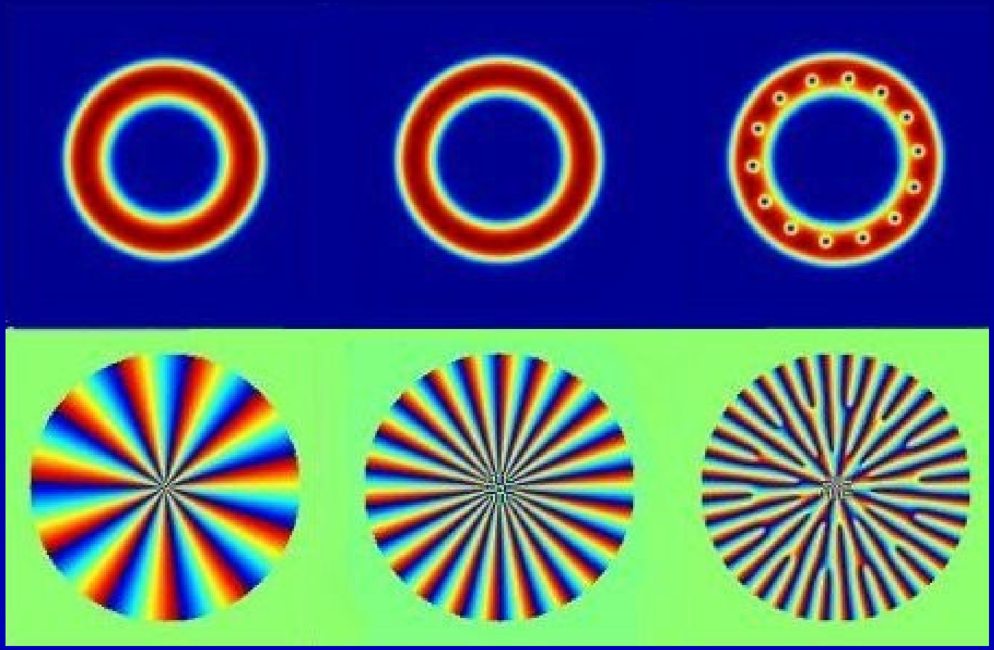
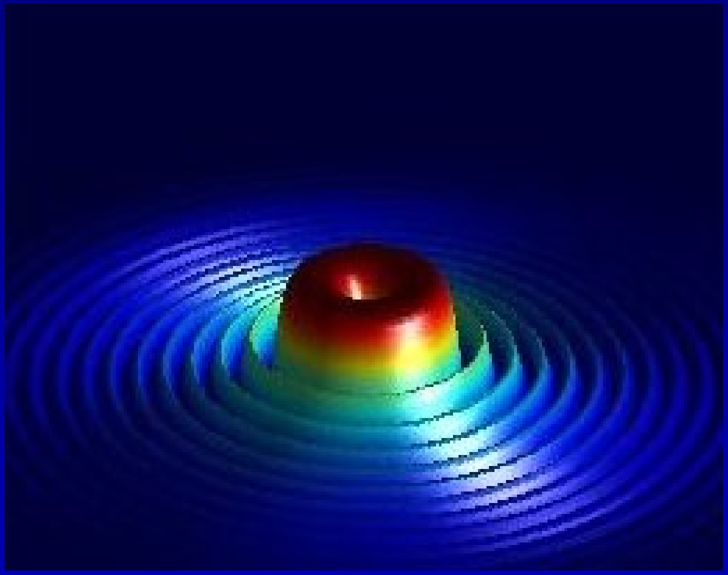
Persistent currents and hysteresis in Bose-Einstein condensates

Brian Jackson





[1] School of Mathematics and Statistics, University of Newcastle

[2] BEC-INFM and Dipartimento di Fisica, Università di Trento, Italy

Topics

- **Vortices in annular condensates**

M. Cozzini [2], B. Jackson [1,2], S. Stringari [2], cond-mat/0510143 (accepted for PRA).

- **Vortex formation in rotating traps**

B. Jackson [1,2] and C. F. Barenghi [1] (in preparation)

[1] School of Mathematics and Statistics, University of Newcastle

[2] BEC-INFM and Dipartimento di Fisica, Università di Trento, Italy

Vortices in annular condensates

Gross-Pitaevskii (GP) equation:

$$\boxed{i\hbar\frac{\partial\Psi}{\partial t}=\left(-\frac{\hbar^2}{2m}\nabla^2+V+g|\Psi|^2\right)\Psi}$$

$\Psi(\mathbf{r},t)$ condensate wavefunction

$$\boxed{g=\frac{4\pi N\hbar^2 a}{m}}$$

$$\int|\Psi(\mathbf{r})|^2 d\mathbf{r}=1$$

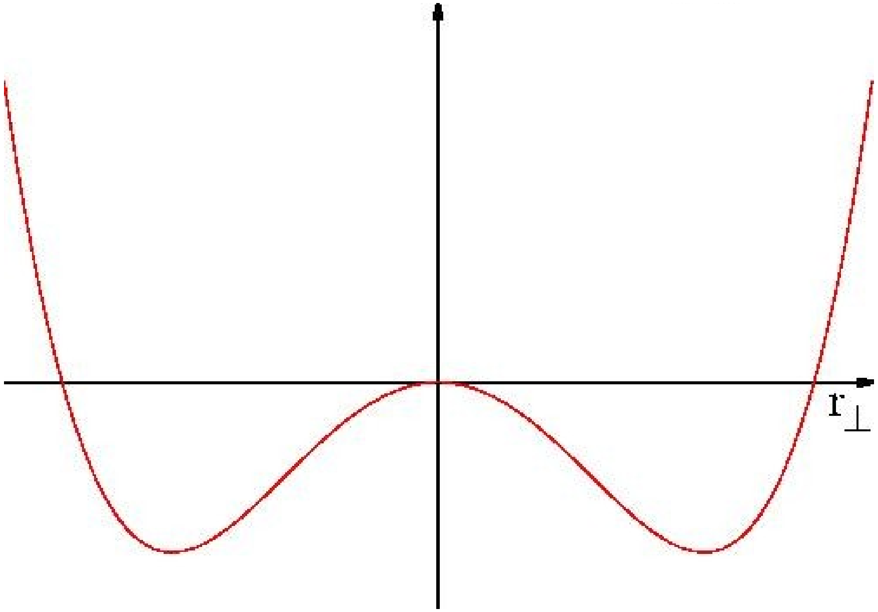
m atomic mass

N number of atoms

a scattering length

Annular condensates:

$$\boxed{V = \frac{\hbar\omega_\perp}{2}\left(-\frac{r^2}{d_\perp^2} + \lambda\frac{r^4}{d_\perp^4}\right)}$$



Dimensionless units:

Distance: $$d_\perp = \left(\frac{\hbar}{m\omega_\perp}\right)^{1/2}$$

Time: $$\omega_\perp^{-1}$$

Energy: $$\hbar\omega_\perp$$

$$\mu\Psi = \left[\cancel{-\frac{1}{2}\nabla^2} + \frac{1}{2}\left(-r^2 + \lambda r^4\right) + g\left|\Psi\right|^2 \right]\Psi$$

$$\boxed{\Psi(\mathbf{r},t) = \Psi(\mathbf{r})e^{-i\mu t}}$$

Thomas-Fermi approximation (no vortices)

$$n(r) \equiv \left|\Psi(r)\right|^2 = \frac{1}{g}\left(\mu + \frac{1}{2}r^2 - \frac{\lambda}{2}r^4\right) \quad \text{for} \quad n(r) > 0$$

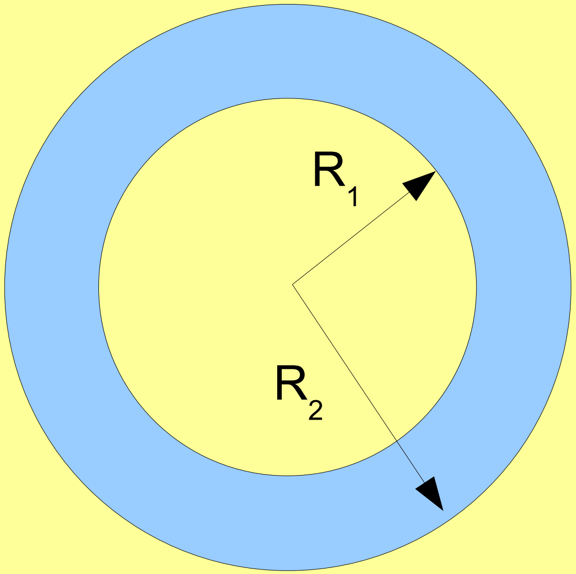
$$n(r) = 0 \quad \text{otherwise}$$

Find where *n(r)=0* → radii of condensate

$$\boxed{\mu > 0} \qquad R^2 = \frac{1+\sqrt{1+8\lambda\mu}}{2\lambda}$$

$$\boxed{\mu < 0} \qquad R_1^2 = \frac{1-\sqrt{1+8\lambda\mu}}{2\lambda}, \quad R_2^2 = \frac{1+\sqrt{1+8\lambda\mu}}{2\lambda}$$

Annular condensate: $\mu < 0$



$$R_{1,2}^2 = \frac{1 \mp \sqrt{1+8\lambda\mu}}{2\lambda}$$

$$\boxed{R_\pm^2 = R_2^2 \pm R_1^2}$$

$$\lambda R_+^2 = 1$$

$$\lambda R_-^2 = \eta \equiv \left(\frac{12 g \lambda^2}{\pi} \right)^{1/3}$$

$$\boxed{\mu = \frac{\eta^2 - 1}{8\lambda}}$$

$\eta < 1$ for annular condensate

$$\boxed{i\frac{\partial\Psi}{\partial t}=\left[-\frac{1}{2}\nabla^2+\frac{1}{2}\left(-r^2+\lambda r^4\right)+g\left|\Psi\right|^2-\Omega\hat{L}_z\right]\Psi}$$

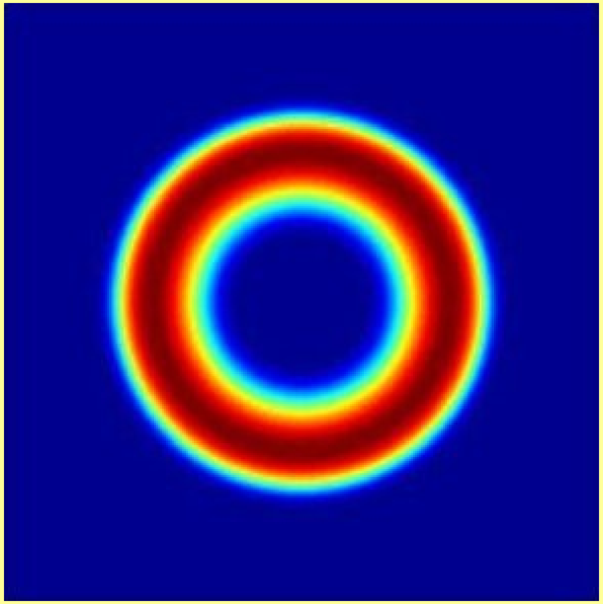
$$\hat{L}_z=-i\frac{\partial}{\partial\phi}=-i\left(x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}\right)$$

$$\boxed{\mu\Psi = \left[-\frac{1}{2}\nabla^2 + \frac{1}{2}\left(-r^2 + \lambda r^4\right) + g|\Psi|^2 - \Omega\hat{L}_z\right]\Psi}$$

$$\hat{L}_z = -i\frac{\partial}{\partial\phi} = -i\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$$

Solve GP equation numerically

plot density $|\Psi|^2$ for $\Omega = 0$



$\boxed{\Omega \neq 0}$ quantized vortices

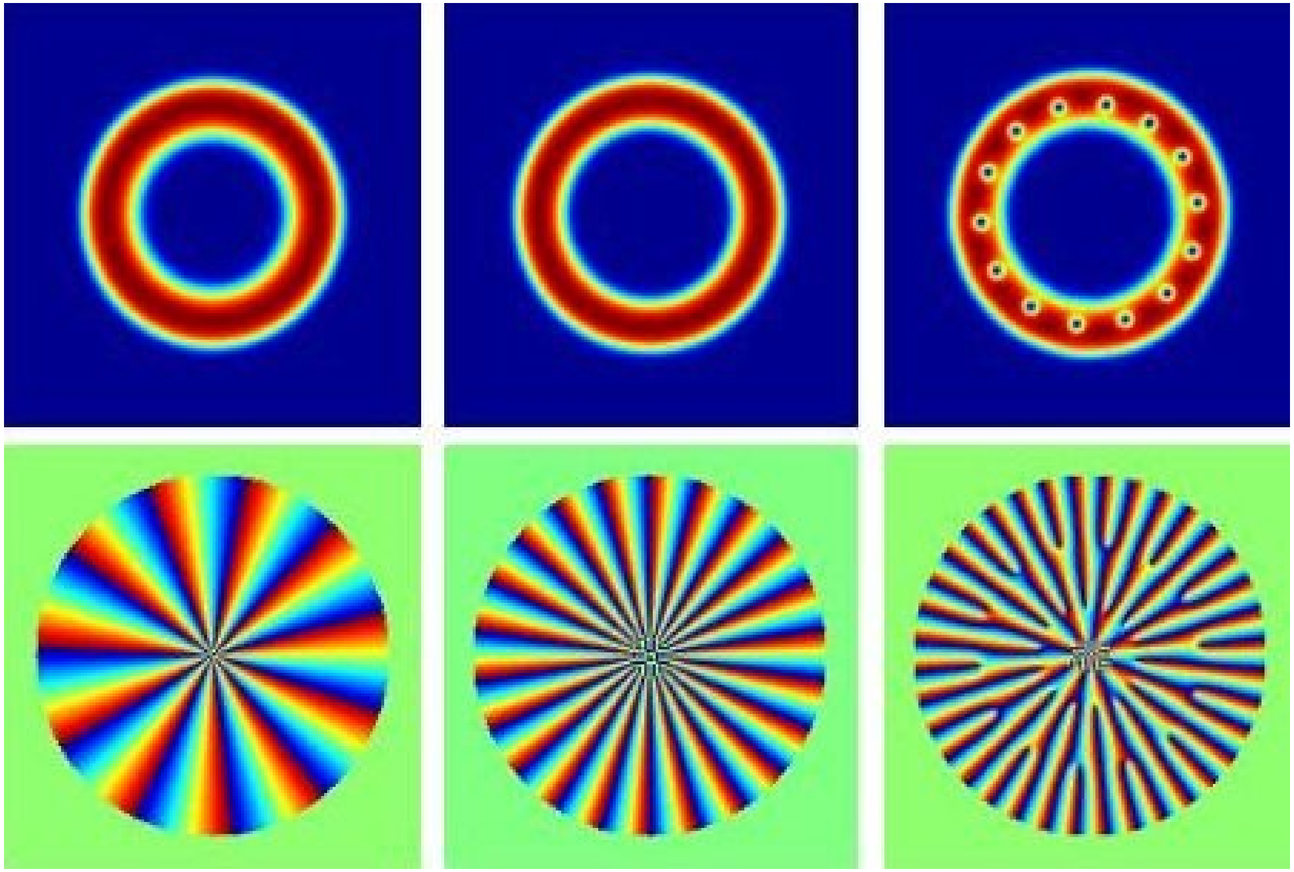
$$\Psi(r,\phi) = |\Psi(r)|\, e^{i\nu\phi} \quad \Rightarrow \quad \hat{L}_z\Psi = \nu\Psi$$

