What is turbulence from nonlinear dynamics standpoint?

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Plan of the talk

1. What is meant by turbulence? Types of turbulence.
2. The main stages in studies of hydrodynamical turbulence.
3. Evolution of views on the origin of turbulence.
4. Nikitin's numerical experiment.
5. How turbulence is developing?
6. Turbulence as a noise-induced phase transition. A parametrically excited noised pendulum as one of the simplest models of such a transition.
7. The main equations and dynamics of a plane jet.
8. The initial development of turbulence in linear approximation. The peculiarities of the numerical solution of non-self-adjoint boundary-value problem with complex eigenvalues.
9. Using the Krylov–Bogolyubov method.

What is meant by turbulence? Types of turbulence.

- Any random processes
- Dynamical chaos (Ruelle and Takens, Manneville)
- Plasma turbulence
- Acoustic turbulence (random acoustic waves excited by noise)
- Hydrodynamical turbulence
  - ⋆ Irregular fluid flows in open systems (e.g. in air or in rivers)
  - ⋆ Irregular fluid flows in closed systems (e.g. between rotating cylinders)

The main stages in studies of hydrodynamical turbulence

- Hagen (1839) — Decreasing the fluid viscosity due to heating.
- Reynolds (1883) — Experiments with stained liquid showing intermittent behavior.
- Rayleigh (1879) — Instability of jets.
- Kolmogorov, Obukhov (1941) — Theory of homogeneous isotropic turbulence.
- Landau (1944) — Theory of the turbulence development (Van der Pol equations).
- Prandtl (1949) — Turbulent viscosity.
- Schlichting (1965) — Boundary layer theory.
- Heisenberg (1924) — Instability of Poiseuille flow.
- Ginevsky, Vlasov (1967) — Control of turbulence in jets by acoustical waves.

The main features of hydrodynamical turbulence:

- Irregularity of flow
- Presence of vortices and moderately regular patterns (coherent structures)
- Intermittent behavior (experiments by Reynolds with stained liquid)

Evolution of views on the origin of turbulence

After the discovery by Andronov of self-oscillations the latter were found in a great number of different systems.

No wonder that Gorelik believed that “turbulence with its threshold of self-excitation, with typical hysteresis in its appearance or disappearance as the flow velocity increases or decreases, with paramount importance of nonlinearity for its developed (stationary) state is self-oscillations.”

Evolution of views on the origin of turbulence

Landau held implicitly the same viewpoint

He assumed that at first oscillations with a single frequency are excited, i.e. a wave of the form $A(t)(\omega t - kx)$, where $A(t)$ is a complex time-dependent amplitude, appears. For this amplitude he wrote an equation similar to the well known van der Pol equation:

$$\dot{A} = (\alpha - \beta A^2)A$$

Landau wrote:

"With further increase of the Reynolds number new periods appear sequentially. As for the newly appeared motions, they have increasingly small scales".

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As a result, multi-frequency self-oscillations with incommensurate frequencies, i.e. quasi-periodic motion, must set in. In the system phase space an attractor in the form of a multi-dimensional torus has to correspond to these self-oscillations.

In essence, almost all researchers of hydrodynamical instability and plasma instability, starting with Heisenberg, use the same approach (the most of the researchers are using this approach till now). They seek a solution of the corresponding linearized equations in the form $u = \exp(-i(\omega t - kx))$, where k is a real and $\omega = \omega_0 + i\delta$ is a complex number. If they find that $\delta > 0$, they conclude that the system is unstable and self-excited.

But this approach is not correct because a crucial role in stability or instability is played by boundary conditions.

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Evolution of views on the origin of turbulence

In the 1970s, after the discovery of deterministic chaos and the realization that a multi-dimensional torus is unstable, the Landau theory became questionable, but the conception of self-oscillations was retained. The difference was only in that, instead of quasi-periodic, they became speak of as chaotic self-oscillations.

According to these new ideas, the onset of turbulence is the sudden birth of a strange attractor in the phase space of certain dynamical variables (Ruelle and Takens).

