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2D enslaving of MHD turbulence

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Abstract. At odds with its name, the classical weak turbulence theory only works for turbulence which is strong enough. Namely, the nonlinear resonance broadening has to be greater than the Fourier mode spacing $\Delta k = 2\pi/L$, where L is the size of the bounding box. I will revise the wave turbulence theory by extending it to the finite-size systems and by generalizing the description, traditionally done for the energy spectra, to higher wave-mode moments and its probability density functions. I show that when the perturbations of the external magnetic field are so small that the nonlinear resonance broadening is smaller than the Fourier mode spacing, the finite $k_\parallel$ modes get slaved to the $k_\parallel = 0$ modes. In other words, evolution in the perpendicular to the external field direction is identical to purely two-dimensional (2D) turbulence, whereas there is no evolution in the parallel direction, i.e. the parallel structure remains the same as in the initial condition (or forcing). In terms of the relative magnetic field perturbations, the condition of such a 2D enslaving is $\tilde{b}/B_0 < \sigma_{2D} = 2\pi k_\parallel^{1/2}/(L_\parallel^{1/2} k_\perp^2 L_\perp)$. The classical weak turbulence works if $k_\perp L_\perp \sigma_{2D} = 2\pi k_\parallel^{1/2}/(L_\parallel^{1/2} k_\perp) < \tilde{b}/B_0 < 1$. In the wide intermediate range of intensities $\sigma_{2D} < \tilde{b}/B_0 < k_\perp L_\perp \sigma_{2D}$ turbulence has a mesoscopic nature such that the wave correlation time remains of the order of the inverse Fourier-mode spacing and independent on the wave amplitude. Both the 2D and the wave components are present in this regime and constancy of the wave correlation time hints at the possibility that the wave dynamics is linear.

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Contents

1. Introduction

Wave turbulence (WT) theory was developed for weak deviations from a strong uniform external magnetic field within the incompressible MHD model in [1]–[3]. It was found to be consistent with experimental observational data [4]. This theory also provided a conceptual framework for further extensions to other MHD systems, i.e. compressible MHD [5], Electron MHD [6] and Hall MHD [7]. As usual in the classical WT theory [8], the description was developed for the wave *energy spectra* assuming that WT is statistically uniform in an *unbounded* coordinate space.

In the present paper, I will extend the weak turbulence approach onto the MHD systems which have *finite* dimensions across and along the external magnetic field. We will see that in this case a new regime of MHD turbulence exists for very small disturbances in which the three-dimensional (3D) motions are enslaved to purely 2D MHD turbulence. The classical WT description of [1] will be valid for much stronger intensities, which ensure that the nonlinear frequency resonance broadening is significantly greater than the spacing between the frequencies of the adjacent wave modes. We will see that there exists a broad range of intermediate intensities where the MHD turbulence is neither in the slaved regime nor in the classical WT regime. This regime exhibits a plateau behaviour for the frequency broadening in which this quantity is insensitive to the turbulence intensity and it remains of the order of the inter-mode frequency spacing.

Another new element of the present paper is the extension of the WT theory (which traditionally deals only with the wave spectrum) to the *probability density functions* (PDF) of the wave amplitudes using the approach developed in [9]–[11]. This will allow us to see that WT theory predicts not only solutions corresponding to Gaussian statistics, but also solutions with 'fat' PDF tails which correspond to an anomalously high probability of waves with greater than average amplitudes.

In this paper, we will consider very anisotropic MHD turbulence with $k_\perp \gg k_\parallel$ which is described by so-called reduced MHD equations [12, 13]. This is because treatment of the case $k_\perp \approx k_\parallel$ is very lengthy, and even the final kinetic equations take two journal pages to write [1]. This situation is not ideal for those who would like to read a first introduction into the WT approach, as well as to understand the main theoretical assumptions and their range of validity. On the other hand, MHD turbulence has a tendency to evolve to states with $k_\perp \gg k_\parallel$ and, therefore, this limit turns out to be the most important one.