$$\boxed{\Psi = \sqrt{n}\, e^{iS}}$$

$\Omega = 0.2$ $\Omega = 0.4$ $\Omega = 0.5$

Density, *n*

Phase, *S*



$\nu = 11$ $\nu = 22$

Thomas-Fermi approximation ($\nu>0$)

$$n(r) \equiv \left|\Psi(r)\right|^2 = \frac{1}{g}\left(\mu - \frac{\nu^2}{2r^2} + \frac{1}{2}r^2 - \frac{\lambda}{2}r^4\right) \quad \text{for} \quad n(r) > 0$$

$$n(r) = 0 \quad \text{otherwise}$$

$$\boxed{\lambda\nu \ll 1}$$

$$\lambda R_+^2 = 1 + \frac{4}{1-\eta^2}(\lambda\nu)^2 + O\left[(\lambda\nu)^4\right]$$

$$\lambda R_-^2 = \eta + \frac{1}{\eta}\left[\frac{4}{\eta^2-1} + \frac{2}{\eta}\ln\left(\frac{1+\eta}{1-\eta}\right)\right](\lambda\nu)^2 + O\left[(\lambda\nu)^4\right]$$

$$\lambda\mu = \frac{\eta^2-1}{8} + \frac{1}{2\eta}\ln\left(\frac{1+\eta}{1-\eta}\right)(\lambda\nu)^2 + O\left[(\lambda\nu)^4\right]$$

Thomas-Fermi approximation ($\nu > 0$)

$$n(r) \equiv |\Psi(r)|^2 = \frac{1}{g}\left(\mu - \frac{\nu^2}{2r^2} + \frac{1}{2}r^2 - \frac{\lambda}{2}r^4\right) \quad \text{for} \quad n(r) > 0$$

$$n(r) = 0 \quad \text{otherwise}$$

Energy per particle:

$$E = \frac{1}{2}\int d\mathbf{r}\, \Psi^* \left(-\nabla^2 - r^2 + \lambda r^4 + gn\right)\Psi$$

$$\Rightarrow \quad \lambda E = -\frac{1}{8} + \frac{3}{40}\eta^2 + E_1(\lambda\nu)^2 + O\left[(\lambda\nu)^4\right]$$

$$E_1 = \frac{3}{4\eta^3}\left[2\eta - (1-\eta^2)\ln\left(\frac{1+\eta}{1-\eta}\right)\right]$$

Find ground state in rotating frame- minimize: $E'(\nu) = E(\nu) - \Omega\nu$

$$E'(\nu) = -\frac{1}{8\lambda} + \frac{3}{40\lambda}\eta^2 + \lambda E_1 \nu^2 - \Omega\nu$$

Treating ν as a continuous variable... $\dfrac{\partial E'}{\partial \nu} = 0$

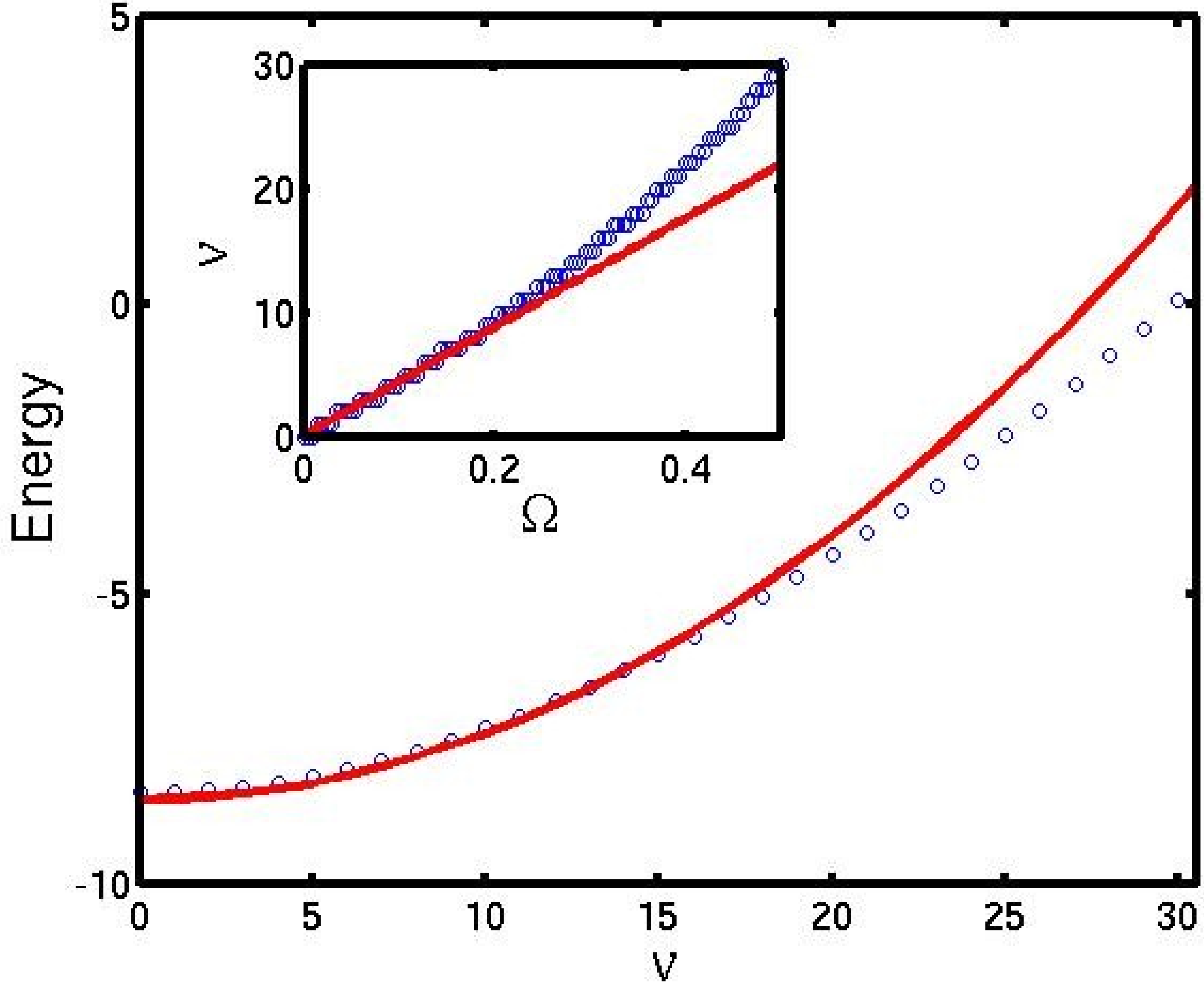
$$\boxed{\lambda\nu = \frac{\Omega}{2E_1}}$$

$$E_1 = \frac{3}{4\eta^3}\left[2\eta - \left(1-\eta^2\right)\ln\left(\frac{1+\eta}{1-\eta}\right)\right]$$

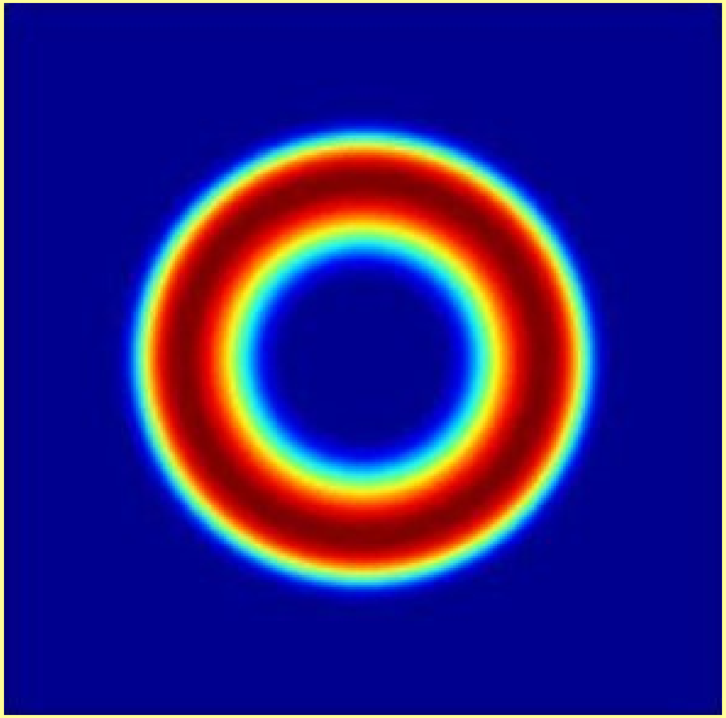
In limit of narrow annulli: $\eta \rightarrow 0$

$$E_1 \approx 1 \qquad \nu \approx \frac{\Omega}{2\lambda}$$

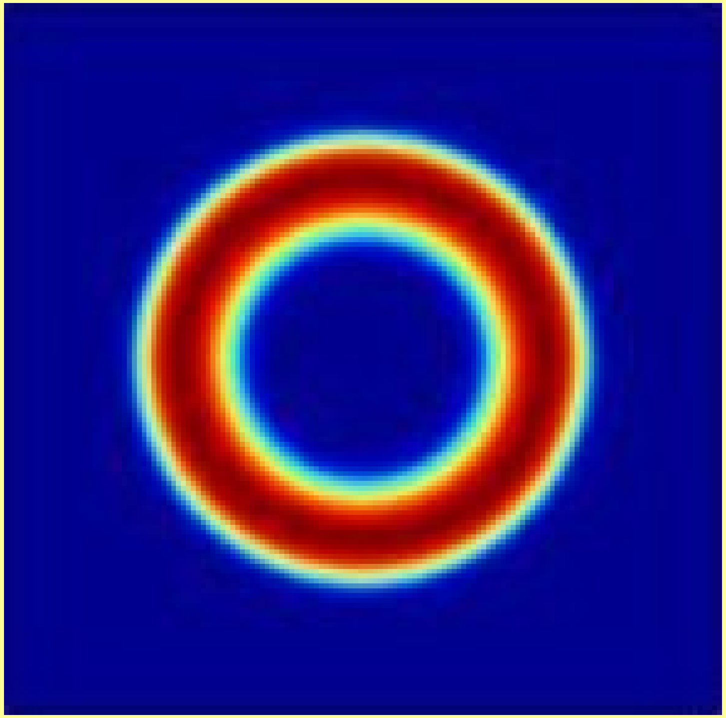
λ=0.01, g=1000



Densities:



$\nu = 0$



$\nu = 11$

Detecting vortices in annular condensates:

- Collective modes
- Expansion
- Momentum distribution

Collective modes

GP equation in rot. frame→ $\Psi = \sqrt{n}\, e^{iS}$ → TF approximation → linearize → ***Hydrodynamic equations***

$$\frac{\partial}{\partial t}\delta n + \left(\frac{\nu}{r^2} - \Omega\right)\frac{\partial}{\partial \phi}\delta n + \nabla \bullet \left(n_0 \nabla \delta S\right) = 0$$

$$\frac{\partial}{\partial t}\delta S + \left(\frac{\nu}{r^2} - \Omega\right)\frac{\partial}{\partial \phi}\delta S + g\delta n = 0$$

$$\boxed{\delta n, \delta S \propto e^{i(m\phi - \omega t)}}$$

$$\boxed{\left[\left(\omega + m\Omega - \frac{m\nu}{r^2}\right)^2 - \frac{m^2}{r^2} g n_0\right]\delta S + \frac{1}{r}\frac{\partial}{\partial r}\left(r g n_0 \frac{\partial}{\partial r}\delta S\right) = 0}$$

Non-rotating case (ν=0), in limit $\eta \to 0$

Introduce new variable: $\zeta = \frac{2}{R_-^2}\left(r^2 - \frac{R_+^2}{2} \right)$

$$\Rightarrow \quad \boxed{\omega^2 \delta S + \frac{\partial}{\partial \zeta}\left[\left(1 - \zeta^2\right) \frac{\partial}{\partial \zeta} \delta S \right] = 0}$$

Legendre's equation, eigenvalues:

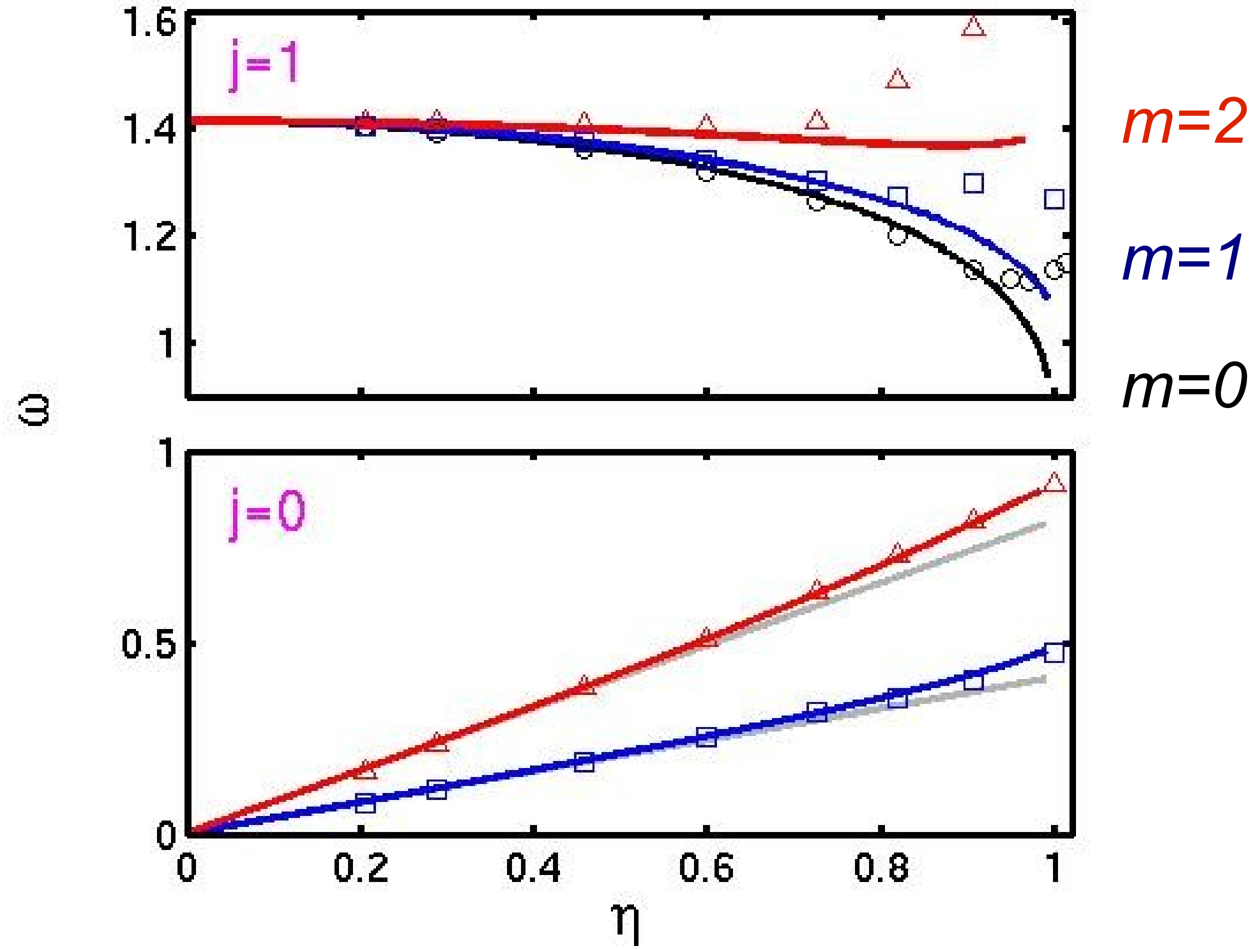
$$\boxed{\omega^2 = j(j+1)}$$

j=*0,1,2,...*where different *m* states are degenerate in this limit

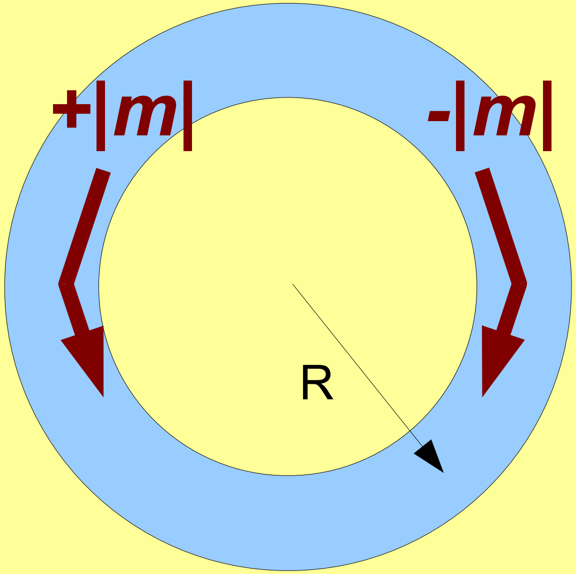
Collective modes

$$\omega(\eta \to 0) = \sqrt{2}$$

$$\omega(\eta \to 0) = 0$$



Low-lying modes (j=0)



Correspond to sound waves traveling around the annulus

For η<<1

$$\omega = cq \qquad q = \frac{m}{R}$$

speed of sound *mean radius*

$$c^2 = N\frac{\partial\mu}{\partial N} \qquad R = \frac{R_+}{\sqrt{2}}$$

$$\boxed{\omega = |m|\frac{\eta}{\sqrt{6}}}$$

without vortex: +*ve* and -*ve* *m* modes are degenerate

with vortex: introduces frequency splitting between |*m*|>*0* modes

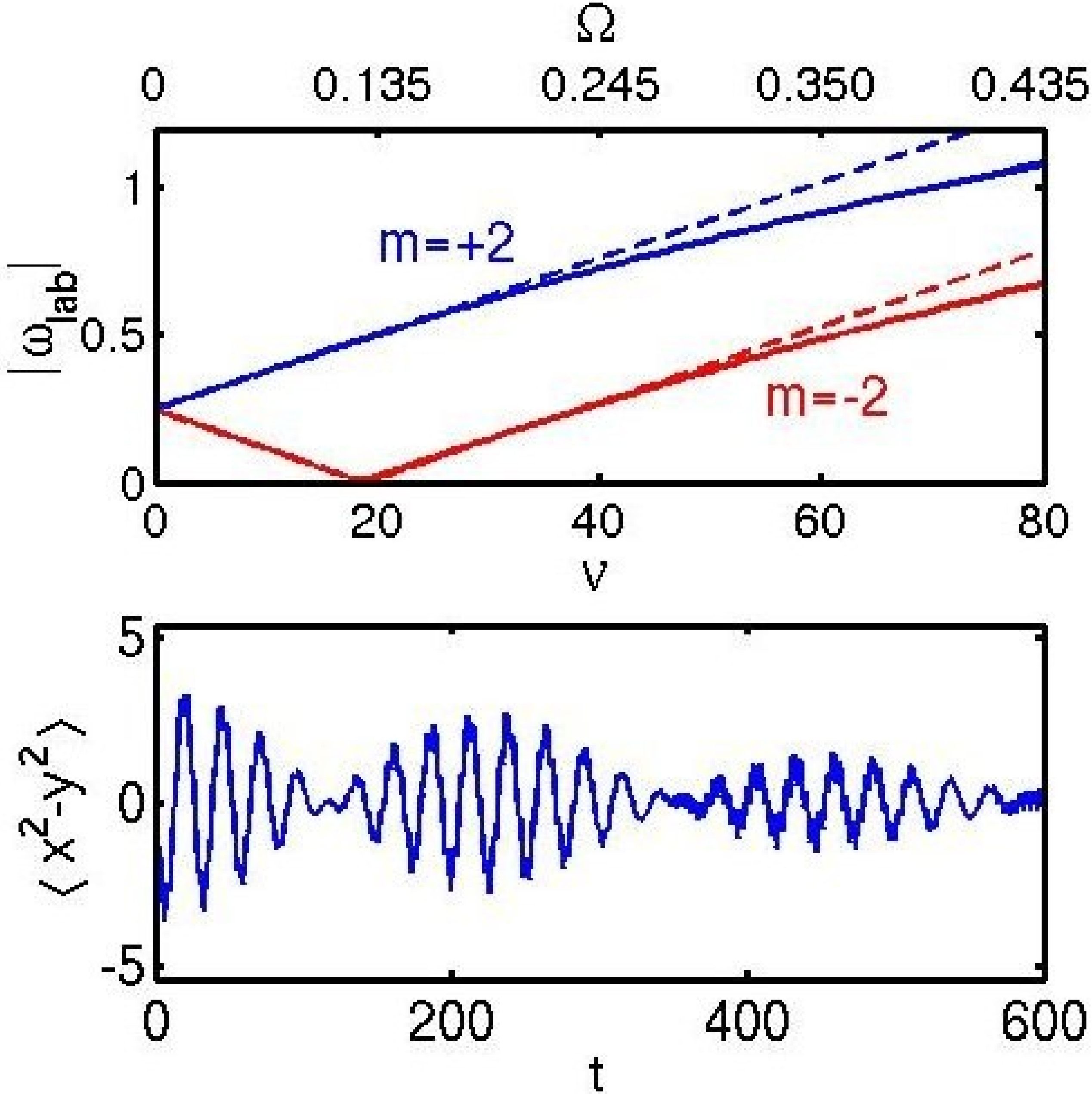
j=0 modes: sum rule calculation

$$\boxed{\Delta\omega = 4|m|\lambda\nu\left[1 + O\left(\eta^2\right)\right]}$$

$$\omega_\pm = |m|\left(\frac{\eta}{\sqrt{6}} \pm 2\lambda\nu\right)$$

If both modes are excited, then leads to “precession” of the oscillation

$$\omega_{pr} = 2\lambda\nu \approx \Omega$$



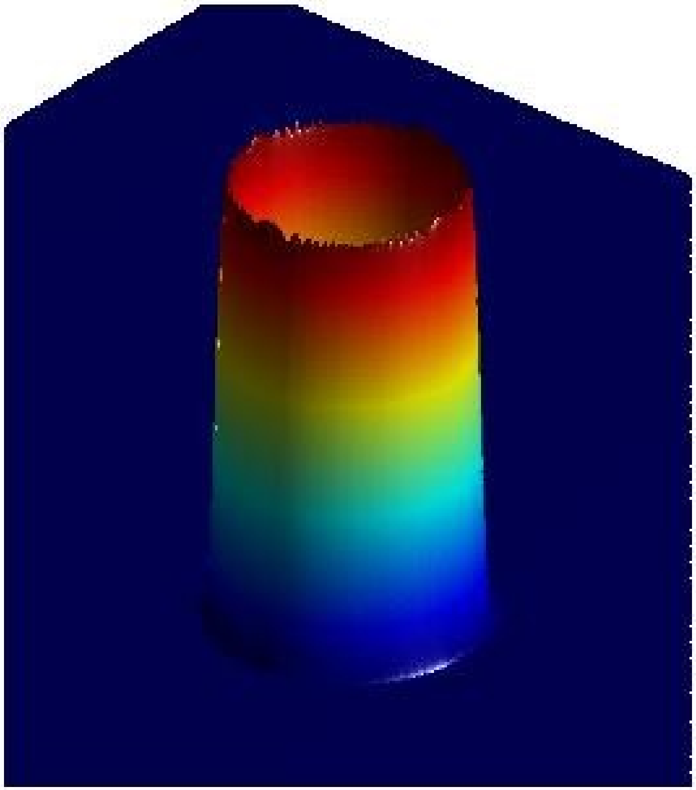
Ω
0
0.135
0.245
0.350
0.435
1
0.5
0
$|\omega_{lab}|$
m=+2
m=-2
0
20
40
60
80
v
5
0
-5
$\langle x^2-y^2 \rangle$
0
200
400
600
t

Detecting vortices in annular condensates:

- Collective modes
- Expansion
- Momentum distribution

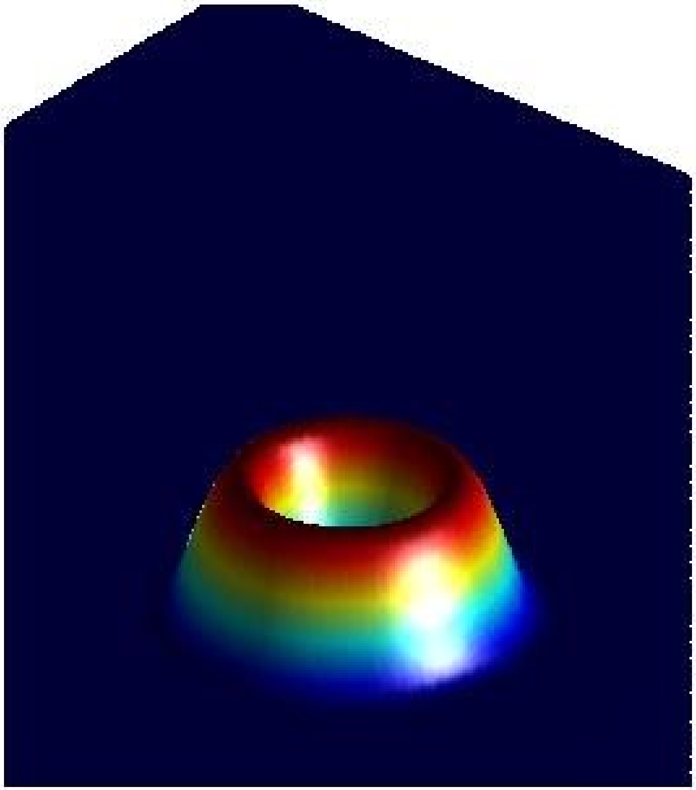
Expansion

$\nu = 0$



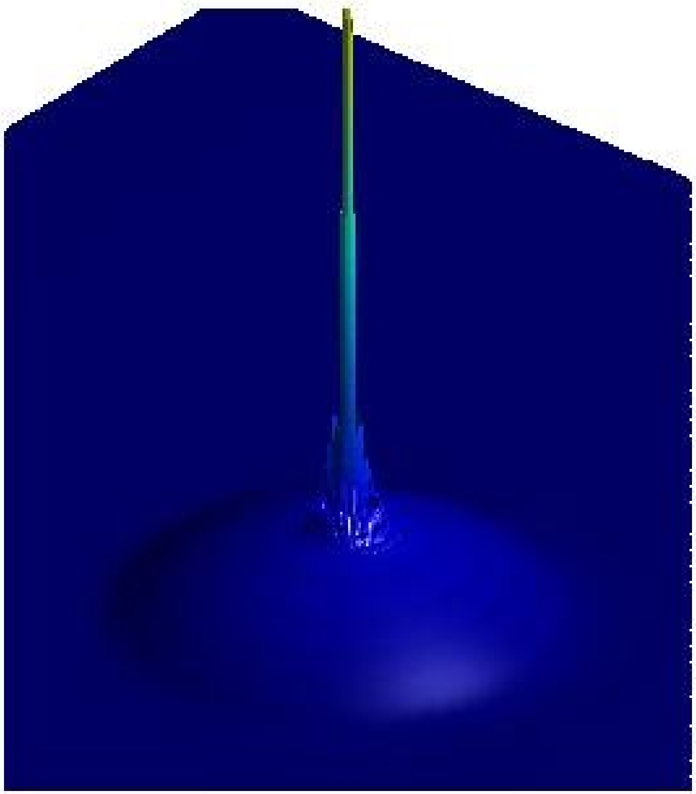
Expansion

$\nu = 0$



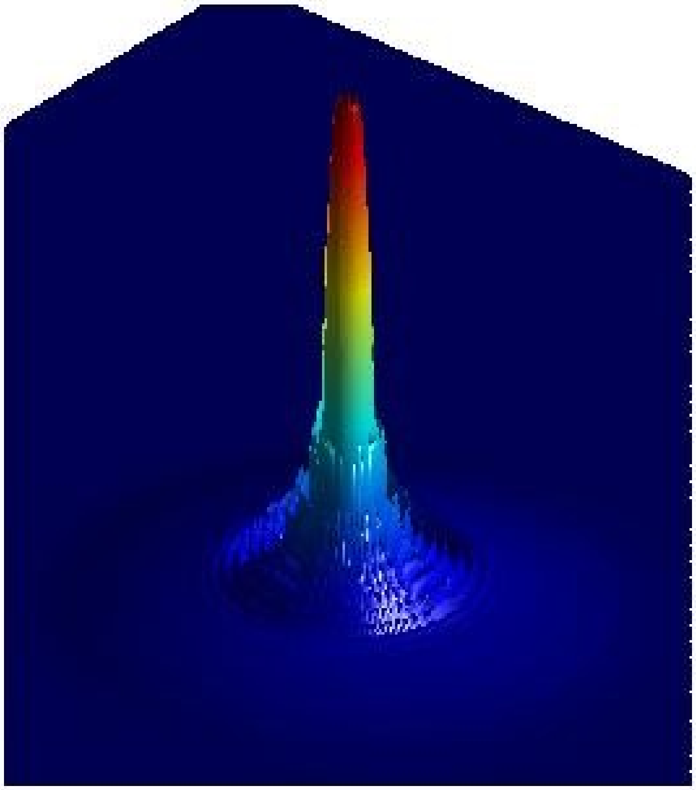
Expansion

$\nu = 0$



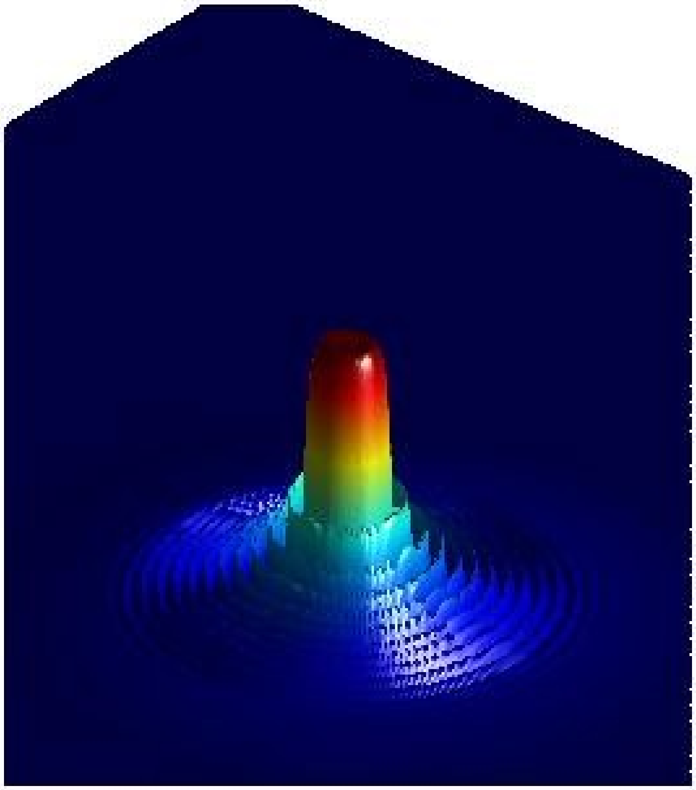
Expansion

$\nu = 0$



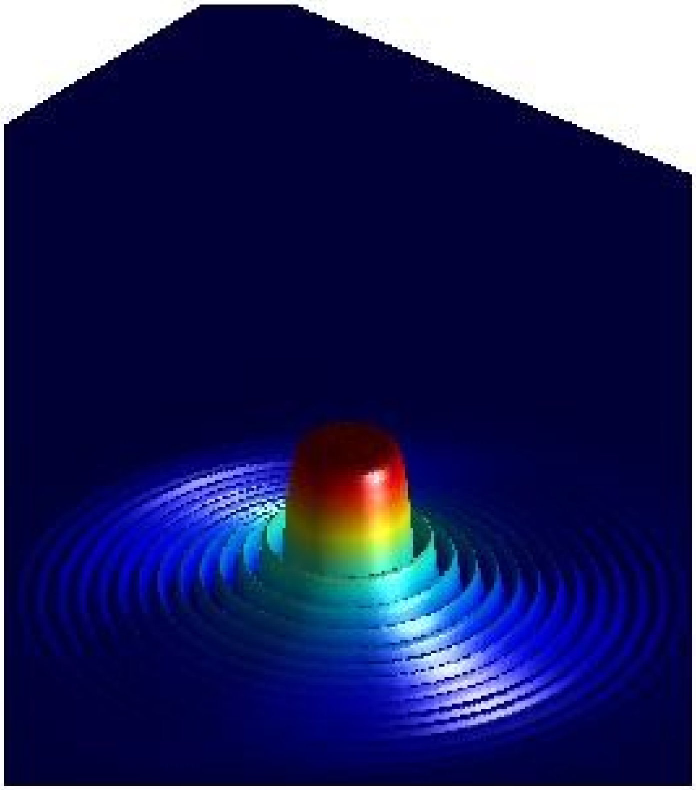
Expansion

$\nu = 0$



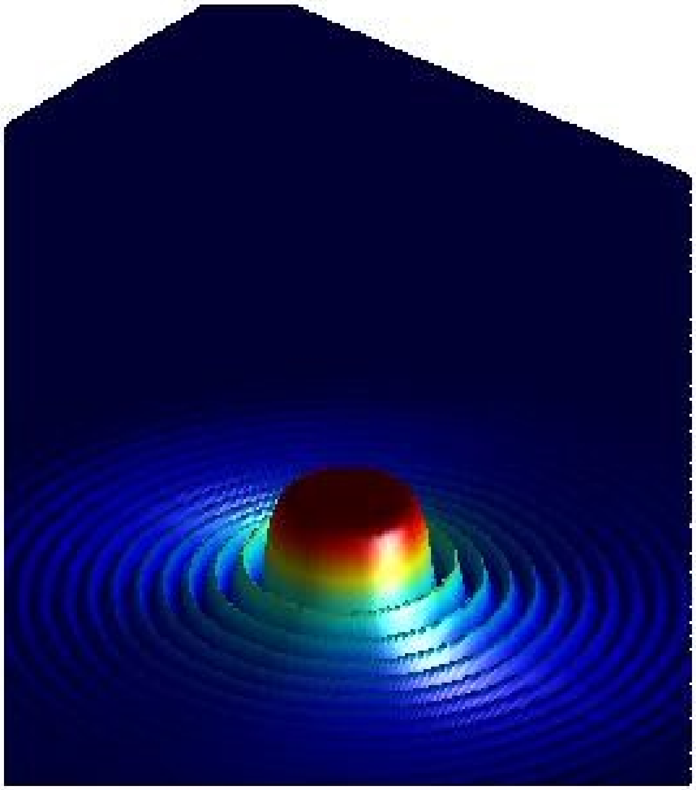
Expansion

$\nu = 0$



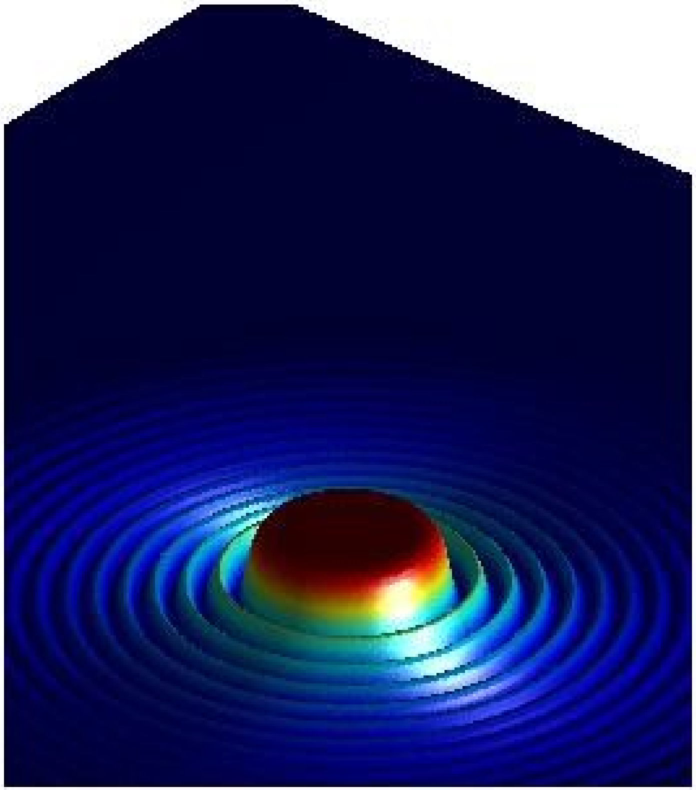
Expansion

$\nu = 0$



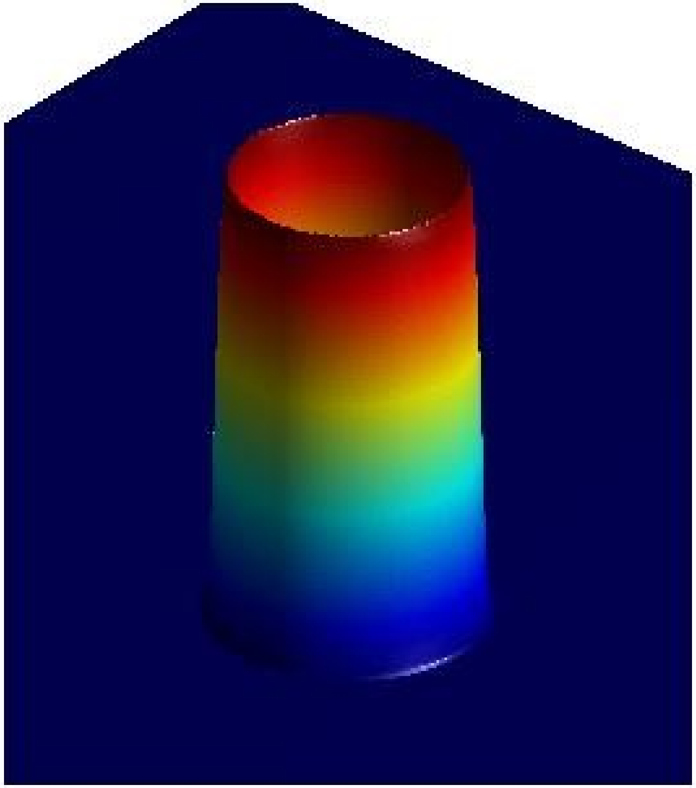
Expansion

$\nu = 0$



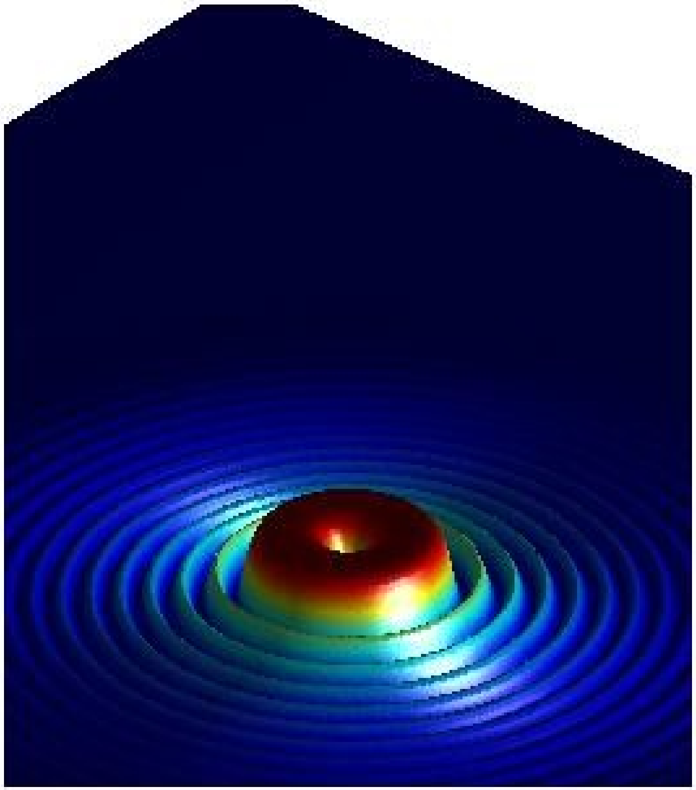
Expansion

$\nu = 1$

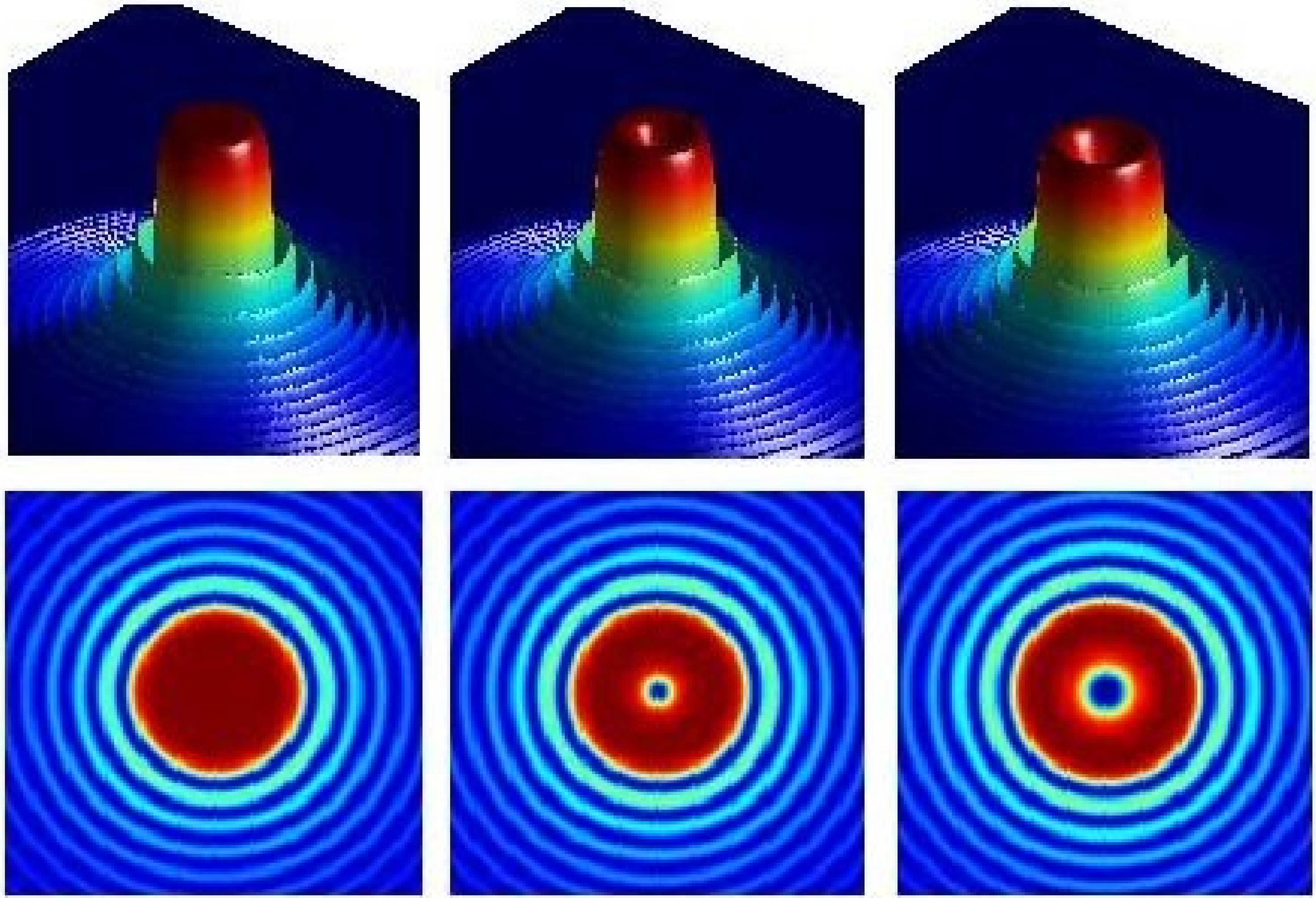


Expansion

$\nu = 1$



$\nu = 0$ $\nu = 1$ $\nu = 2$



Expansion with no interactions (g=0):

$$\Psi(\mathbf{r},t) = e^{-i\pi/2}\frac{m}{2\pi\hbar t}\int d\mathbf{r}'\,\Psi(\mathbf{r}',0)e^{im(\mathbf{r}-\mathbf{r}')^2/2\hbar t}$$

At large times $t >> \dfrac{mR_+^2}{\hbar}$

$$\left|\Psi\left(\mathbf{r},t\to\infty\right)\right|^2 = \left(\frac{m}{\hbar t}\right)^2\left|\Phi\left(\frac{m\mathbf{r}}{\hbar t}\right)\right|^2$$