This view is reflected in the new editions of “Hydrodynamics” by Landau and Lifshits (this addition was made by M. Rabinovich) and of “Statistical Fluid Mechanics: Mechanics of Turbulence” by Monin and Yaglom.

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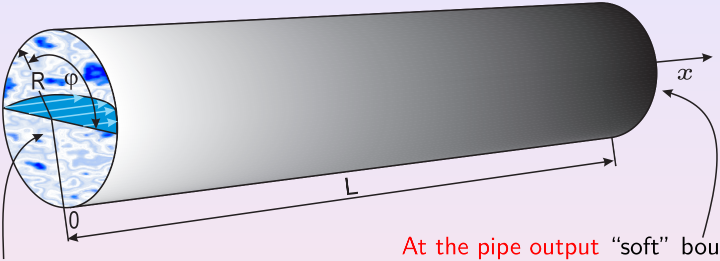
Nikitin's numerical experiment

The fact that turbulence is not a self-oscillatory process is supported by numerical experiments by Nikitin (MSU).



At the pipe input:

$$u(0,r) = u_0 \left(1 - \frac{r^2}{R^2}\right) + A\cos\omega t,$$

where $\omega = 0.36 u_0/R$, i.e. $\mathrm{St} \approx 0.11$. The flow velocity u_0 was set so that $\mathrm{Re} = 2u_0 R/\nu = 4000$.

At the pipe output "soft" boundary conditions:

$$\frac{\partial^2 u(L,r)}{\partial x^2} = \frac{\partial^2 \Omega_r(L,r)}{\partial x^2} = \frac{\partial^2 \Omega_\varphi(L,r)}{\partial x^2} = 0.$$

Under these conditions a reflected wave apparently does not appear, or is very weak.

At initial moment:

$\mathbf{V}(x,r)|_{t=0} = \left\{u_0\left(1 - r^2/R^2\right), 0, 0\right\}$

Nikitin's numerical experiment

As the amplitude of the harmonic force A exceeded a certain critical value, random high-frequency pulsations appeared in the lower part of the pipe from $x = x_0$, where x_0 was the less the greater was A.

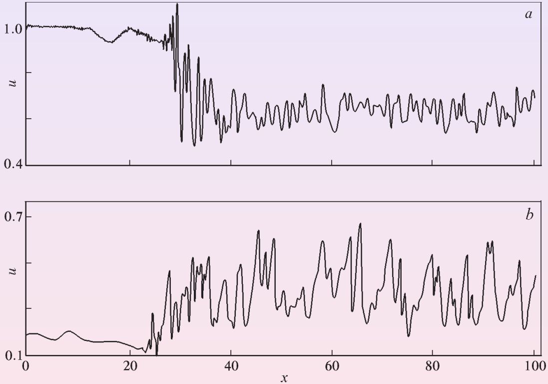


Figure: Instantaneous distributions of the longitudinal velocity component in a steady regime for $A/u_0 = 0.04$: (a) along the pipe axis $(r/R = 0.02)$ and (b) near the pipe wall $r/R = 0.93$

Nikitin's numerical experiment

As the amplitude A gradually decreased, the turbulent region drifted progressively downstream and disappeared at a certain value of A.

It is known that Poiseuille flow in a circular pipe, in contrast to that in a plane channel, possesses the property that laminar flow is stable with respect to small perturbations for any Reynolds number.

However in the case of sufficiently large Reynolds numbers such a flow is unstable with respect to perturbations of a finite value. If an attractor, corresponding to the turbulent mode, existed, and if the role of the harmonic disturbance was to lead phase trajectories into the attractor basin, then turbulence should not disappear following cessation of the harmonic disturbance.

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Nikitin's numerical experiment

Possible counter-arguments against the above ideas lie in the fact that the results of numerical simulations with periodic boundary conditions are very close to those observed experimentally. But the data obtained by Nikitin in his numerical experiment are also visually close to numerical data for periodic boundary conditions.

