Weak Turbulence for an Elastic Plate

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PLAN

- Introduction to weak turbulence & "wave condensation"
- Weak Turbulence Theory for an Elastic Plate
- Numerical simulations of nonlinear plate equations
- Conclusions & Perspectives

Wave Turbulence

It is a thing which you can easily explain twice before anybody knows what you are talking about

A. Milne "The house at Pooh Corner"

$$\frac{dA^s_{\boldsymbol{k}}}{dt} + is\omega_{\boldsymbol{k}} A^s_{\boldsymbol{k}} = \sum_{s_1 s_2 s_3} \int L_{\boldsymbol{k}\boldsymbol{k}_1\boldsymbol{k}_2\boldsymbol{k}_3} A_1^{s_1} A_2^{s_2} A_3^{s_3} \delta(\boldsymbol{k}_1 + \boldsymbol{k}_2 + \boldsymbol{k}_3 - \boldsymbol{k}) d^{(D)}\boldsymbol{k}_{123}$$

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for a weak amplitude deformation

$$A^s_k = \epsilon a^s_k e^{-is\omega_k t}$$

$$\frac{da^s_{\boldsymbol{k}}}{dt} = \epsilon^2 \sum_{s_1 s_2 s_3} \int L_{\boldsymbol{k}\boldsymbol{k}_1\boldsymbol{k}_2\boldsymbol{k}_3} a_1^{s_1} a_2^{s_2} a_3^{s_3} e^{i(s\omega_{\boldsymbol{k}} - s_1\omega_1 - s_2\omega_2 - s_3\omega_3)t} \delta^{(D)}(\boldsymbol{k}_1 + \boldsymbol{k}_2 + \boldsymbol{k}_3 - \boldsymbol{k}) d^D \boldsymbol{k}_{123}$$

A multi-scale analysis shows that, “wave turbulence” has a natural asymptotic -over long times- closure for higher moments: the fast oscillation drives the system close to gaussian statistics and higher moments are written in terms of the second order moment

Benney & Saffman (1966), Zakharov, L'vov & Falkovich (1992), Newell, Nazarenko & Biven (2001).

$$\langle a_{k_1} a^*_{k_2} \rangle = n_{k_1} \delta^{(D)}(k_1 + k_2)$$

$$\langle a_{k_1} a_{k_2} a^*_{k_3} a^*_{k_4} \rangle = \langle a_{k_1} a^*_{k_3} \rangle \langle a_{k_2} a^*_{k_4} \rangle + \langle a_{k_1} a^*_{k_4} \rangle \langle a_{k_2} a^*_{k_3} \rangle$$

&c.

Weak Turbulence Kinetic Equation for 4-wave resonance

$$\langle a_{k_1} a^*_{k_2} \rangle = n_{k_1} \delta^{(D)}(k_1 + k_2)$$

$$\begin{aligned}\frac{d}{dt} n(p_1) &= \epsilon^4 sgn(t) \sum_{s_1 s_2 s_3} \int |J_{-p_1 \mathbf{k}_1 k_2 k_3}|^2 n_{k_1} n_{k_2} n_{k_3} n_{p_1} \left(\frac{1}{n_{p_1}} + \frac{1}{n_{k_1}} - \frac{1}{n_{k_2}} - \frac{1}{n_{k_3}} \right) \\ &\times \delta(\omega(p_1) + \omega(k_1) - \omega(k_2) - \omega(k_3)) \delta(k_1 + k_2 - k_3 - p_1) d^2 k_{123}\end{aligned}$$

Remarks:

-The mechanism for energy exchange is wave resonance

- sgn(t) & reversibility

Conserved quantities

- "Mass" Conservation $N = \int n_k(t) d^D k$
- Energy Conservation $K = \int \omega_k n_k(t) d^D k$
- Momentum Conservation $\boldsymbol{P} = \int \boldsymbol{k} n_{\boldsymbol{k}}(t) d^D \boldsymbol{k}$
- H-Theorem $d\mathcal{S}/dt \geq 0$ $\mathcal{S}(t) = \int \ln(n_k)\, d^D k$ t>0

NB. on irreversibility

Stationary solutions

- Equilibrium: Rayleigh-Jeans distribution

$$n_k^{eq} = \frac{T}{\omega_k - v \cdot k - \mu}$$

- Non-equilibrium: Kolmogorov-Zakharov spectra

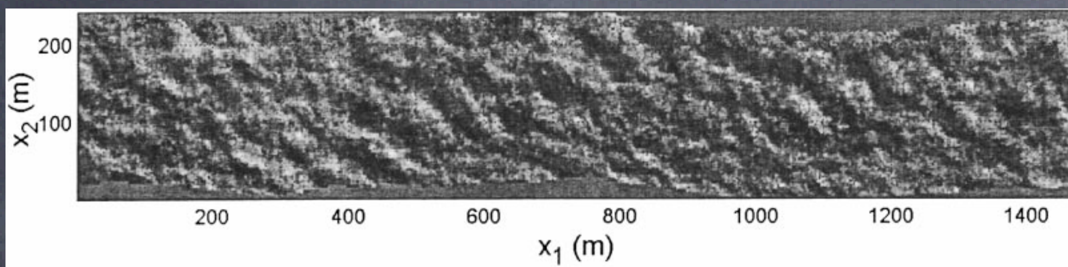
$$n_k^{KZ} = \frac{Q^{1/3}}{k^{(2\beta+D-\alpha)/3}}$$