$\Phi(\mathbf{k})$ = momentum distribution – F.T. of $\Psi(\mathbf{r})$

Detecting vortices in annular condensates:

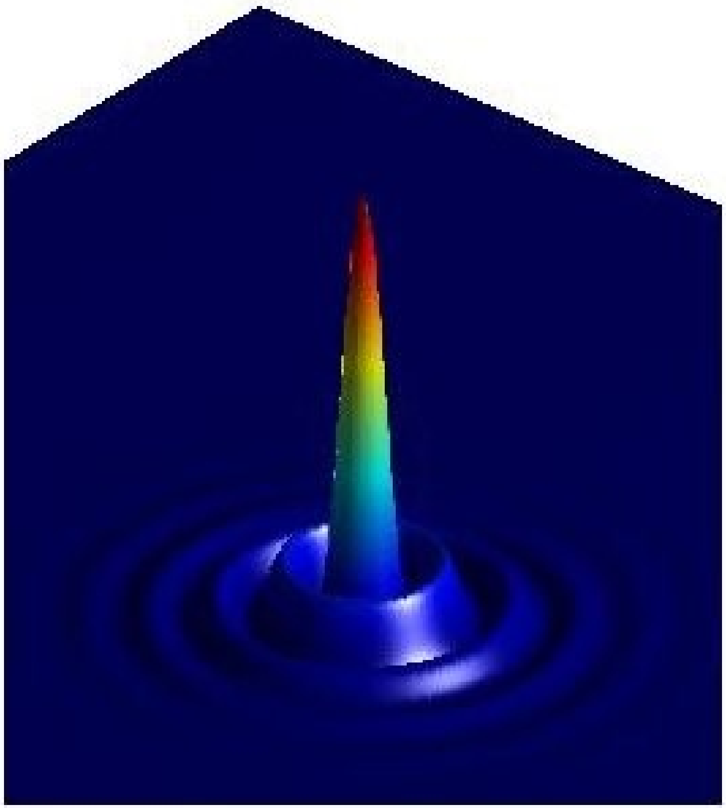
- Collective modes
- Expansion
- Momentum distribution

Momentum distribution

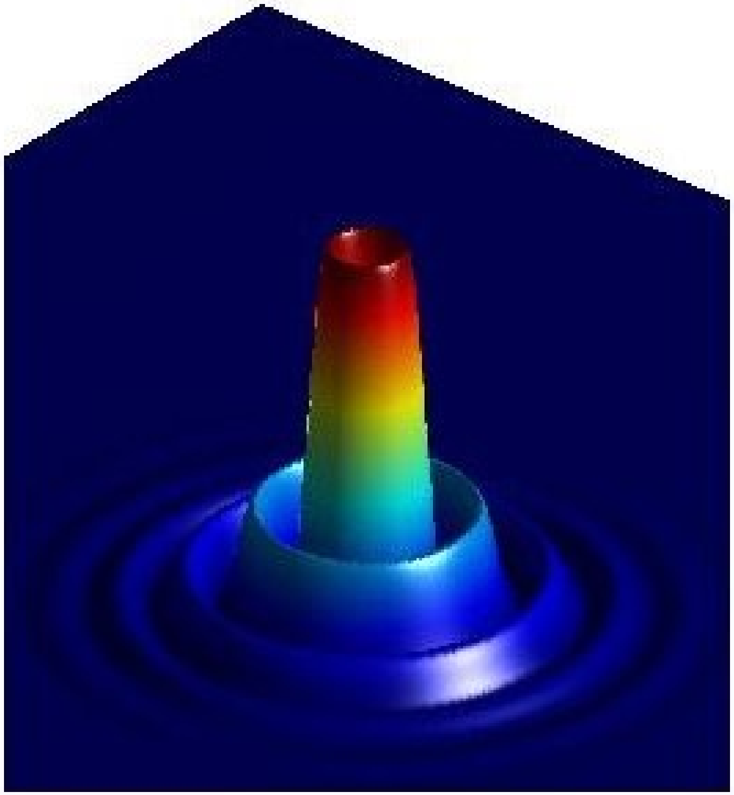
with single vortex: $\Psi(\mathbf{r}) = \left|\Psi(r)\right| e^{i\nu\phi}$

$$\left|\Phi(k)\right| = \int_0^\infty dr\, r\left|\Psi(r)\right| J_\nu(kr)$$

Momentum Distribution



$\nu = 0$



$\nu = 1$

Momentum distribution

with single vortex: $\Psi(\mathbf{r}) = \left|\Psi(r)\right| e^{i\nu\phi}$

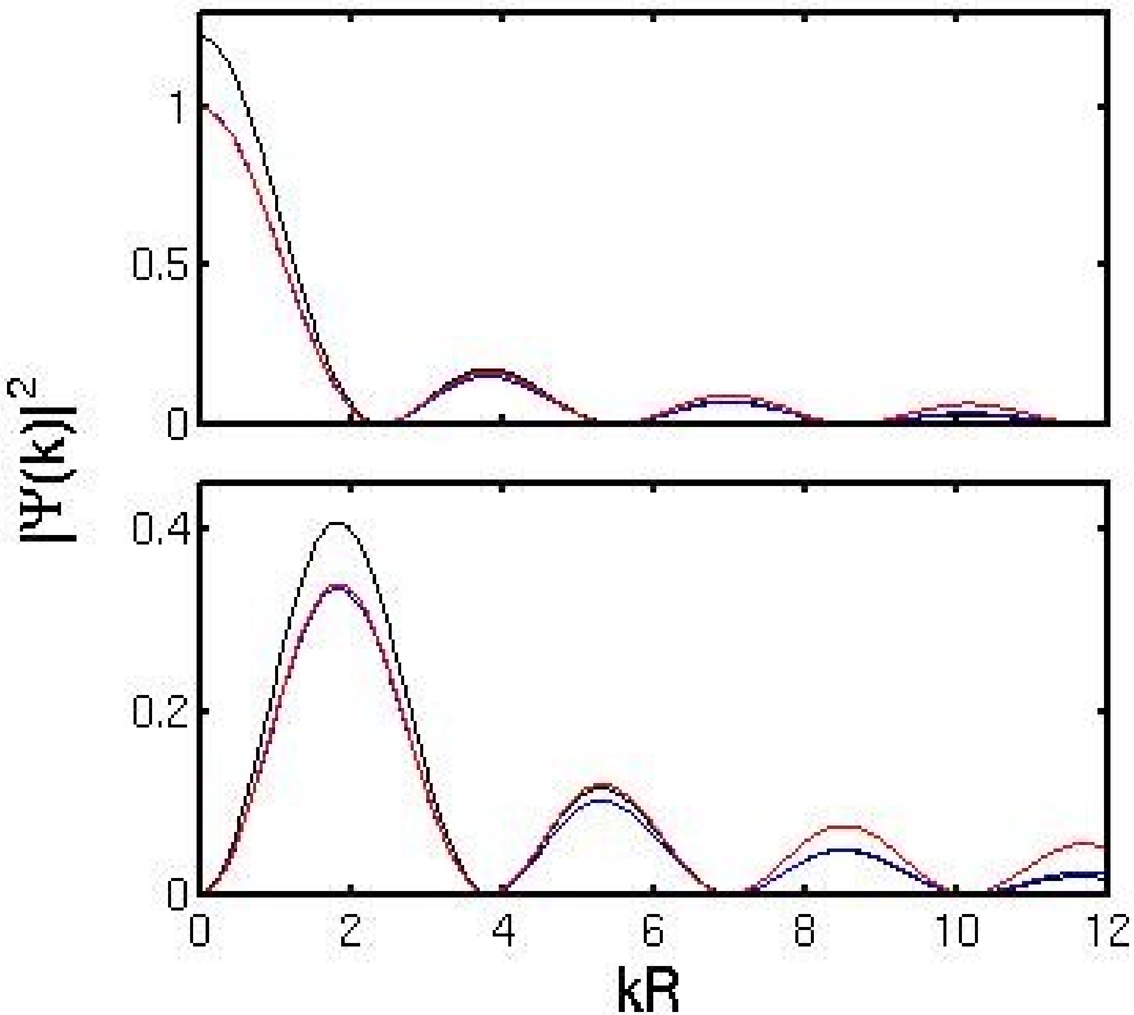
$$\left|\Phi(k)\right| = \int_0^\infty dr\, r\left|\Psi(r)\right| J_\nu(kr)$$

TF approximation in thin annulus limit $(\eta \rightarrow 0)$:

$$\left|\Phi(k)\right| = \sqrt{\frac{3\pi\eta}{128\lambda}}\left[J_\nu(kR) + O\left(\eta^2\right)\right]$$

radius of annulus: $R = \frac{R_+}{\sqrt{2}} = \frac{1}{\sqrt{2\lambda}}$

Momentum Distribution

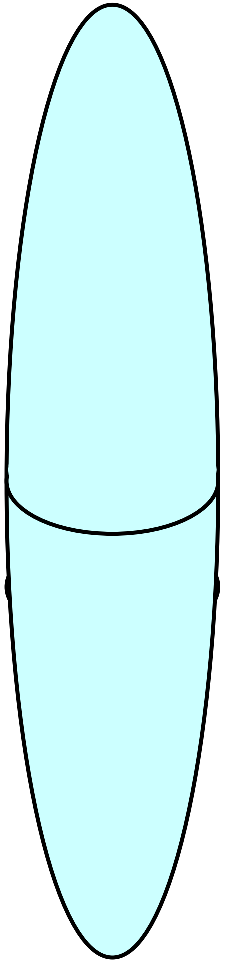


$\nu = 0$

$\nu = 1$

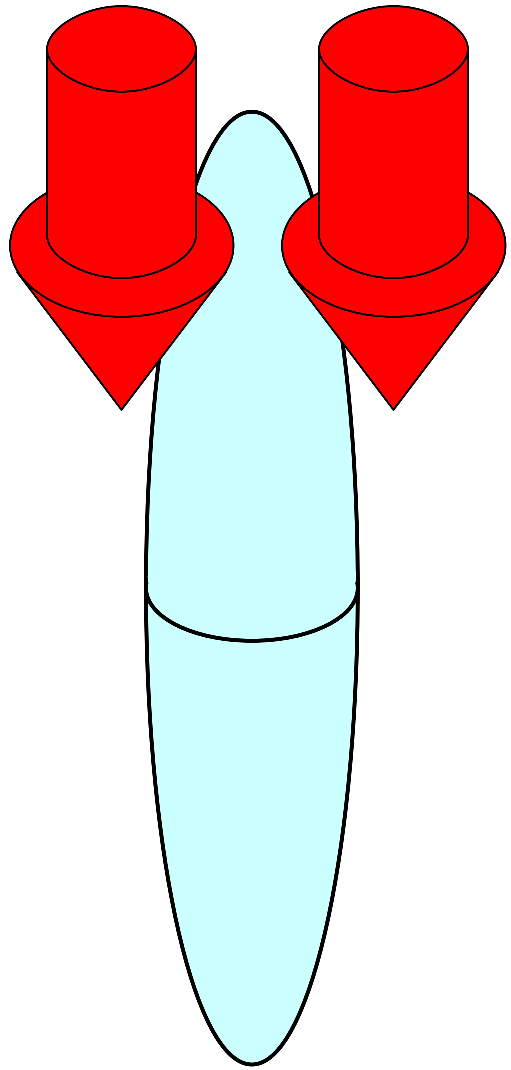
Vortex formation in rotating condensates