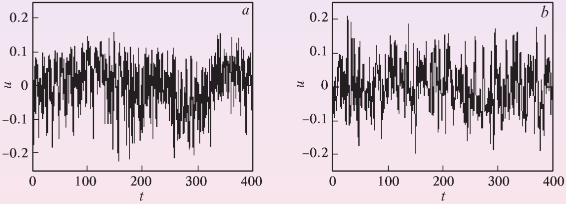
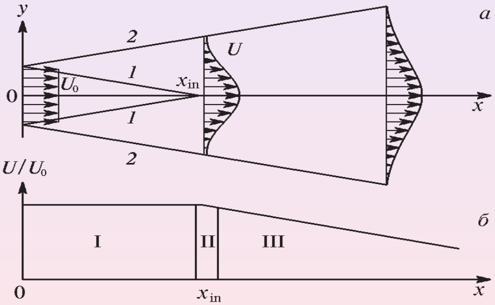


Figure: Shape of turbulent velocity pulsations in a pipe. (a) with periodic boudary conditions, and (b) with "soft" boundary conditions

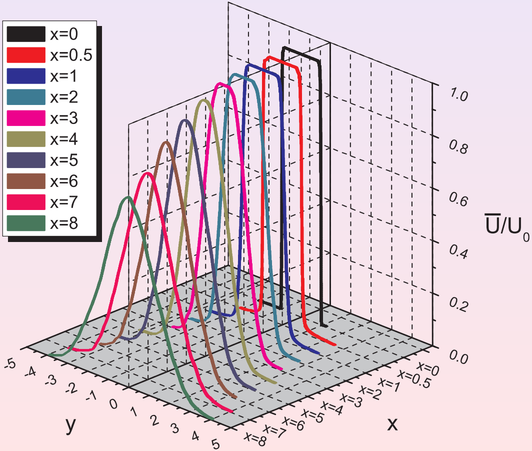
The main properties of jet flows

Issuing from a nozzle, a fluid jet always noticeably diverges. This is associated with the fact that owing to viscosity increasingly more neighboring fluid layers are involved in the motion. This phenomenon has come to be known as the entrainment.



The main properties of jet flows

The profile of mean flow velocity changes essentially in the process. At the nozzle exit it is nearly rectangular, whereas away from the nozzle it becomes bell-shaped.



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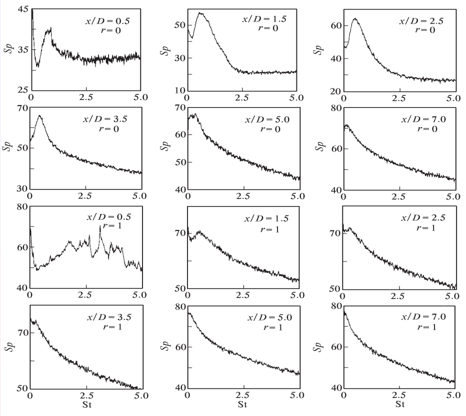

Figure: Evolution of spectral density *Sp* (in decibels) of the velocity pulsations *u* with increasing distance from the nozzle exit x/D along the jet axis ($r = 0$) and along a line offset by R from the axis ($r = 1$) ($\mathrm{St} = fD/U_0$ is the Stroulial number

The main properties of jet flows

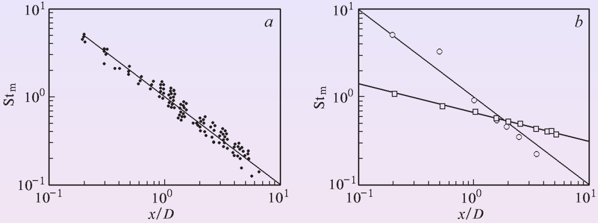

Figure: The experimental dependences of the Strouhal number St_m, on the relative distance x/D from the jet nozzle exit along the jet axis and within the mixing layer: (a) Petersen's data for the mixing layer; and (b) our data. In (b) the dependence on distance along the jet axis, and along a line offset by R from the axis, are shown by squares and circles, respectively. The solid lines show the dependences $St_m = C_1 x^{-1/3}$ and $St_m = C_1 x^{-1}$, where $C_1 \approx 0.67$ and $C_2 \approx 1$

An explanation of the spectrum behavior

Nonlinear feedback

$$\frac{xf_m}{U_v} + \frac{xf_m}{a} = N,$$

where U_v is the velocity of the vortices (it follows from visual observations and measuring spatial-temporal correlations that $U_v \approx 0.5 - 0.7U_0$), a is the sound velocity, and N is an integer.