$$n_k^{KZ} = \frac{P^{1/3}}{k^{2\beta/3+D}}$$

$$J \sim k^{\beta} \qquad \omega_k \sim k^{\alpha}$$

Experimental evidence of Weak Turbulence in gravity (ocean) waves

Y. Toba (1973); Hwang et al. (2000).



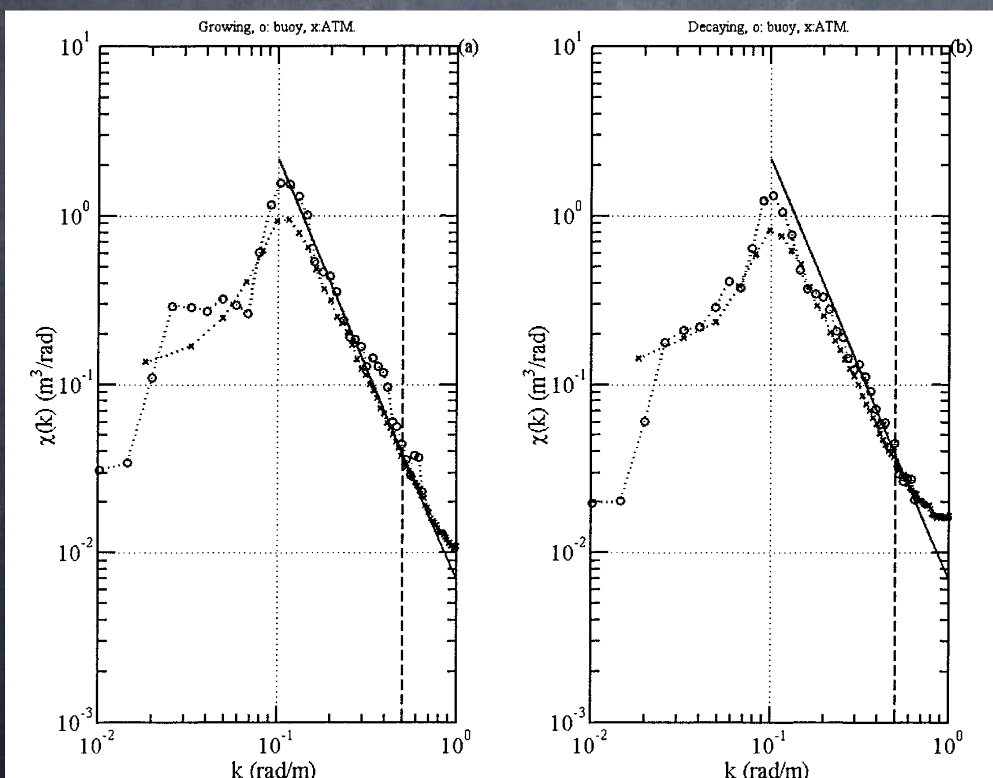


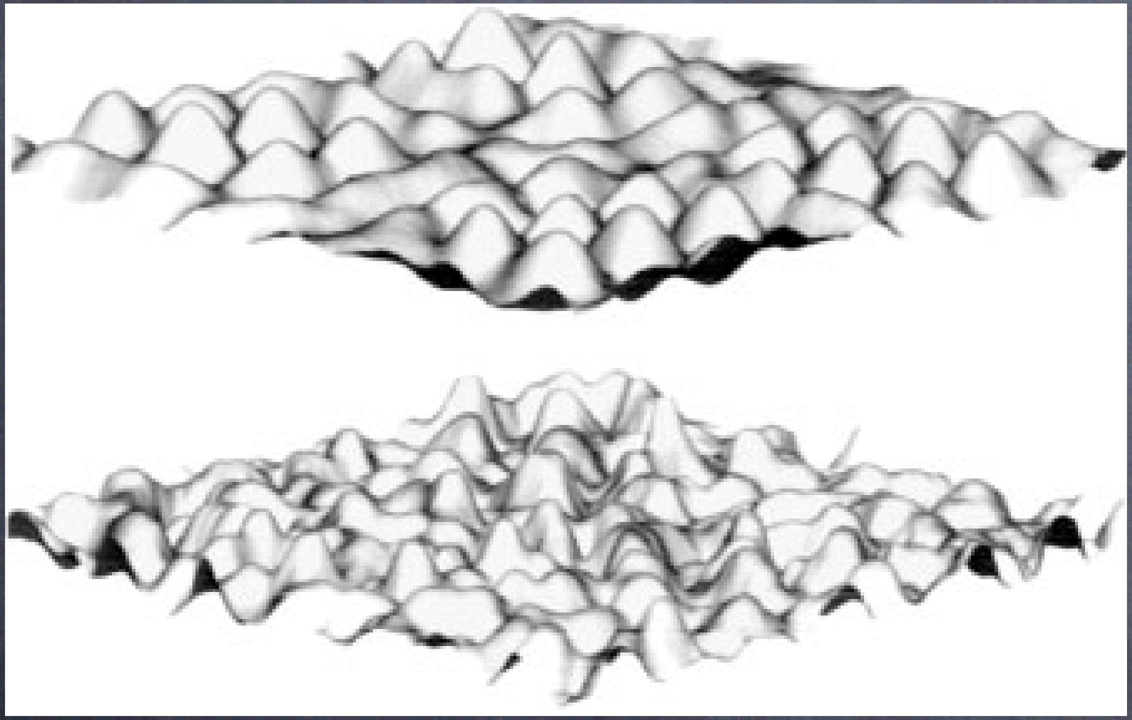
FIG. 8. A comparison of the omnidirectional spectra measured by ATM (crosses) and offshore buoy (ID 44014) (circles). (a) Average of the first 2 hours of data—quasi-steady condition, and (b) average of the last 2 hours of data—decaying wave field. Solid curves: $\chi(k) = 0.06u_* g^{-0.5}k^{-2.5}$ (Phillips 1985).

$$\langle |\zeta_{\boldsymbol{k}}|^2 \rangle = \frac{P^{1/3} g^{1/2}}{k^{5/2}}$$

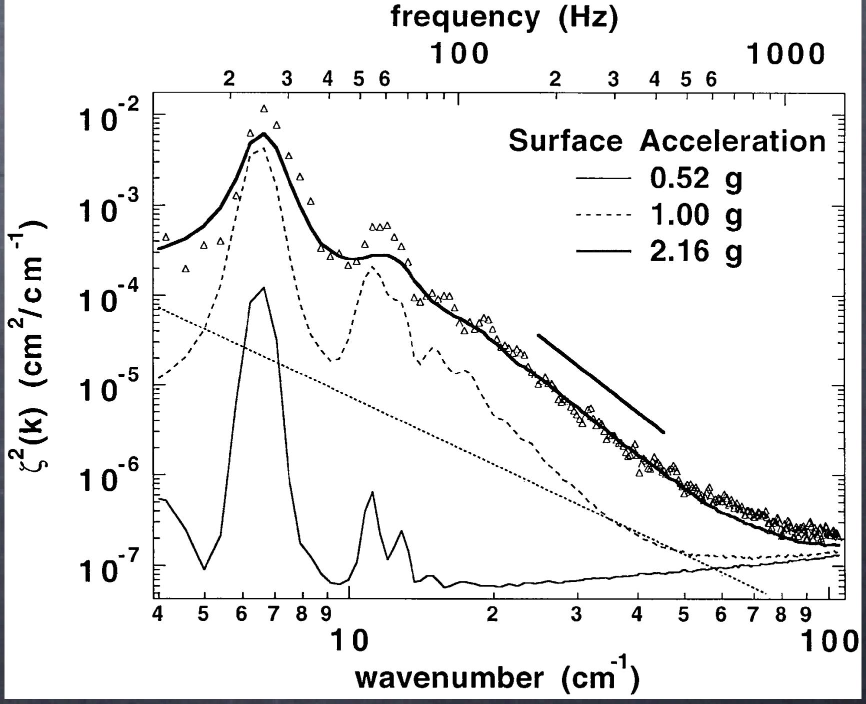
slope: -5/2

Experimental evidence of Weak Turbulence in capillary waves

Wright, Budakian & Putterman (1996); Henry, Alstrøm & Levinsen (2000); Brazhnikov, Kolmakov, Levchenko & Mezhov-Deglin (2002).

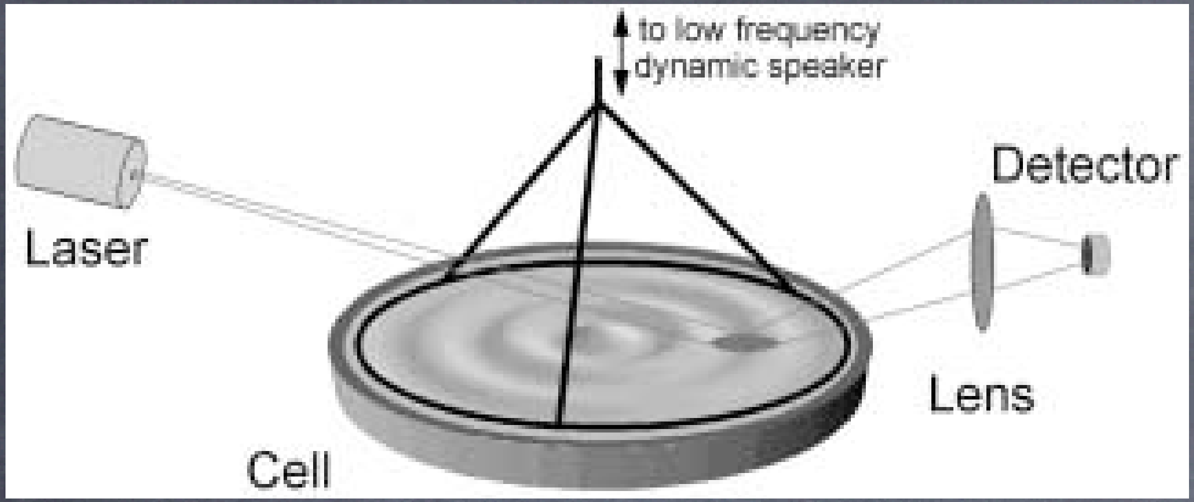


$$\langle|\zeta_{\boldsymbol{k}}|^2\rangle = C\frac{P^{1/2}\rho^{1/4}}{\sigma^{3/4}}\frac{1}{k^{17/4}}$$



d'après Wright et al.