Trapped BEC



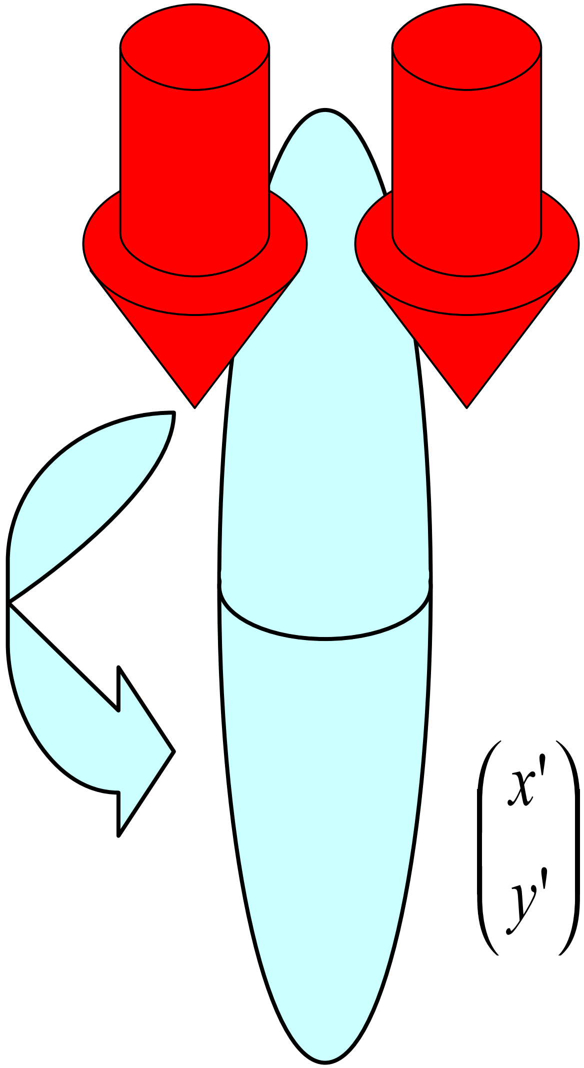
$$V = \frac{1}{2} m \left[\omega_\perp^2 \left(x^2 + y^2 \right) + \omega_z^2 z^2 \right]$$

Trapped BEC



$$V = \frac{1}{2} m \left[\omega_\perp^2 \left((1+\varepsilon) x^2 + (1-\varepsilon) y^2 \right) + \omega_z^2 z^2 \right]$$

Trapped BEC

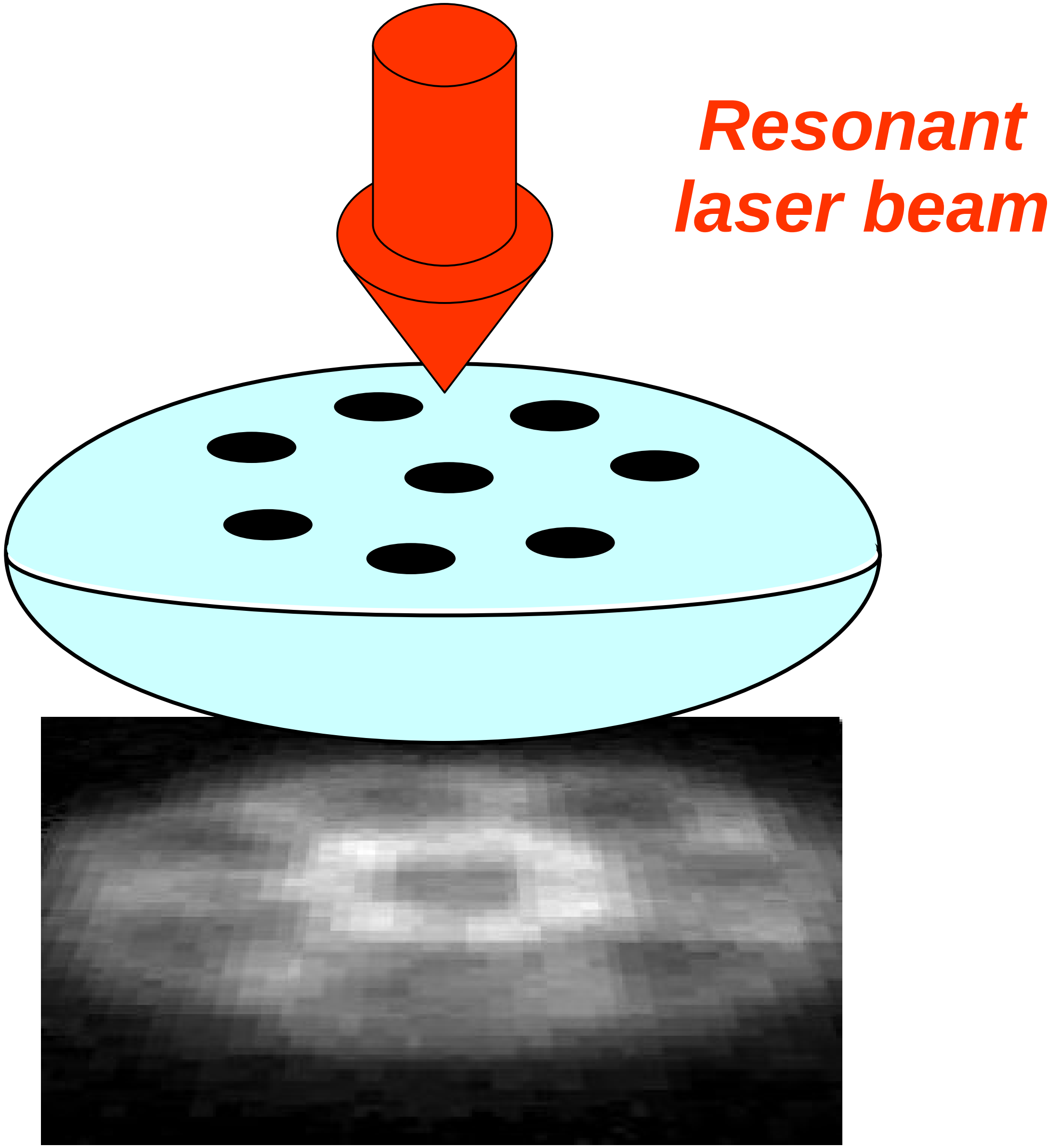


Ω

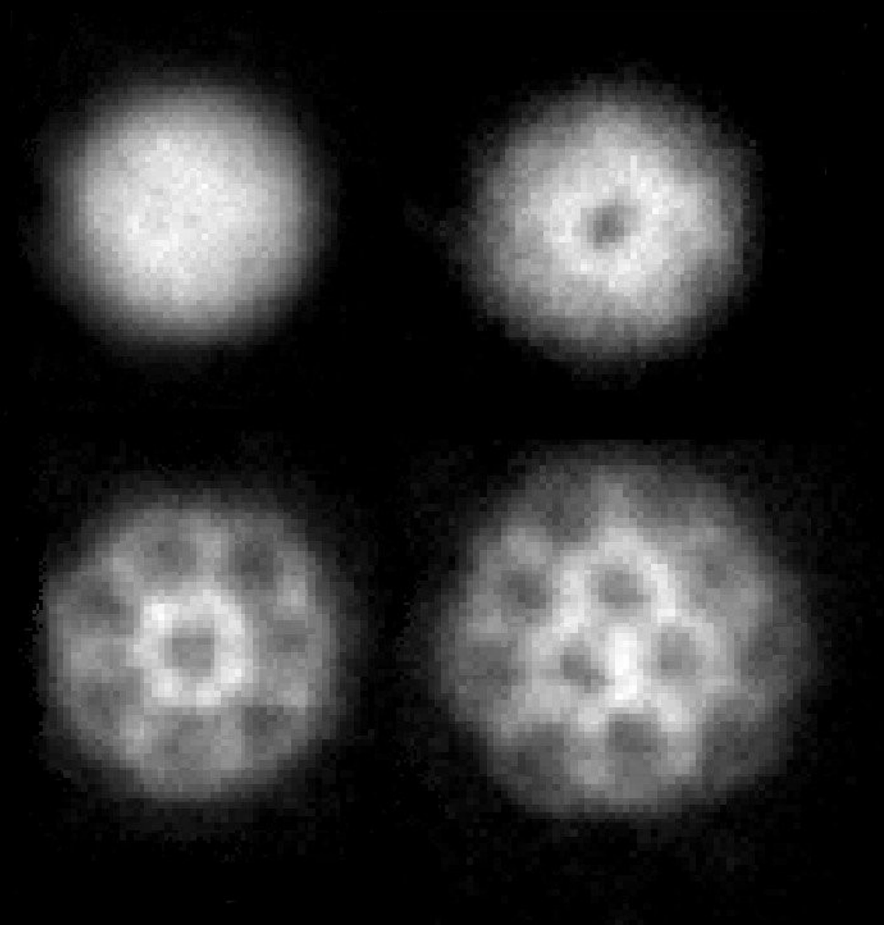
$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos(\Omega t) & \sin(\Omega t) \\ -\sin(\Omega t) & \cos(\Omega t) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$V = \frac{1}{2} m \left[\omega_\perp^2 \left((1+\varepsilon) x'^2 + (1-\varepsilon) y'^2 \right) + \omega_z^2 z^2 \right]$$

Trapped BEC



Trapped BEC



Source: *ENS*

2-dim Gross-Pitaevskii eqn.

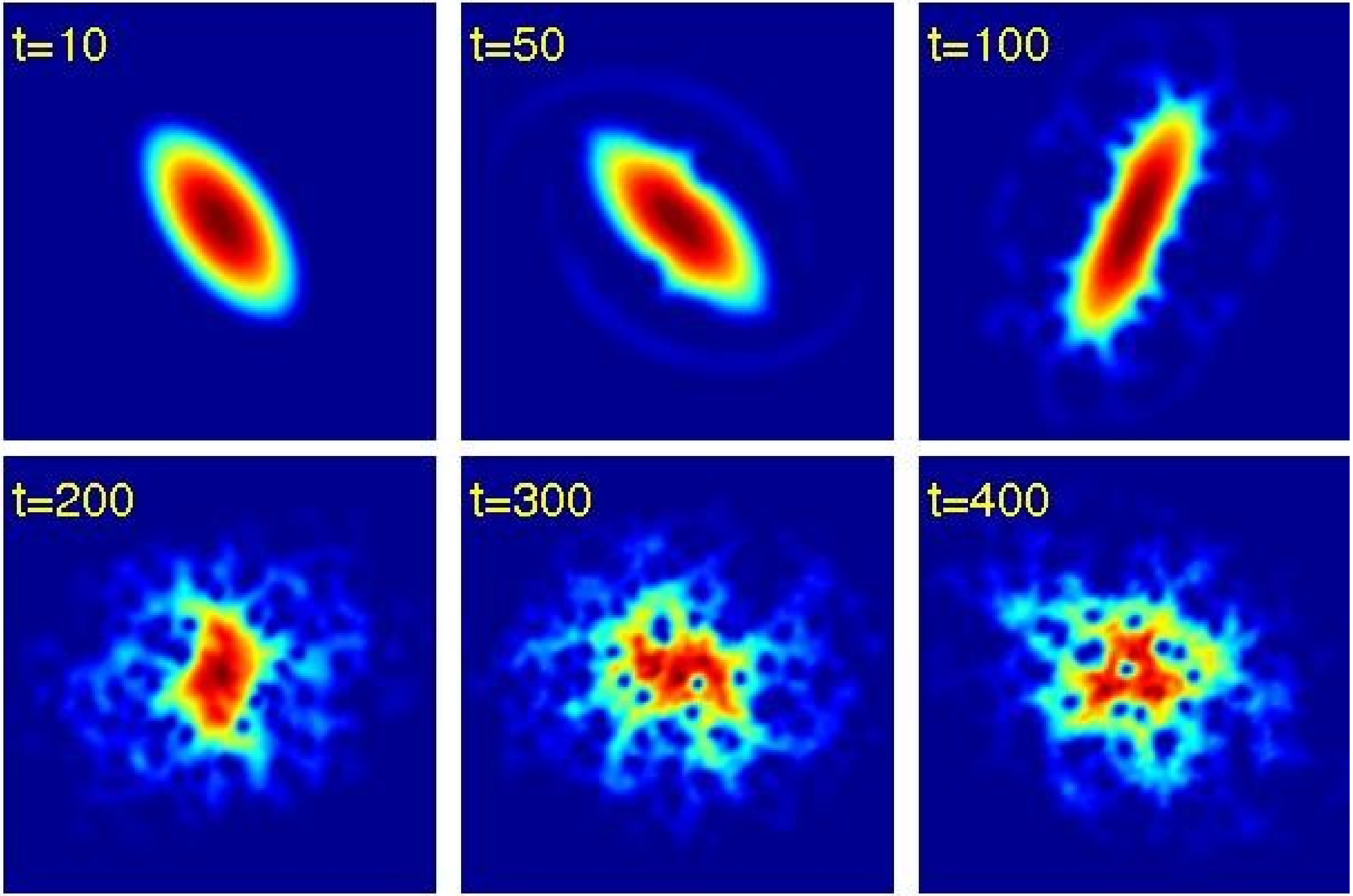
$$\boxed{i\frac{\partial\Psi}{\partial t}=\left[-\frac{1}{2}\nabla^2+\frac{1}{2}\left[(1+\varepsilon)x'^2+(1-\varepsilon)y'^2\right]+g|\Psi|^2\right]\Psi}$$

$$\begin{pmatrix}x'\\y'\end{pmatrix}=\begin{pmatrix}\cos(\Omega t) & \sin(\Omega t)\\-\sin(\Omega t) & \cos(\Omega t)\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$$