Turbulence as a noise-induced phase transition.

In a number of works (Landa, Zaikin) we have offered a hypothesis that turbulence in nonclosed fluid flows is caused by noise, and is a result of a peculiar noise-induced phase transition.

A parametrically excited noised pendulum as one of the simplest models of such a transition

Because the study of any phenomenon is expedient by using the simplest model where this phenomenon may occur, we considered such a transition in a pendulum with a randomly vibrated suspension axis described by the equation

$$\ddot{\varphi} + 2\beta(1 + \alpha\dot{\varphi}^2)\dot{\varphi} + \omega_0^2(1 + \xi(t))\sin\varphi = 0$$

We have added the nonlinear friction to avoid 2π turn of the pendulum. It turns out that the behavior of such a pendulum in many respects is similar to behavior of a jet.

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The main equations and dynamics of a plane jet

We used the approximation of incompressible fluid and started from the Navier Stokes equations in terms of the stream function $\Psi(t,x,y)$ and vorticity $\Omega(t,x,y)$. In dimensionless coordinates these equations are

$$
\begin{aligned}
&\Omega(t,x,y) = \Delta\Psi(t,x,y)\ , \\
&\frac{\partial\Omega(t,x,y)}{\partial t} - \frac{\partial\Psi(t,x,y)}{\partial x}\frac{\partial\Omega(t,x,y)}{\partial y} \\
&+\frac{\partial\Psi(t,x,y)}{\partial y}\frac{\partial\Omega(t,x,y)}{\partial x} - \frac{2}{\mathrm{Re}}\Delta\Omega(t,x,y) = 0\ ,
\end{aligned}
$$

where Δ is the Laplacian, $\mathrm{Re} = 2U_0 d/\nu$ is the Reynolds number and U_0 is the longitudinal velocity component in the center of the nozzle, ν is the kinematic viscosity, x is the coordinate along the jet axis, and y is the transversal coordinate.

The main equations and dynamics of a plane jet

$$\Omega(t,x,y) = \Delta\Psi(t,x,y) \,,$$
$$\frac{\partial\Omega(t,x,y)}{\partial t} - \frac{\partial\Psi(t,x,y)}{\partial x}\frac{\partial\Omega(t,x,y)}{\partial y} + \frac{\partial\Psi(t,x,y)}{\partial y}\frac{\partial\Omega(t,x,y)}{\partial x} - \frac{2}{\mathrm{Re}}\Delta\Omega(t,x,y) = 0 \,,$$

The stream function $\Psi(t,x,y)$ is related to the longitudinal (U) and transversal (V) components of the flow velocity by

$$U(t,x,y) = \frac{\partial\Psi}{\partial y}, \quad V(t,x,y) = -\frac{\partial\Psi}{\partial x}$$

The main equations and dynamics of a plane jet

In accordance with our idea,

the onset of turbulence is caused mainly by random disturbances (noise) at the nozzle exit section

Therefore we set

$$
\begin{aligned}
U(t,x,y) &= u_d(x,y) + u(t,x,y),\\
V(t,x,y) &= v_d(x,y) + v(t,x,y),
\end{aligned}
$$

where $u_d(x,y)$ and $v_d(x,y)$ are dynamical constituents of longitudinal and transversal velocity components, respectively, $u(x,y)$ and $v(x,y)$ are stochastic constituents caused by the input noise.

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We set

$$u_d(x, y) = \frac{1}{2}\left[1 - \tanh\left(q\,\frac{|y| - 1}{\delta_0(x)} - r\right)\right],$$

and find $v_d(x, y)$ and $\Omega_d(x, y)$.