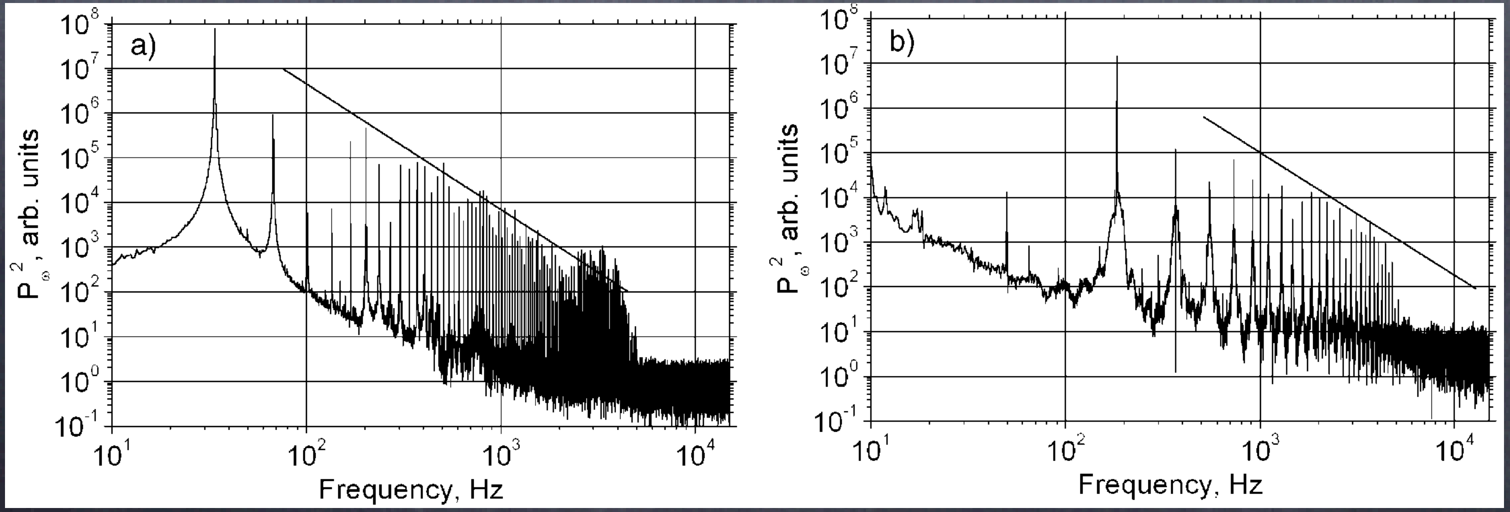
slope: -4.2



$$\langle|\zeta_{\boldsymbol{k}}|^2\rangle = C\frac{P^{1/2}\rho^{1/4}}{\sigma^{3/4}}\frac{1}{k^{17/4}}$$

$$\langle|\zeta_\omega|^2\rangle = C\frac{P^{1/2}\sigma^{1/6}}{\rho^{2/3}}\frac{1}{\omega^{17/6}}$$

Brazhnikov, Kolmakov, Levchenko & Mezhov-Deglin (2002).



slope: -17/6

Wave Condensation

C. Connaughton, C. Josserand, A. Picozzi, Y. Pomeau & SR (2005)

The nonlinear Schrödinger equation

$$i\partial_t \psi = -\Delta\psi + |\psi|^2\psi$$

Weak turbulence theory for NLS

$$\begin{aligned}\frac{dn_{\boldsymbol{p}_1}}{dt} &= \frac{4\pi}{(2\pi)^D}\int (n_{\boldsymbol{k}_2}n_{\boldsymbol{k}_3}n_{\boldsymbol{k}_1} + n_{\boldsymbol{k}_2}n_{\boldsymbol{k}_3}n_{\boldsymbol{p}_1} - n_{\boldsymbol{k}_2}n_{\boldsymbol{k}_1}n_{\boldsymbol{p}_1} - n_{\boldsymbol{k}_3}n_{\boldsymbol{k}_1}n_{\boldsymbol{p}_1}) \\ &\times \delta(\omega_{\boldsymbol{p}_1} + \omega_{\boldsymbol{k}_1} - \omega_{\boldsymbol{k}_2} - \omega_{\boldsymbol{k}_3})\delta^{(D)}(\boldsymbol{p}_1 + \boldsymbol{k}_1 - \boldsymbol{k}_2 - \boldsymbol{k}_3)d^2\boldsymbol{k}_{123}\end{aligned}$$

with $\omega_k = k^2$

Equilibrium solution $$n_k^{eq} = \frac{T}{k^2 - \mu},$$

where the Lagrange multiplier are given by the initial condition for E & N:

Equilibrium solution

$$n_k^{eq} = \frac{T}{k^2 - \mu},$$

where the Lagrange multiplier are given by the initial condition for E & N:

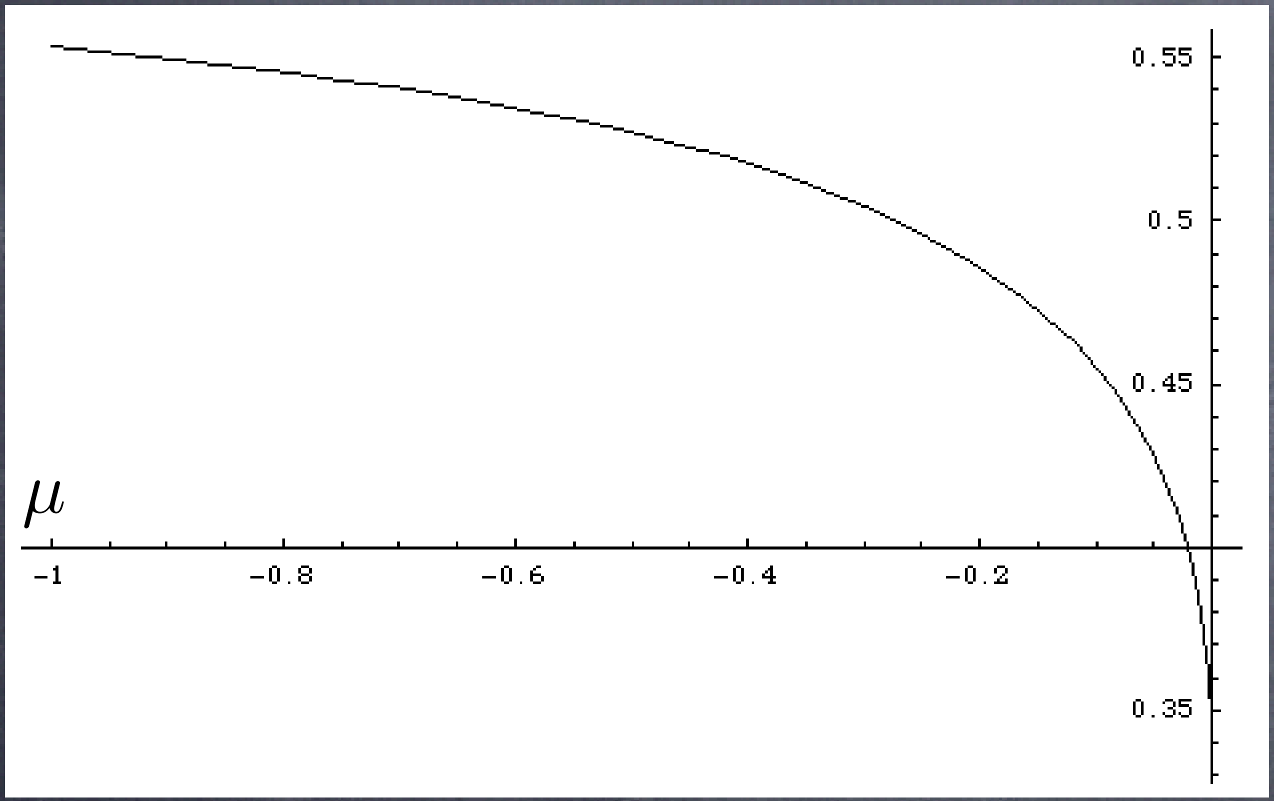
$$\frac{N}{V} = 4\pi T k_c \left[1 - \frac{\sqrt{-\mu}}{k_c} \arctan\left(\frac{k_c}{\sqrt{-\mu}}\right)\right]$$

$$\frac{E}{V} = \frac{4\pi T k_c^3}{3} \left[1 + 3\frac{\mu}{k_c^2} + 3\left(\frac{-\mu}{k_c^2}\right)^{\frac{3}{2}} \arctan\left(\frac{k_c}{\sqrt{-\mu}}\right)\right]$$

Warning solution needs a ultraviolet cut-off!!!