Initial state: $\varepsilon=0$ and $\Omega=0$ **no vortices**

For $|\varepsilon|>0$ and $\Omega_1<\Omega<\Omega_2 \rightarrow$ **vortex formation**

$\Omega=0.78$



2-dim Gross-Pitaevskii eqn.

$$\boxed{i\frac{\partial \Psi}{\partial t} = \left[-\frac{1}{2}\nabla^2 + \frac{1}{2}\left[(1+\varepsilon)x'^2 + (1-\varepsilon)y'^2\right] + g|\Psi|^2 \right]\Psi}$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos(\Omega t) & \sin(\Omega t) \\ -\sin(\Omega t) & \cos(\Omega t) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

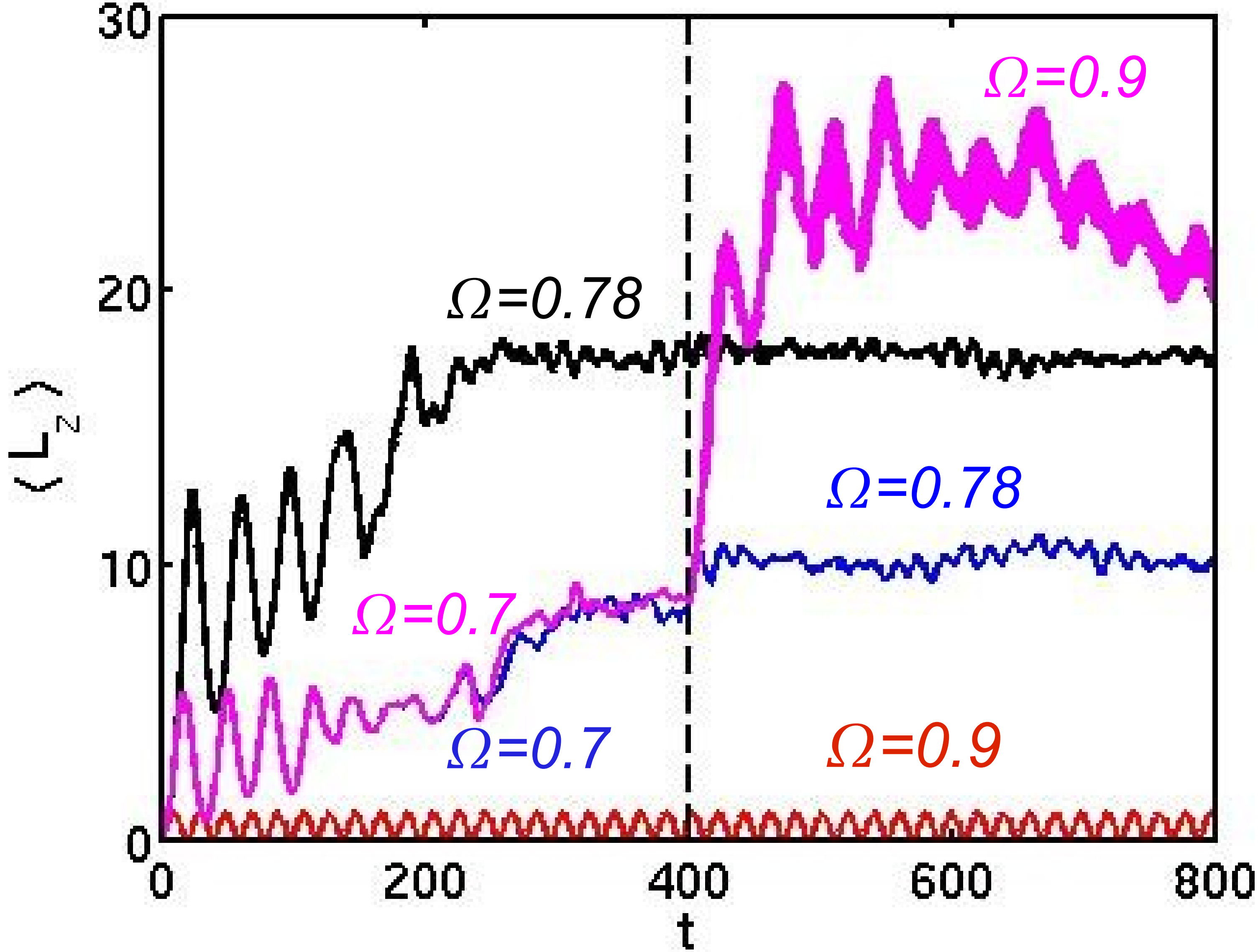
Initial state: ε=0 and $\Omega=0$ **no vortices**

For $|\varepsilon|$>0 and $\Omega_1 < \Omega < \Omega_2 \rightarrow$ **vortex formation**

Number of final vortices depends upon history

$\boxed{\Omega_0=0}$

$$\langle L_z \rangle = \int d\mathbf{r}\, \Psi^* \hat{L}_z \Psi$$



Different initial states:

$$\boxed{\mu\Psi = \left[-\frac{1}{2}\nabla^2 + \frac{1}{2}r^2 + g\left|\Psi\right|^2 - \Omega_0 \hat{L}_z \right]\Psi}$$

- Solve in imaginary time for different Ω_0 to obtain various initial states

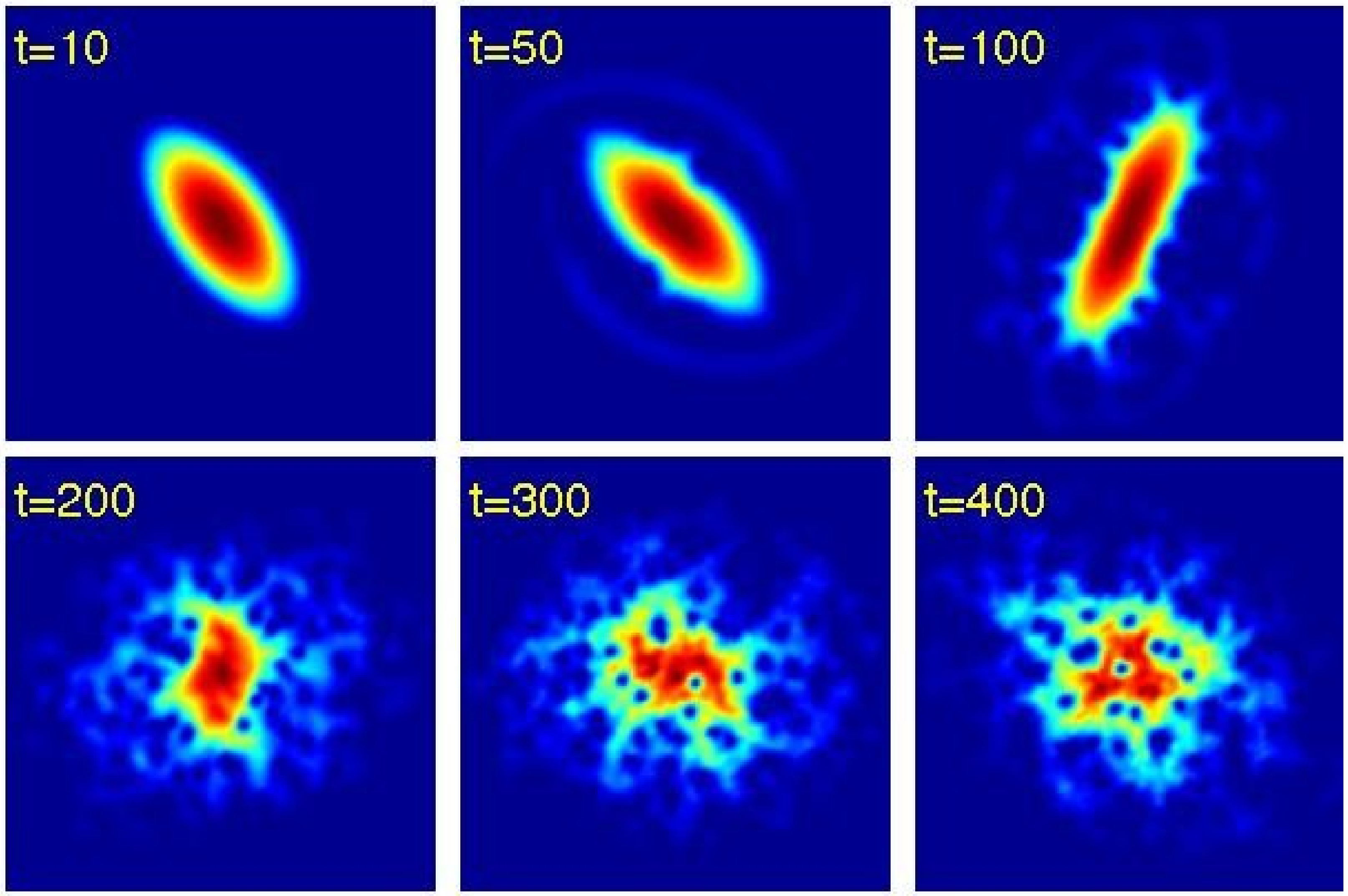
Different initial states:

$$\boxed{i\frac{\partial\Psi}{\partial t}=\left[-\frac{1}{2}\nabla^2+\frac{1}{2}\left[\left(1+\varepsilon\right)x'^2+\left(1-\varepsilon\right)y'^2\right]+g\left|\Psi\right|^2\right]\Psi}$$

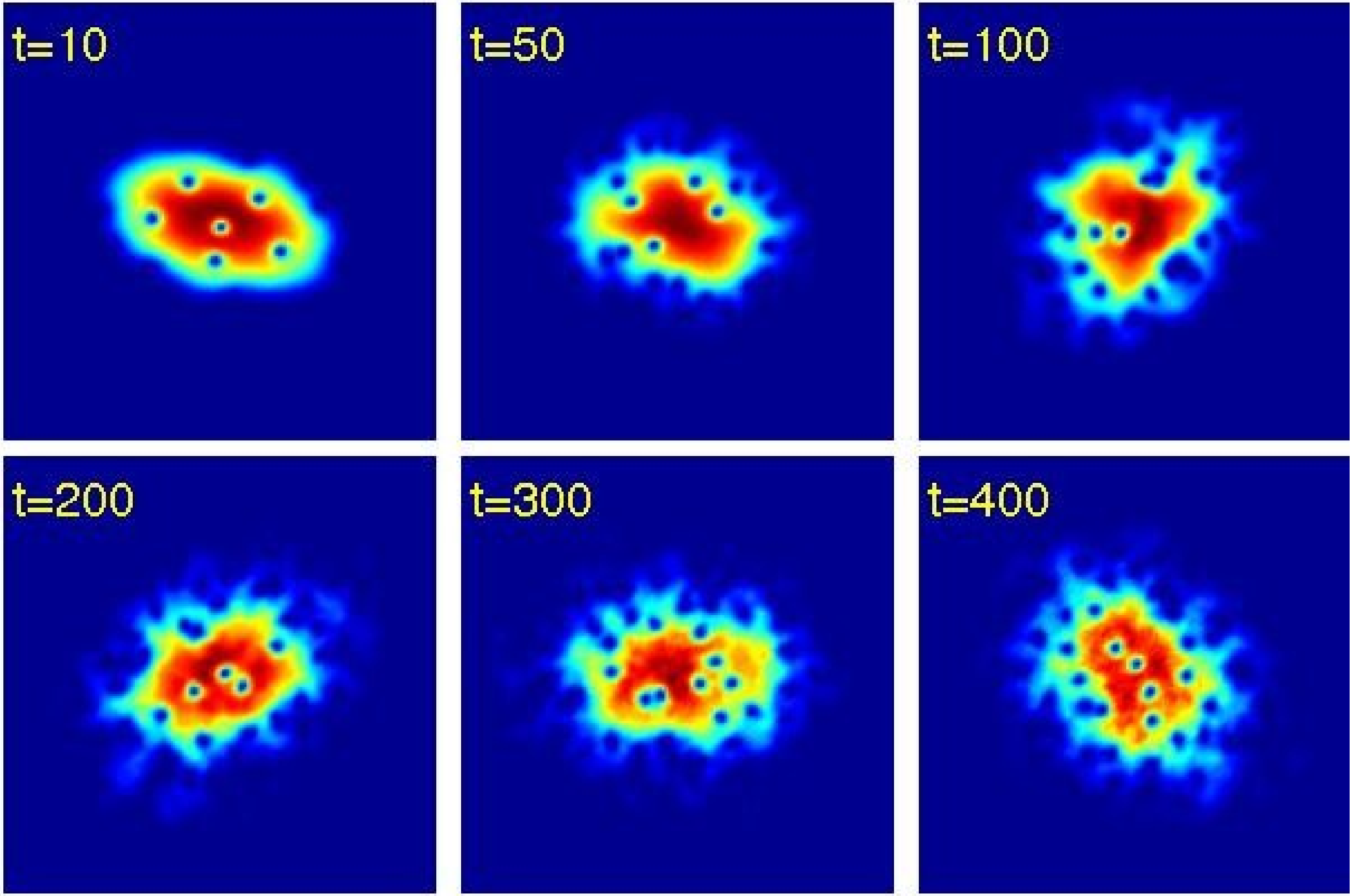
- Solve in imaginary time for different Ω_0 to obtain various initial states

- Propagate in real time at fixed Ω (=0.78) for different initial states and observe subsequent vortex formation and evolution of $\langle \mathbf{L_z} \rangle$