For large $|y|$

$$v_d(x, \pm\infty) \approx \mp\frac{16qr}{3\delta_0(x)\mathrm{Re}},$$

$$\Omega_d(x, \pm\infty) \approx \mp\frac{256q^3 r}{9\delta_0^2(x)\mathrm{Re}^2}.$$

The main equations and dynamics of a plane jet

Dynamical constituents

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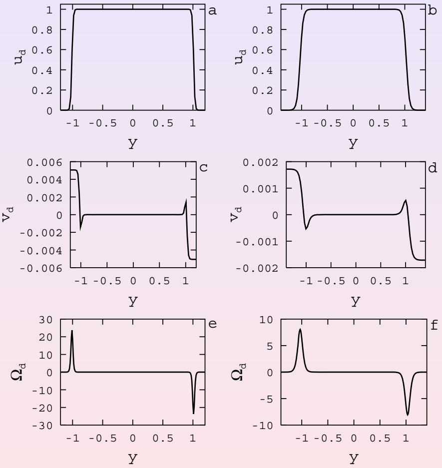
Plots of various quantities versus y for $b_0 = 0.1$, $q = 3$, $r = 0.5$, $\mathrm{Re} = 25000$:

(a) $u_d(0, y)$; (b) $u_d(8, y)$;
(c) $v_d(0, y)$, (d) $v_d(8, y)$;
(e) $\Omega_d(0, y)$; (f) $\Omega_d(8, y)$.



The main equations and dynamics of a plane jet

Equations for stochastic constituents

$$
\begin{aligned}
&\Omega - \Delta\Psi = 0,\\
&\frac{\partial\Omega}{\partial t} - u_d(x,y)\,\frac{\partial\Omega}{\partial x} + v_d(x,y)\,\frac{\partial\Omega}{\partial y} - \Omega_{dy}(x,y)\,\frac{\partial\psi}{\partial x}\\
&+\Omega_{dx}(x,y)\,\frac{\partial\psi}{\partial y} - \frac{2}{\mathrm{Re}}\,\Delta\Omega = \frac{\partial\psi}{\partial x}\,\frac{\Omega}{\partial y} - \frac{\partial\psi}{\partial y}\,\frac{\Omega}{\partial x}\,,
\end{aligned}
\tag{1}
$$

where

$$
\Omega_{dx}(x,y) = \frac{\partial\Omega_d(x,y)}{\partial x}, \quad \Omega_{dx}(x,y) = \frac{\partial\Omega_d(x,y)}{\partial y}.
$$

The boundary conditions:

$$
\left.\frac{\partial\psi}{\partial y}\right|_{x=0} = \xi_1(t,y), \quad \left.\frac{\partial\psi}{\partial x}\right|_{x=0} = -\xi_2(t,y)
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Using the Krylov–Bogolyubov method

Zero approximation

In zero approximation we consider linearized Navier-Stokes equations for stochastic components.

We seek a partial solution of these equations as a Fourier integral

$$\psi_0(t,x,y) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} \psi_0^{(S)}(t,x,y)\, dS, \qquad (2)$$

where

$$\psi_0^{(S)}(t,x,y) = f(S,x,y) \exp\left[i \left(St - \int\limits_0^x Q(S,x)\, dx \right) \right], \qquad (3)$$

S is the frequency of the wave on our time scale $Q(S,x)$ is the complex wave number.

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Zero approximation

Taking into account that the jet diverges slowly,

we can represent the function $f^{(S)}(x, y)$ and the wave number $Q(S, x)$ as series in a conditional small parameter $\mu \sim 1/\sqrt{\text{Re}}$:

$$
\begin{aligned}
f^{(S)}(x, y) &= f_0(S, x, y) + \mu f_1(S, x, y) + \dots \,, \\
Q(S, x) &= Q_0(S, x) + \mu Q_1(S, x) + \dots \,,
\end{aligned} \tag{4}
$$

where $f_0(S, x, y)$, $f_1(S, x, y)$, $\dots$ are unknown functions vanishing, along with their derivatives, at $y = \pm\infty$.