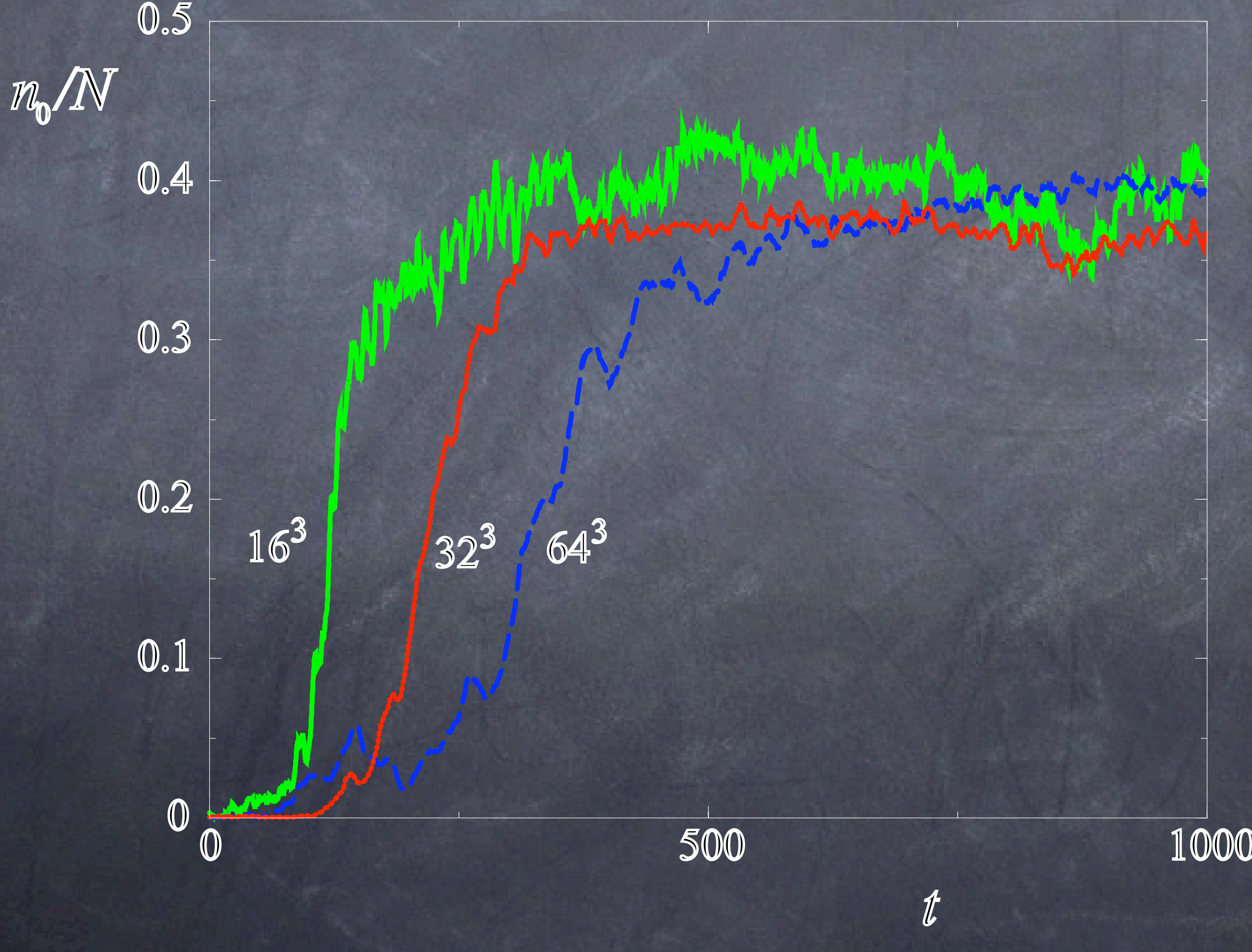
Condensation criteria in 3D

$$\frac{E_c}{N k_c^2}$$



Condensation arises if $\mu = 0$ or $\frac{E_c}{N k_c^2} = \frac{1}{3}$

Numerical Evidence of wave-condensation in 3D



Elasticity of Plates

Energy/h ~ h^2 $(\text{bending})^2$ + $(\text{stretching})^2$

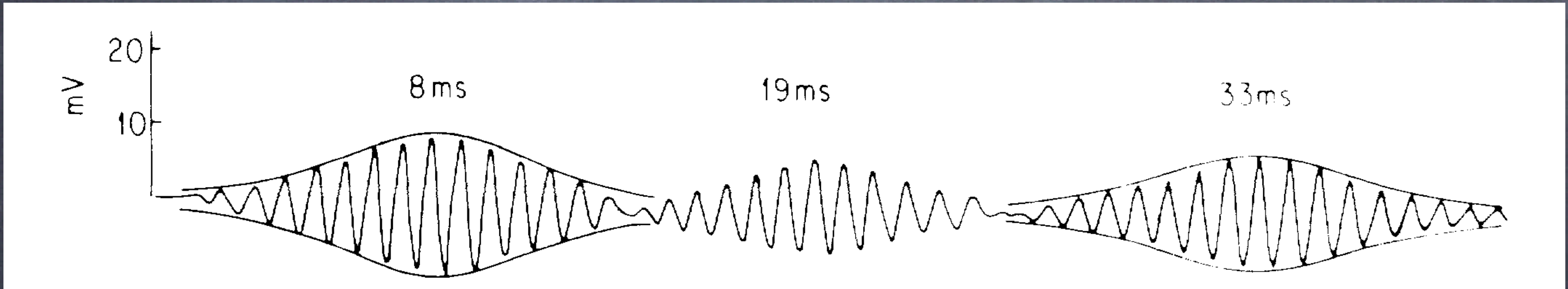
Lord Rayleigh, Theory of Sound

bending ~ linear in deformation

stretching ~ quadratic in deformation

Soliton envelope in a cilindrical shell

Wu, Wheatley, Putterman & Rudnick (1988);



Elasticity of Plates

$$
\begin{aligned}
h\rho\partial_{tt}\zeta &= -\frac{Eh^3}{12(1-\sigma^2)}\Delta^2\zeta + \{\zeta,\chi\}; \\
\frac{1}{Eh}\Delta^2\chi &= -\frac{1}{2}\{\zeta,\zeta\}.
\end{aligned}
$$

Here: E is Young modulus and h the thickness of the plate &

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\end{aligned}
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Here: E is Young modulus and h the thickness of the plate &

$$\{f,g\} \equiv f_{xx}g_{yy} + f_{yy}g_{xx} - 2f_{xy}g_{xy}.$$

and $\{\zeta,\zeta\}/2 = \zeta_{xx}\zeta_{yy} - \zeta_{xy}^2$ is the Gaussian curvature

Properties

-Center of mass conservation: {f,g} = div(something)

$$\frac{\partial^2}{\partial t^2}\int \zeta(x,y,t)dxdy = 0$$

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$$\frac{\partial^2}{\partial t^2}\int \zeta(x,y,t)dxdy = 0$$

-Hamiltonian evolution

$$\begin{aligned} H[\zeta_k, p_k]/h &= \int \left[\frac{1}{2\rho}|p_k|^2 + \frac{Eh^2k^4}{24(1-\sigma^2)}|\zeta_k|^2\right] d^2k \\ &+ \frac{1}{4}\int V_{k_1,k_2;k_3,k_4}\zeta_{k_1}\zeta_{k_2}\zeta_{k_3}\zeta_{k_4}\delta^{(2)}(k_1+k_2+k_3+k_4)d^2k_{1,2,3,4} \end{aligned}$$

where

$$V_{12;34} = \frac{E}{2(2\pi)^2}\left(\frac{1}{2|\boldsymbol{k}_1+\boldsymbol{k}_2|^4} + \frac{1}{2|\boldsymbol{k}_3+\boldsymbol{k}_4|^4}\right)|\boldsymbol{k}_1\times\boldsymbol{k}_2|^2|\boldsymbol{k}_3\times\boldsymbol{k}_4|^2$$

Properties (cont)

-Poisson Bracket $$[\zeta_k, p_{k'}] = \delta^{(2)}(k - k')$$

-Bending waves $$\omega_k = \sqrt{\frac{Eh^2}{12\rho(1-\sigma^2)}}k^2 = hck^2$$

Canonical Variables

If

$$\zeta_k = \frac{X_k}{\sqrt{2}}(A_k + A^*_{-k})$$
$$p_k = -\frac{i}{\sqrt{2}X_k}(A_k - A^*_{-k})$$

one has

$$[A_k, A^*_{k'}] = i\delta^{(2)}(k - k')$$

and with $X_k = \frac{1}{\sqrt{\omega_k \rho}}$

$$H[A_k, A^*_k]/h = \int \omega_k |A_k|^2 d^2k + \frac{1}{4} \sum_{s_1,s_2,s_3,s_4} \int T_{k_1,k_2;k_3,k_4} A^{s_1}_{k_1} A^{s_2}_{k_2} A^{s_3}_{k_3} A^{s_4}_{k_4} \delta^{(2)}(k_1+k_2+k_3+k_4)$$

