respectively, in the horizontal plane, and μ_3 is dilatation in the vertical direction. The other transformations $\hat{G}_{1,2}$ are constructed in the analogous way.

As follows from (45), (46)

$$W_{\mathbf{k}\mathbf{m}_1\mathbf{m}_2\mathbf{m}_3} = W_{\hat{G}_3\mathbf{k}_2\hat{G}_3\mathbf{k}_3\hat{G}_3\mathbf{k}\hat{G}_3\mathbf{k}_1} = \lambda_3^{2c-2-a}\mu_3^{2d-1-b}W_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}. \quad (54)$$

For the Jacobian of such change of variables we get:

$$|\hat{G}_3| = \lambda_3^2\mu_3. \quad (55)$$

The collision integral may be represented as a sum of four replicas of itself in the form:

$$I^{(4)}\left[N(\mathbf{k})\right] = \frac{1}{4}\int d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3\, W_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}\left(f_{\mathbf{k}} + \lambda_1^{\alpha} f_{\hat{G}_1\mathbf{k}} + \lambda_2^{\alpha} f_{\hat{G}_2\mathbf{k}} + \lambda_3^{\alpha} f_{\hat{G}_3\mathbf{k}}\right), \tag{56}$$

where the scale factors are $\lambda_i = \frac{|\mathbf{k}|}{|\mathbf{k}_i|},\ i = 1, 2, 3.$

If solutions are sought in the power-law form
$N(\mathbf{k}) = N(p,q) \sim p^{-x} q^{-y}$, then

$$
\begin{aligned}
f_{\mathbf{k}\mathbf{m}_1\mathbf{m}_2\mathbf{m}_3} &= N(\mathbf{k})N(\mathbf{k}_1)N(\mathbf{k}_2)N(\mathbf{k}_3)\left(N^{-1}(\mathbf{k}) + N^{-1}(\mathbf{k}_1) - N^{-1}(\mathbf{k}) - N^{-1}(\mathbf{k} \right. \\
&= (pp_1p_2p_3)^{-x}(qq_1q_2q_3)^{-y}\left(p^x q^y + p_1^x q_1^y - p_2^x q_2^y - p_3^x q_3^y\right) \qquad (5
\end{aligned}
$$

and after e.g. the $\hat{G}_3$ transformation

$$f_{\mathbf{k}\mathbf{m}_1\mathbf{m}_2\mathbf{m}_3} = f_{\hat{G}_3\mathbf{k}_3\hat{G}_3\mathbf{k}_3\hat{G}_3\mathbf{k}\hat{G}_3\mathbf{k}_1} = \lambda_3^{-3x}\mu_3^{-3y} f_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3}. \qquad (58)$$

By using (53), (54), (55), (57) we can represent the collision integral (56) in the factorized form

$$I^{(4)} = \frac{1}{4p^r q^s} \int d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 W_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3} f_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3} \left(p^r q^s + p_1^r q_1^s - p_2^r q_2^s - p_3^r q_3^s\right) \tag{59}$$

where, cf. (45), (46),

$$r = 8 + 2c - 2 - a - 3x, \quad s = 4 + 2d - 1 - b - 3y. \tag{60}$$

The power-law solutions annihilating the collision integral follow from the symmetry of the last factor in $I^{(4)}$ and the corresponding delta-function.

As usual, exploiting the frequency delta-function in the collision integral results in the spectrum related to the constant energy flux. Thus, if $r = a$, $s = b$, then $I^{(4)}$ vanishes since the last factor coincides with the argument of the delta-function. We thus get the first Kolmogorov-like distribution

$$N(p,q) \sim P p^{-x_1} q^{-y_1}, \quad (x_1, y_1) = (6 + 2c - 2a, 3 + 2d - 2b)/3 \quad (61)$$

which corresponds to the constant energy flux P through the spectrum.

If $r = 0$, $s = 0$, then the last factor in the collision integral vanishes. Such solution exploits the symmetry properties of the collision term, which corresponds, as usual, to the conservation of the wave action. We thus get the second Kolmogorov-like spectrum

$$N(p,q) \sim Qp^{-x_2}q^{-y_2}, \quad (x_2, y_2) = (6 + 2c - a, 3 + 2d - b)/3 \quad (62)$$

which corresponds to the constant flux of the wave action Q.

Again, as usual, we have the generalized Rayleigh-Jeans solutions of the form

$$N \sim \frac{1}{\omega_k - \mathbf{k} \cdot \mathbf{a} - \mu}, \tag{63}$$

where $\mathbf{a}$ is a constant vector and μ is an arbitrary constant.

5.2 Weak turbulence and power-law spectra of NIW

As follows from (48), (44) and (43) the matrix element and the argument of the delta function (which we shorthand as $\Delta\omega$) in the collision integral (47) have the following scaling exponents:

$$W(\lambda p, \mu q) = \lambda^4 \mu^0 W(p,q), \quad \Delta\omega(\lambda p, \mu q) = \lambda^2 \mu^{-2} \Delta\omega(p,q). \qquad (64)$$

Therefore we can apply the results of the previous subsection to this particular case and find the following power-law solutions of the kinetic equation :

1. corresponding to the constant energy flux:

$$N(p,q) \sim Pp^{-6/3}q^{-7/3},$$

2. corresponding to the constant wave-action flux:

$$N(p,q) \sim Qp^{-8/3}q^{-5/3},$$

3. corresponding to the zero flux

$$N(p,q) \sim Cp^{-2}q^{2}.$$

The energy spectra, in the first approximation, coincide with the N - spectra up to the constant f.

6 Conclusions

We obtained stationary power-law energy spectra for NIW. These waves are necessarily long in the horizontal direction and short in the vertical direction. NIW are ubiquitous in the atmosphere and oceans and, from the practical point of view, our predictions, when confronted with observations, should allow to simultaneously test both the validity of the random-phase approximation which is at the basis of the wave turbulence theory, and the physics behind the data.

From the theoretical point of view, we see that, like in Medvedev & Zeitlin, 2005, where anisotropic scale-invariant corrections to the acoustic-type dispersion of the short equatorial waves were considered, the inertia-gravity waves in the atmosphere and ocean provide a new example of applications of anisotropic scaling laws in the wave turbulence.