Using the Krylov–Bogolyubov method

Zero approximation

Substituting (2), in view of (3) and (4), into Eq. (1) and retaining only terms containing first derivatives with respect to x we obtain the following equations for $f_0(\mathrm{S}, x, y)$ and $f_1(\mathrm{S}, x, y)$:

$$L_0(Q_0)f_0 = 0, \tag{5}$$

$$L_0(Q_0)f_1 = iQ_1L_1(Q_0)f_0 - L_2(Q_0)f_0, \tag{6}$$

Using the Krylov–Bogolyubov method

Zero approximation

where

$$
\begin{aligned}
L_0(Q_0) &= i\Big(\mathsf{S} - u_{\mathrm{d}}(x,y)Q_0\Big)\left(\frac{\partial^2}{\partial y^2} - Q_0^2\right) \\
&+ v_{\mathrm{d}}(x,y)\left(\frac{\partial^3}{\partial y^3} - Q_0^2\,\frac{\partial}{\partial y}\right) + iQ_0\,\Omega_{\mathrm{d}y}(x,y) + \Omega_{\mathrm{d}x}(x,y)\,\frac{\partial}{\partial y} \\
&- \frac{2}{\mathrm{Re}}\left(\frac{\partial^4}{\partial y^4} - 2Q_0^2\,\frac{\partial^2}{\partial y^2} + Q_0^4\right), \qquad (7)
\end{aligned}
$$

$$
\begin{aligned}
L_1(Q_0) &= u_{\mathrm{d}}(x,y)\left(\frac{\partial^2}{\partial y^2} - 3Q_0^2\right) + 2\mathsf{S}Q_0 - 2iQ_0 v_{\mathrm{d}}(x,y)\,\frac{\partial}{\partial y} \\
&- \Omega_{\mathrm{d}y}(x,y) + \frac{8iQ_0}{\mathrm{Re}}\left(\frac{\partial^2}{\partial y^2} - Q_0^2\right), \qquad (8)
\end{aligned}
$$

Using the Krylov–Bogolyubov method

Zero approximation

$$
\begin{aligned}
L_2(Q_0) = \mathsf{S}\left(2Q_0\,\frac{\partial}{\partial x} + \frac{\partial Q_0}{\partial x}\right) + u_{\mathrm{d}}(x,y)\left[\frac{\partial^3}{\partial x \partial y^2} - 3Q_0\left(Q_0\,\frac{\partial}{\partial x} + \frac{\partial Q_0}{\partial x}\right)\right] \\
- i v_{\mathrm{d}}(x,y)\left(2Q_0\,\frac{\partial^2}{\partial x \partial y} + \frac{\partial Q_0}{\partial x}\,\frac{\partial}{\partial y}\right) - \Omega_{\mathrm{d}y}(x,y)\,\frac{\partial}{\partial x} \\
+ \frac{4i}{\mathrm{Re}}\left[2Q_0\,\frac{\partial^3}{\partial x \partial y^2} + \frac{\partial Q_0}{\partial x}\,\frac{\partial^2}{\partial y^2} - Q_0^2\left(2Q_0\,\frac{\partial}{\partial x} + 3\,\frac{\partial Q_0}{\partial x}\right)\right].
\end{aligned}
\tag{9}
$$

Using the Krylov–Bogolyubov method

Zero approximation

Eq. (5), with the boundary conditions for function f_0 and its derivatives so as to be vanishing at $y = \pm\infty$, describes a non-self-adjoint boundary-value problem, where Q_0 plays the role of an eigenvalue. We have to find a solution of this equation having a given behavior for large $|y|$.

According to Fredholm's well known theorem, Eq. (6) has a nontrivial solution only if

$$iQ_1 \int_{-\infty}^{\infty} \overline{\chi}(S,x,y) L_1(Q_0) f_0(S,x,y)\, dy - \int_{-\infty}^{\infty} \overline{\chi}(S,x,y) L_2(Q_0) f_0(S,x,y)\, dy = 0, \tag{10}$$

where $\overline{\chi}(S,x,y)$ is a complex conjugate eigenfunction of the adjoint boundary-value problem described by the equation:

$$i\left(\frac{\partial^2}{\partial y^2} - Q_0^2\right)\left[\left(S - u_{\mathrm{d}}(x,y)Q_0\right)\overline{\chi}\right] - \left(\frac{\partial^3}{\partial y^3} - Q_0^2\frac{\partial}{\partial y}\right)\left(v_{\mathrm{d}}(x,y)\overline{\chi}\right)$$
$$+ iQ_0\Omega_{\mathrm{d}y}(x,y)\overline{\chi} - \frac{\partial\left(\Omega_{\mathrm{d}x}(x,y)\overline{\chi}\right)}{\partial y} - \frac{2}{\lambda^2}\left(\frac{\partial^4\overline{\chi}}{\partial y^4} - 2Q_0^2\frac{\partial^2\overline{\chi}}{\partial y^2} + Q_0^4\overline{\chi}\right) = 0.$$

The condition (10) allows us to find the small correction $Q_1(\mathrm{S}, x)$ to the eigenvalue $Q_0(\mathrm{S}, x)$.