Weak Turbulence Theory for Elastic Plates

G.Düring, C. Josserand & SR. (2005).

$$\frac{d}{dt}n_{\boldsymbol{p}_1} = 12\pi\epsilon^4 sgn(t)\int |J_{\boldsymbol{p}_1\mathbf{k}_1\boldsymbol{k}_2\boldsymbol{k}_3}|^2\delta(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3+\boldsymbol{p}_1)n_{k_1}n_{k_2}n_{k_3}n_{p_1}$$
$$\times \sum_{s_1 s_2 s_3}\left(\frac{1}{n_{p_1}}+\frac{s_1}{n_{k_1}}+\frac{s_2}{n_{k_2}}+\frac{s_3}{n_{k_3}}\right)\delta(\omega_{p_1}+s_1\omega_{k_1}+s_2\omega_{k_2}+s_3\omega_{k_3})\,d^2\boldsymbol{k}_{123}$$

Energy Conservation $K=\int \omega_k n_k(t) d^D k$

H-Theorem $\mathcal{S}(t)=\int \ln(n_k)\,d^D k$ $\quad d\mathcal{S}/dt\geq 0$

Isotropic distributions

$$S_{k_1,k_2,k_3,k_4} = \frac{1}{(2\pi)^3}\int |J_{\boldsymbol{k}_1\boldsymbol{k}_2\boldsymbol{k}_3\boldsymbol{k}_4}|^2 \delta^{(2)}(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3+\boldsymbol{p}_1)d\varphi_2 d\varphi_3 d\varphi_4$$
$$= \frac{1}{(2\pi)^3}\int \frac{|J_{\boldsymbol{k}_1\boldsymbol{k}_2\boldsymbol{k}_3\boldsymbol{k}_4}|^2}{|\boldsymbol{k}_2\times\boldsymbol{k}_3|} d\varphi_4$$

For a power law distribution

$$n_k = Ak^{-2x}$$

one has: $$Coll = 3Coll_{2+2} + Coll_{3+1}$$

$$
\begin{aligned}
\mathcal{C}oll_{2+2} &= \pi A^3 k^{4-6x} \int_{\Omega_{up}} k_2 dk_2 k_3 dk_3 S_{kk_1k_2k_3} k_1^{-2x} k_2^{-2x} k_3^{-2x} k^{-2x} \\
&\times \left(k^{2x} + k_1^{2x} - k_2^{2x} - k_3^{2x}\right) \times \left(k^{6x-4} + k_1^{6x-4} - k_2^{6x-4} - k_3^{6x-4}\right)
\end{aligned}
$$

where $$k_1^2 = k_2^2 + k_3^2 - k^2$$

$$
\begin{aligned}
\mathcal{C}oll_{3+1} &= \pi A^3 k^{4-6x} \int_{\Omega_{down}} k_2 dk_2 k_3 dk_3 S_{kk_1k_2k_3} k_1^{-2x} k_2^{-2x} k_3^{-2x} k^{-2x} \\
&\times \left(k^{2x} - k_1^{2x} - k_2^{2x} - k_3^{2x}\right) \times \left(k^{6x-4} - k_1^{6x-4} - k_2^{6x-4} - k_3^{6x-4}\right)
\end{aligned}
$$

where $$k_1^2 = k^2 - k_2^2 - k_3^2$$

Stationary Solutions

- Equilibrium: Rayleigh-Jeans distribution

$$n_k^{eq} = \frac{T}{\omega_k} = \frac{T}{hck^2}$$

- Non-equilibrium: Kolmogorov-Zakharov spectra

$$n_k^{KZ} = C\frac{P^{1/3}}{k^2}$$

However C=0, thus there is a ln(k) correction

Stationary Solutions

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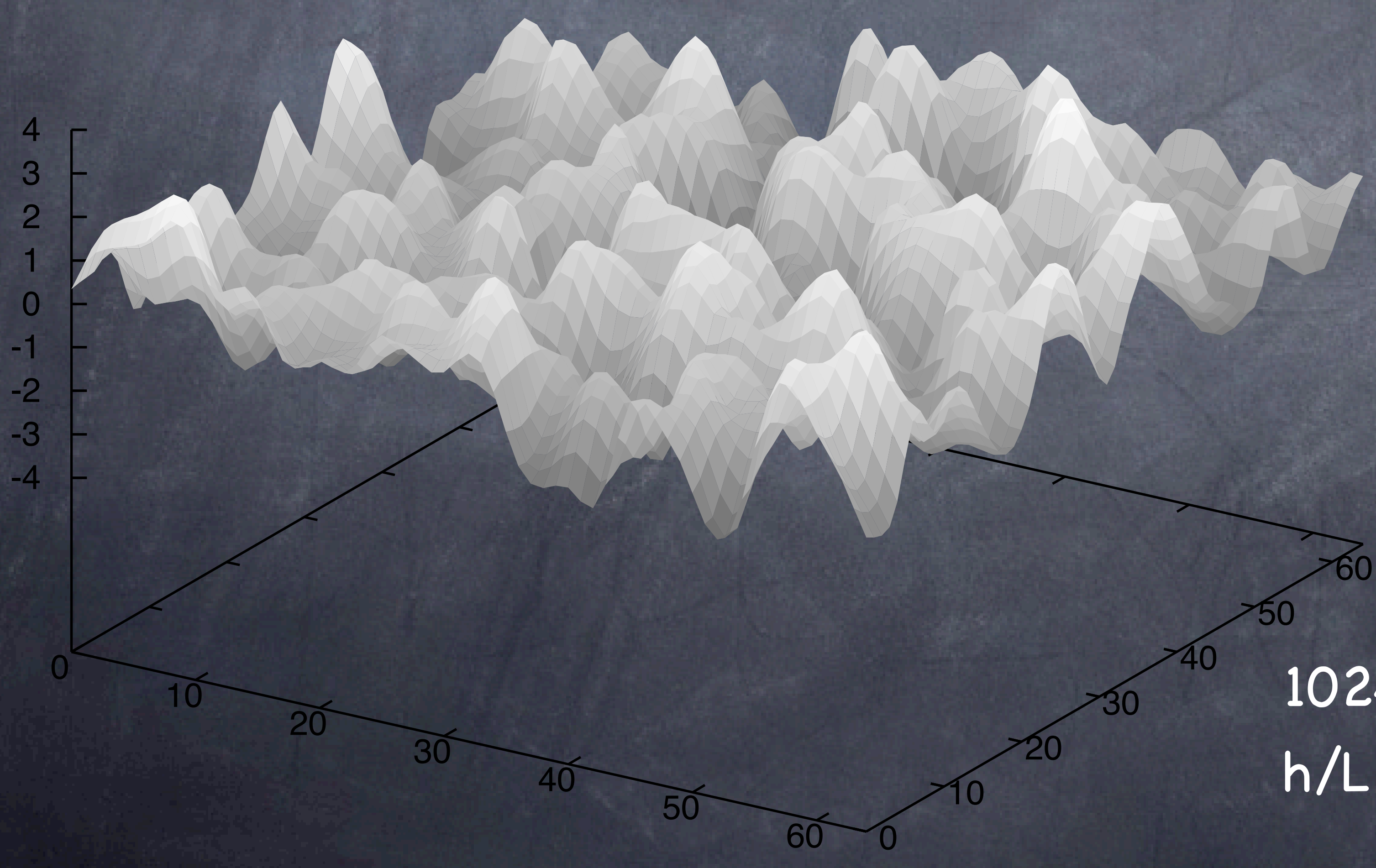
$$n_{\boldsymbol{k}}^{eq} = \frac{T}{\omega_k} = \frac{T}{hck^2}$$