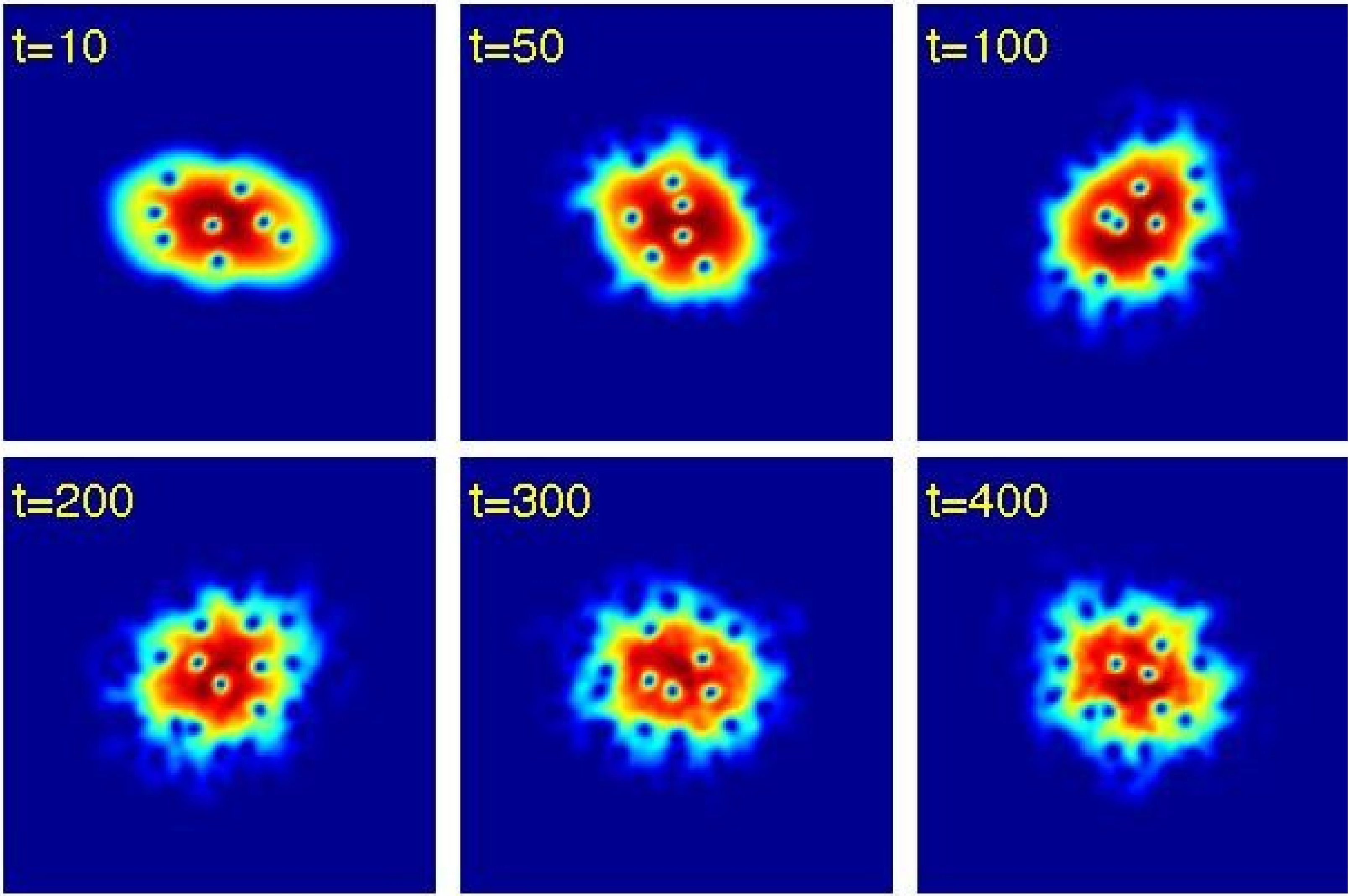
$\Omega_0=0$



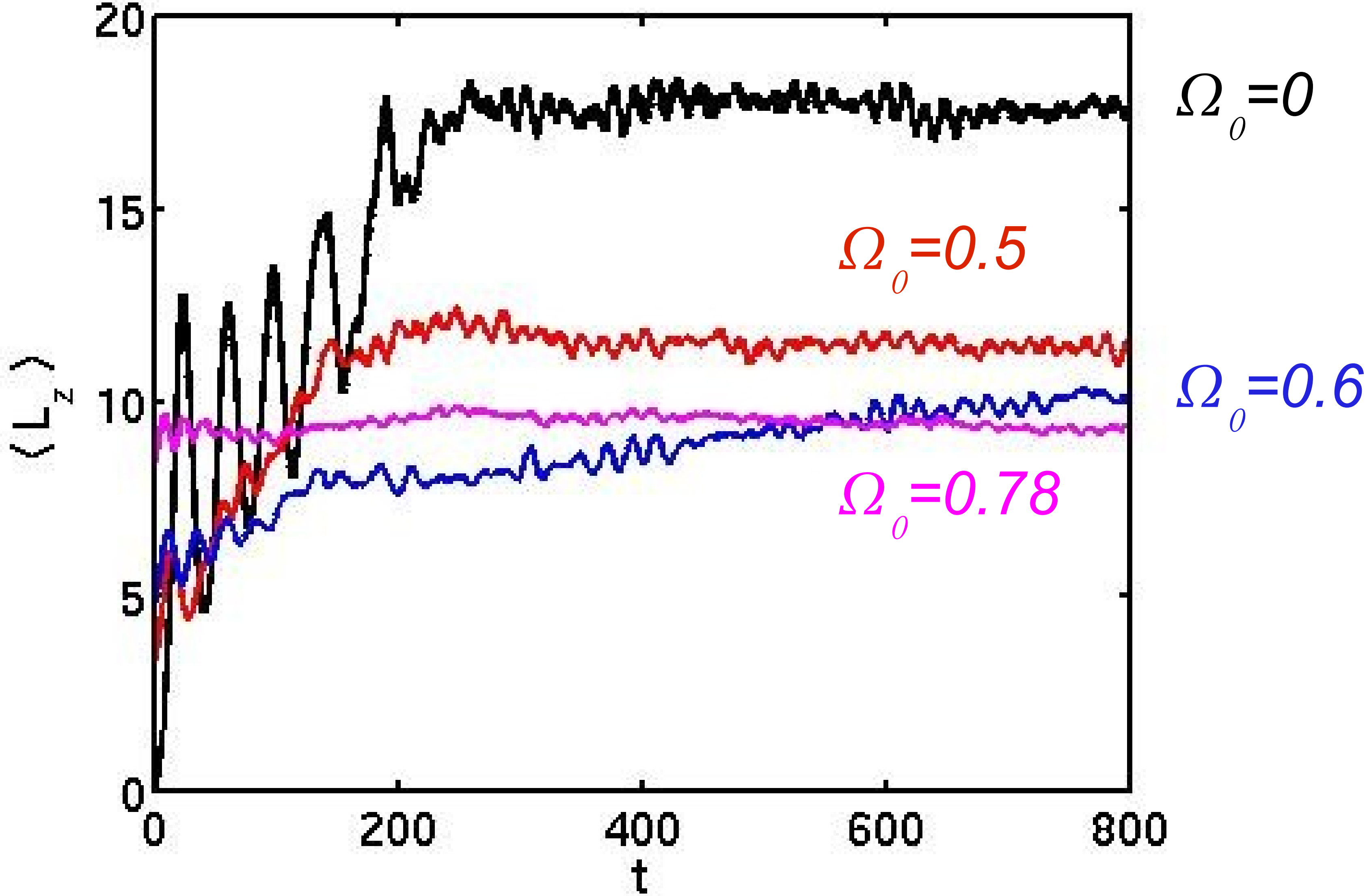
Ω_0=0.5



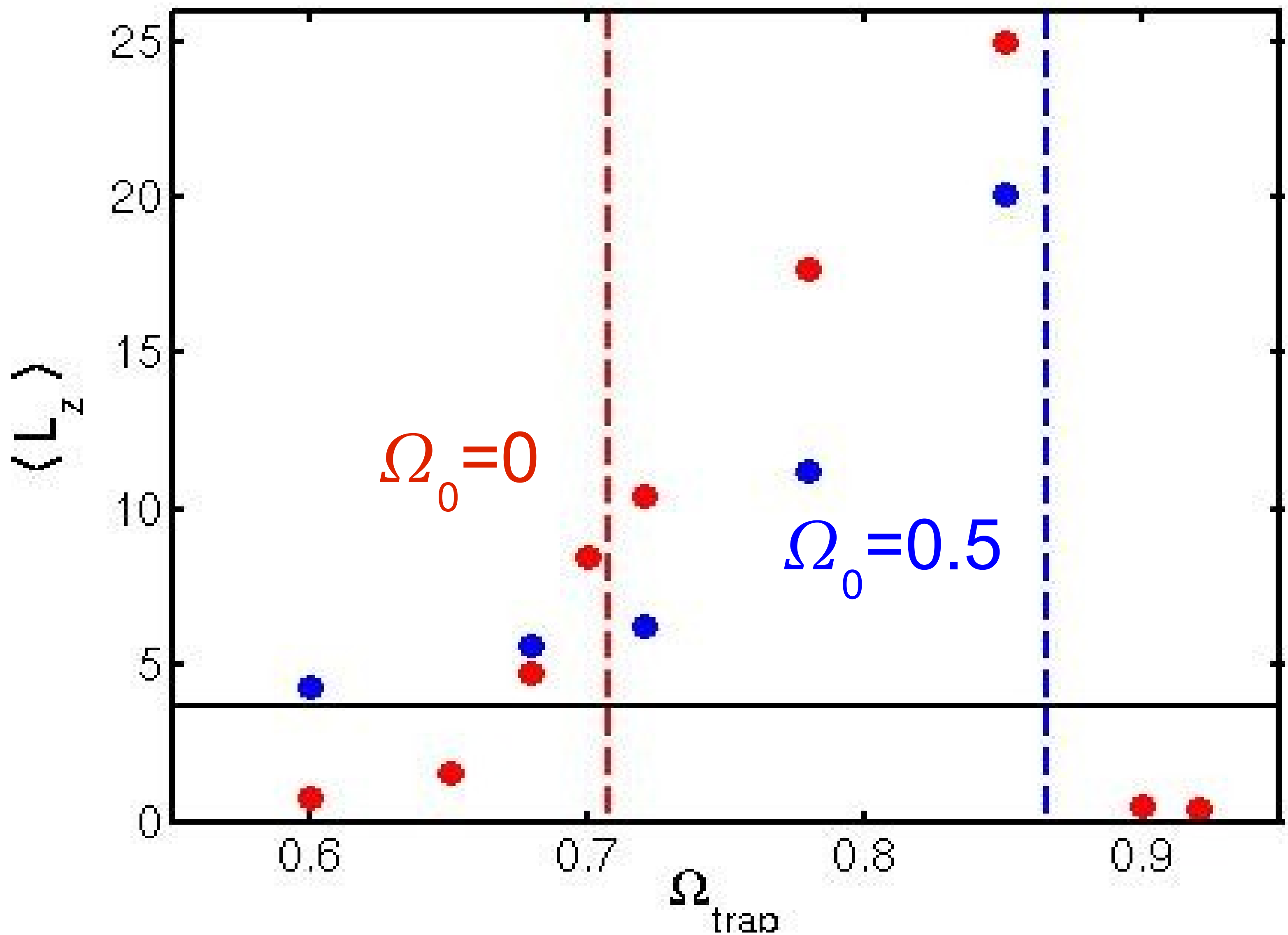
Ω_0=0.6



Ω=0.78



Angular momentum at t=400

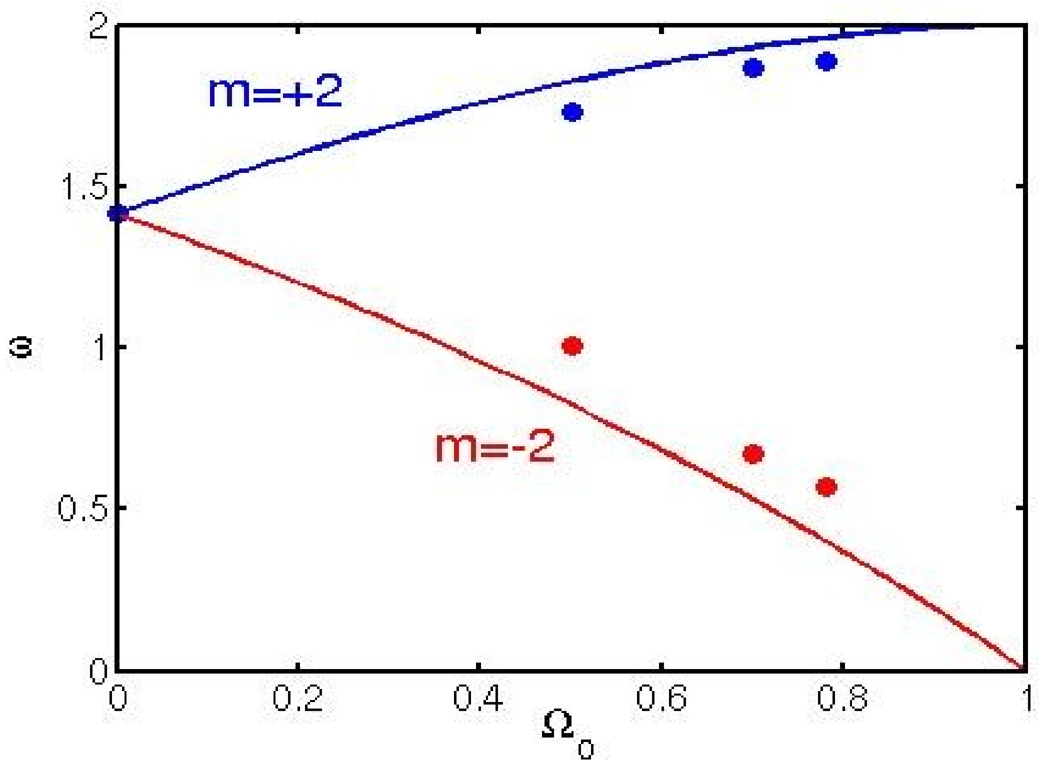


Mode frequencies

Within T.F. Approx. and for large numbers of vortices and ε=0:

$$\omega_{\pm} = \sqrt{2 - \Omega_0^2} \pm \Omega_0$$

Cozzini and Stringari



Conclusions

- **Vortices in annular condensates**

  - Stable multiply-quantized vortices can exist in annular condensates

  - Vortices may be detected by probing collective modes, expansion, or measuring the momentum distribution

- **Vortex formation in rotating traps**

  - Number of vortices (and angular momentum) attained depends upon the vortices already in the condensate

  - Final state depends on rotation history- hysteresis ?