Setting $Q(\mathrm{S}, x) = \Gamma(\mathrm{S}, x) + iK(\mathrm{S}, x)$, we find the gain factor Γ and wave number K as functions of the Strouhal number St and of the distance x from the nozzle exit section by solving the boundary value problem. The wave number allows us to calculate the wave phase velocity $v_{ph} = \mathrm{S}/K$. The results are shown below.

Using the Krylov–Bogolyubov method

Zero approximation

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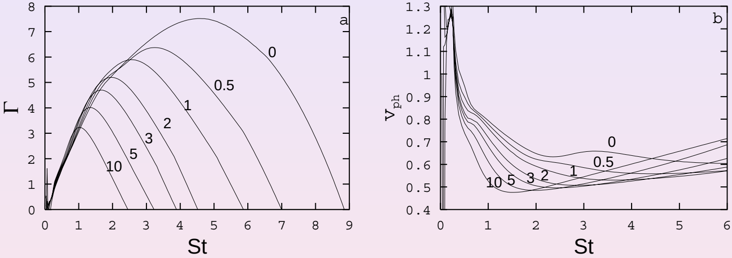

Figure: The dependences on the Strouhal number St for $b_0 = 0.1$, $q = 3$, $r_0 = 0.5$, $\mathrm{Re} = 25000$ and different x of: (a) the gain factor Γ and (b) the wave phase velocity $v_{ph} = \mathrm{S}/K$ The value of x is indicated near the corresponding curve in each case.

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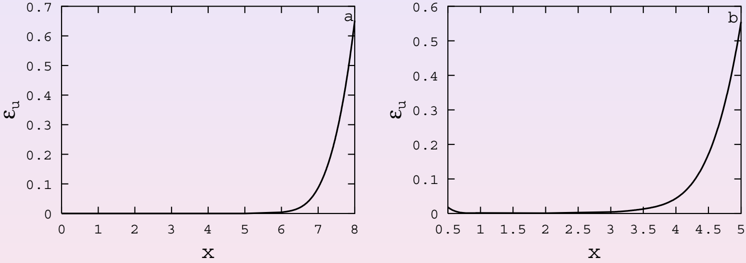

Figure: Plots of the mean square values of turbulent velocity pulsations versus x for (a) $y = 0$ and (b) $y = 0.7$.

Using the Krylov–Bogolyubov method

First and second approximations

In the first approximation

we can find the change of mean velocity due to turbulent pulsations (the mean velocity decreases when the intensity of velocity pulsations increases).

The second approximation

allows us to find nonlinear saturation of hydrodynamical waves.

Conclusions

- The change of the velocity pulsation variance with distance from the nozzle closely resembles changing order parameter with increasing temperature for a noised second order phase transition. That is why we guess that the onset of turbulence can be considered as a specific noise-induced phase transition similar to that for a pendulum with a randomly vibrated suspension axis.
- Our studies showed that the shift of velocity pulsation power spectra to the low-frequency domain is caused mainly by the jet divergence, not pairing of vortices, so that it can therefore be calculated within the linear approximation.
- The transformation of the mean velocity profile can be found without the use of the concept of turbulent viscosity.

Conclusions

- The influence of nonlinearity close to the jet symmetry plane ($y = 0$) is very small within the initial part of the jet, but increases significantly as we approach the boundary layer.
- The intensity of random disturbances at the nozzle exit necessary for the onset of turbulence may be very small. Our quasi-linear theory is valid only for such small intensities. For larger disturbance intensities, the development of turbulence is from the very outset an essentially nonlinear process.