- Non-equilibrium: Kolmogorov-Zakharov spectra

$$n_k^{KZ} = C\frac{P^{1/3}}{k^2}\ln(k)^z$$

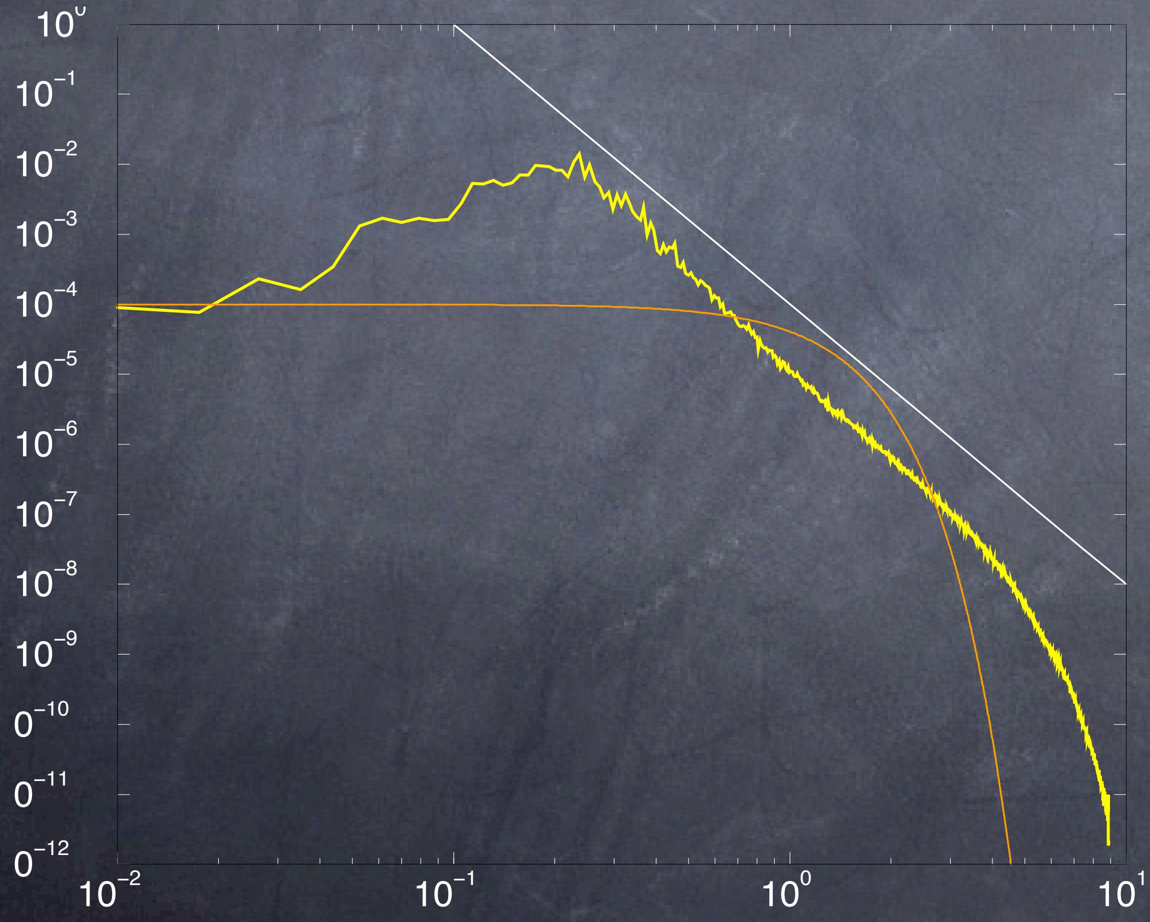
Numerical simulation of Föppl equation



1024 x 1024 $k_c = 2\pi$

$h/L = 0.001$

Numerical Evidence of the equilibrium distribution



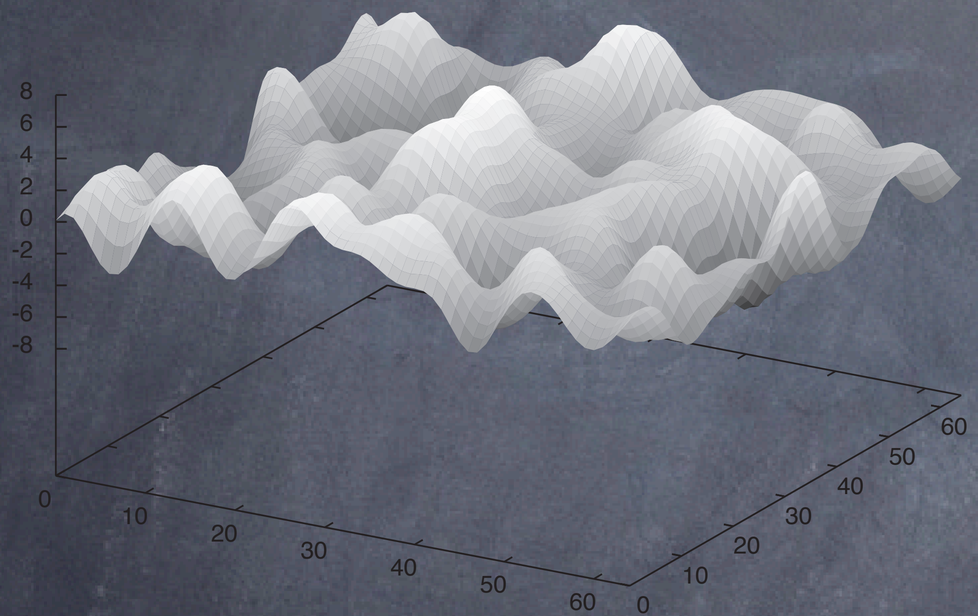
$$\langle|\zeta_k|^2\rangle = \frac{1}{\rho\omega_k} n_k^{eq} = \frac{T}{\rho\omega_k^2} = \frac{T}{\rho c^2 h^2 k^4}$$

slope: -4

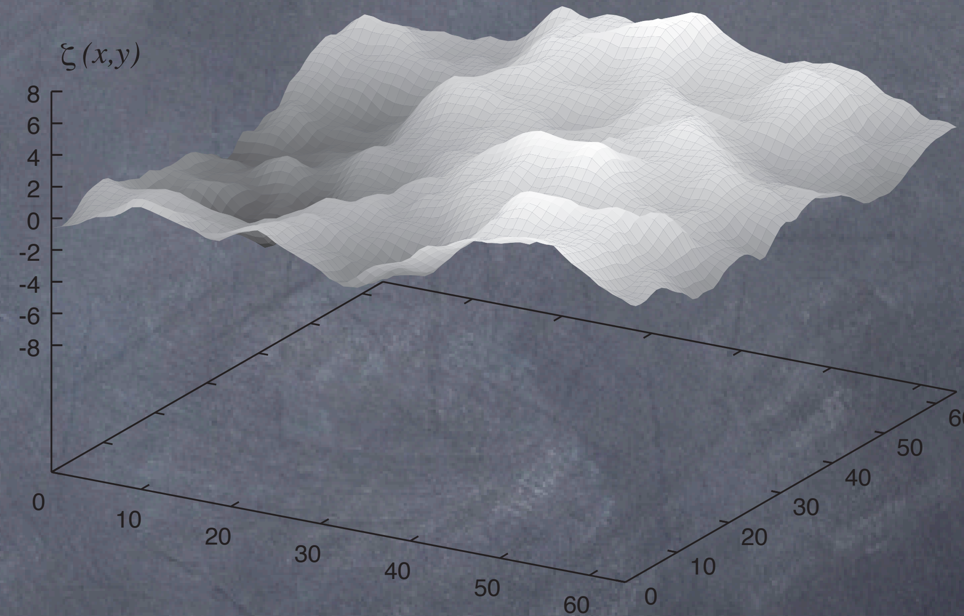
1024 x 1024 kc = 2π

h/L = 0.001

Initial condition

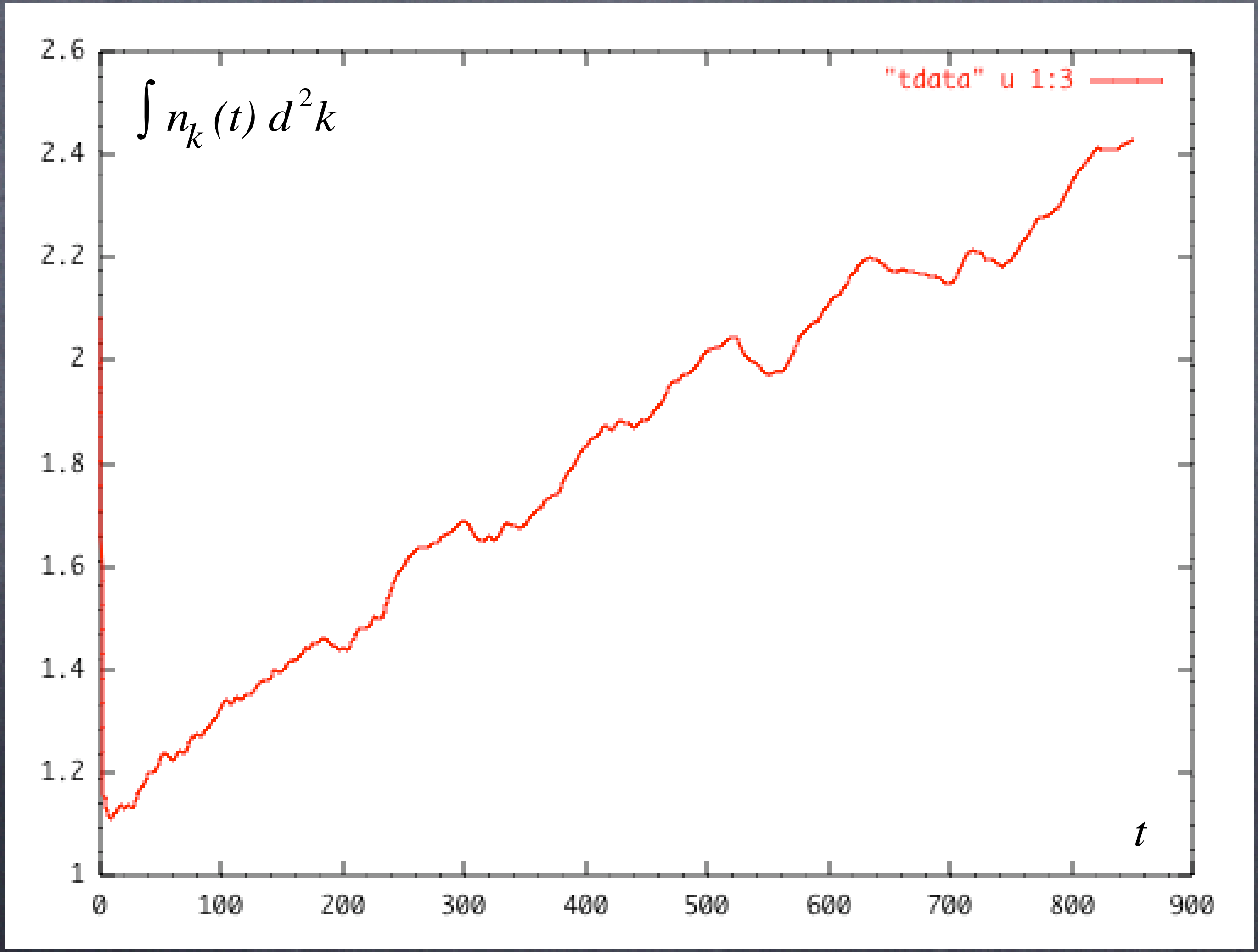


Later evolution

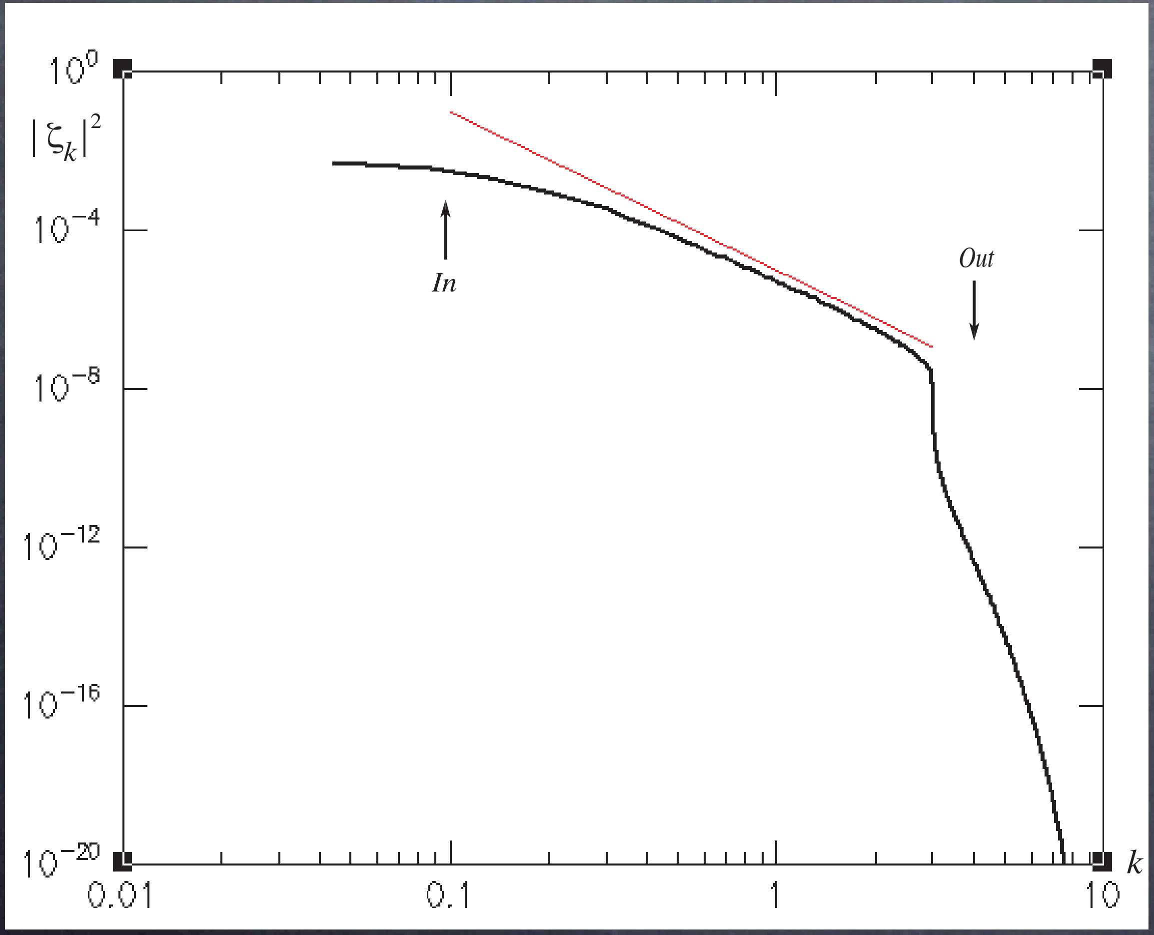


NB. Plot3D of a partial zone of the full plate.

Wave action variation in time



Numerical Evidence of the Kolmogorov-Zakharov spectrum



$$\langle |\zeta_k|^2 \rangle = \frac{1}{\rho \omega_k} n_k^Z$$

$$= C \frac{P^{1/3}}{\rho \omega_k k^2} \ln(k)^z \sim \frac{P^{1/3}}{\rho h c k^4} \ln(k)^z$$

slope: -4

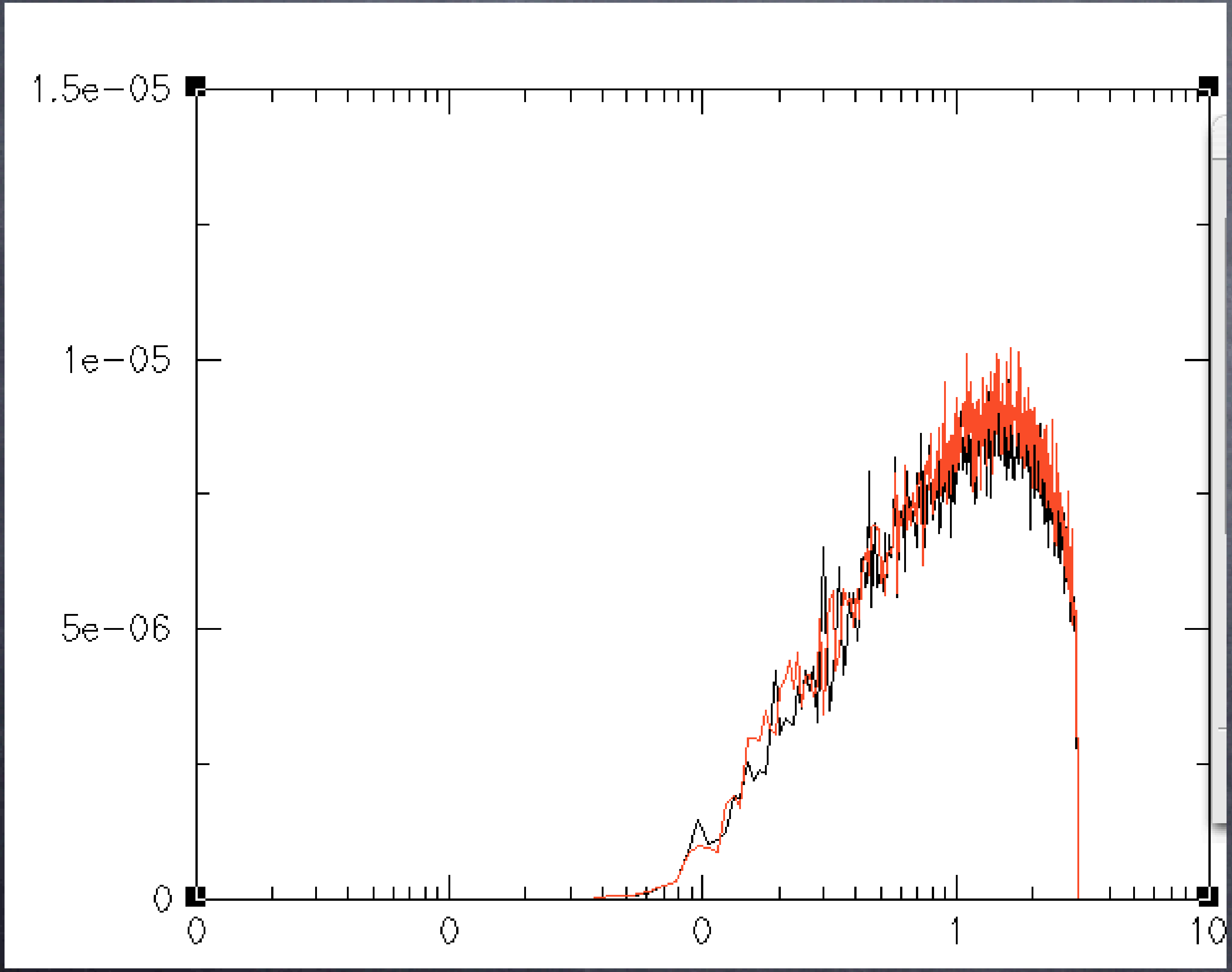
1024 x 1024 kc = 2π

h/L = 0.001

t = 630

The log-correction to the KZ spectrum

$\langle|\zeta_k|^2\rangle\, k^4$



$$\langle|\zeta_k|^2\rangle\, k^4 \sim \ln k$$

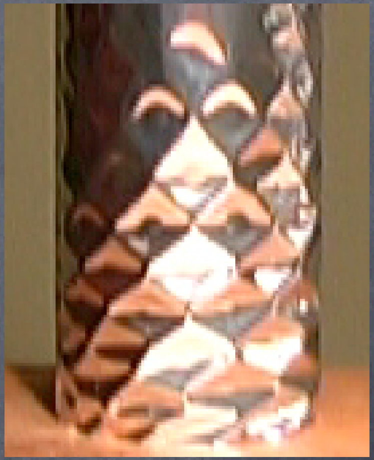
k

Conclusions

- Small oscillations of a vibrating elastic plate are described in long time by a weak turbulence theory.
- Equilibrium distribution is the Rayleigh-Jeans distribution ~ T/k^2. Numerical evidence.
- Although there is no formal a wave action flux spectrum there is a weak inverse cascade.
- Kolmogorov-Zakharov spectrum ~ $Ln^z(k)/k^2$ is observed in elastic bending waves.

Comments & Perspectives

- Larger elastic deformations (ridges d-cones) can arises easily breaking weak turbulence.

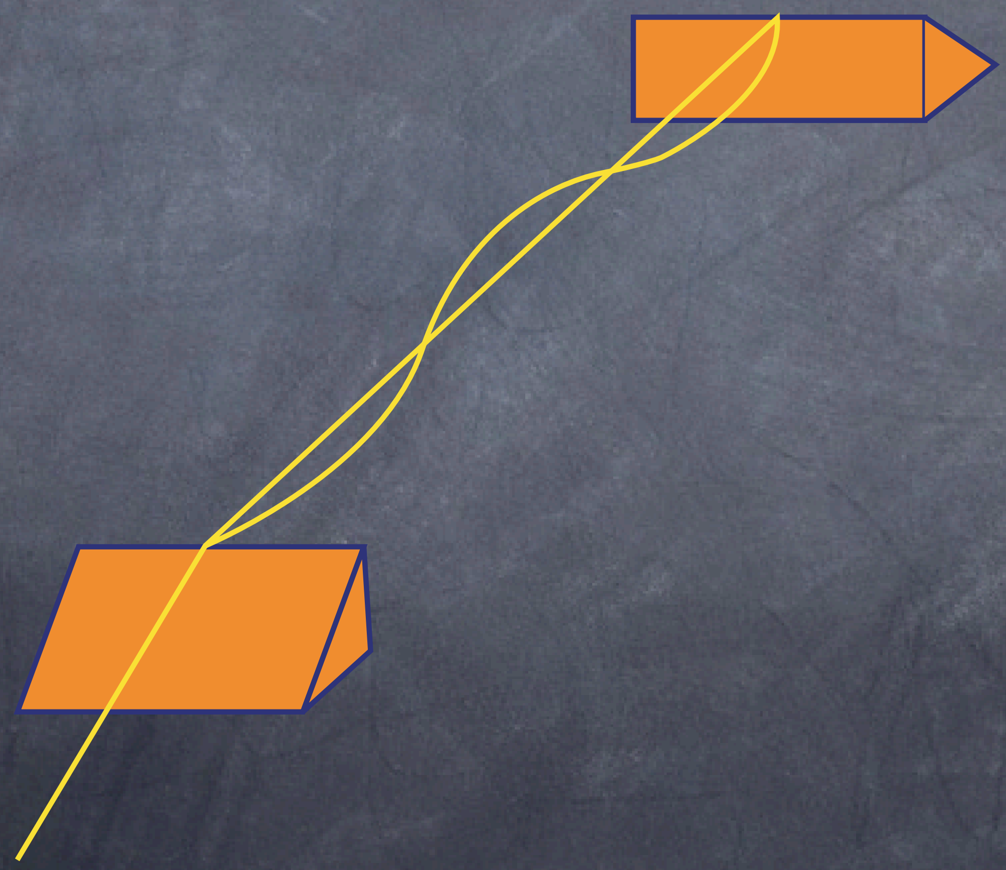


- Measurements of Kolmogorov-Zakharov spectrum in a elastic plate or tube.
- Measurements of Kolmogorov-Zakharov spectrum in a bass or piano string.

Perhaps... One can hear a Kolmogorov spectrum!

Experiment on a bass string

In collaboration with C. Brown (U. of Chicago), L. Oyarte(PUC), E. Cerda & R. Labbé (U of Santiago).



Thick String equation

$$\sigma\partial_{tt}\zeta - \tau\partial_{xx}\zeta - I\rho\partial_{xxtt}\zeta + EI\partial_{xxxx}\zeta + 2\tau\zeta_x^2\partial_{xx}\zeta = 0$$

where ρ & E are the mass density and Young modulus of the material, $I = \frac{\pi}{4}R^4$ is the moment of inertia of the rod, $\sigma = \pi R^2\rho$ is the linear mass density and τ the tension imposed to the string

Dispersion Relation

$$\omega_k = k\sqrt{\frac{\tau/\sigma + \frac{EI}{\sigma}k^2}{1 + \frac{(kR)^2}{4}}}$$