2. Reduced MHD model

The Reduced MHD (RMHD) model relies on the fact that in the presence of a strong external magnetic field, $\mathbf{B} = B_0\,\mathbf{e}_z$, the transverse gradients are much greater than the parallel ones. In this regime, a reduced system of MHD equations was obtained by Kadomtsev and Pogutse [12] and, independently, by Strauss [13],

$$\dot{\psi} + \{\phi, \psi\} = \phi_z, \tag{1}$$

$$\nabla_\perp^2 \dot{\phi} + \{\phi, \nabla_\perp^2 \phi\} = \nabla_\perp^2 \psi_z + \{\psi, \nabla_\perp^2 \psi\}, \tag{2}$$

where 'dot' and subscript z mean the time and the z-derivative respectively and ϕ and ψ are the velocity and the magnetic stream-functions respectively,

$$\mathbf{u}_\perp = \mathbf{e}_z \times \nabla_\perp \phi, \tag{3}$$

$$\tilde{\mathbf{B}}_\perp = \mathbf{e}_z \times \nabla_\perp \psi, \tag{4}$$

and the curly bracket means the 2D Jacobian, i.e. $\{\phi, \psi\} = \mathbf{e}_z \cdot (\nabla_\perp \phi \times \nabla_\perp \psi)$. For simplicity, we put $B_0 = 1$.

In this section, we will re-write the RMHD model in a different but totally equivalent form without making any extra approximations or assumptions. Namely, we will pass to Fourier space and we will introduce variables which will be convenient for the analysis in the following sections. Let us first introduce the Elsasser stream-functions as

$$\eta^\pm = \phi \mp \psi. \tag{5}$$

Then equations (1) and (2) can be rewritten as

$$\nabla_\perp^2(\dot{\eta}^\pm \pm \eta_z^\pm) = -\tfrac{1}{2}[\{\eta^-, \nabla_\perp^2 \eta^+\} + \{\eta^+, \nabla_\perp^2 \eta^-\} \pm \nabla_\perp^2 \{\eta^+, \eta^-\}]. \tag{6}$$

In the linear limit we have wave solutions

$$\eta^\pm \propto \mathrm{e}^{\mathrm{i}\mathbf{k}\times\mathbf{x} - \mathrm{i}\omega^\pm t},$$

where frequencies $\omega^\pm$ are related to the wavevector $\mathbf{k} = (\mathbf{k}_\perp, k_\parallel)$ as

$$\omega^\pm = \pm k_\parallel.$$

Thus, as we see, superscripts '+' and '−' correspond to the waves propagating along and against the external field respectively. Let us consider a system in a 3D periodic box with dimensions $L_\perp \times L_\perp \times L_\parallel$ and denote by $\hat{\eta}_k^\pm$ the Fourier transform of $\eta^\pm$

$$\hat{\eta}_k^\pm = L_\perp^{-2} L_\parallel^{-1} \int_0^{L_\perp} \int_0^{L_\perp} \int_0^{L_\parallel} \eta^\pm(\mathbf{x}) \mathrm{e}^{-\mathrm{i}\mathbf{k}\times\mathbf{x}}\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z$$

with wavevector $\mathbf{k}$ taking values on a 3D lattice, $\mathbf{k} = (2\pi N_x/L_\perp, 2\pi N_y/L_\perp, 2\pi N_z/L_\parallel) N_x, N_y, N_z \in Z$. Let us introduce the interaction representation action variables

$$a_k^\pm = \mathrm{i}k_\perp \hat{\eta}_k^\pm \mathrm{e}^{\mathrm{i}\omega^\pm t}/\epsilon, \tag{7}$$

where ϵ is a positive real number which will help us later to keep track of the nonlinearity order. Note that introducing ϵ is a purely formal procedure for any value of ϵ which does not involve any approximations. In the following sections we will consider the case $a^\pm \sim 1$ and $\epsilon \ll 1$ which corresponds to the disturbances which are weak compared to the external field, and we will perform expansions in small ϵ. In such a weak nonlinear system the variables $a^\pm$ will be useful because they change in time much slower than the linear oscillations, which will allow us to separate the linear and the nonlinear timescales. Thus, we re-write our equations in terms of these variables (without making any assumption about ϵ yet)

$$\dot{a}_k^\pm = \epsilon \sum_{1,2} V_{k12} \mathrm{e}^{\pm 2\mathrm{i}k_{\parallel 1} t} a_1^\mp a_2^\pm \delta_{12}^k, \tag{8}$$

where for brevity

$$a_{1,2}^\pm = a^\pm(\mathbf{k}_{1,2}),$$

$$V_{k12} = V(\mathbf{k}_\perp, \mathbf{k}_{\perp 1}, \mathbf{k}_{\perp 2}) = \frac{(\mathbf{k}_\perp \times \mathbf{k}_{\perp 2})(\mathbf{k}_{\perp 1} \times \mathbf{k}_{\perp 2})_\parallel}{\mathbf{k}_\perp \mathbf{k}_{\perp 1} \mathbf{k}_{\perp 2}} \tag{9}$$

is the interaction coefficient and δ_{12}^k is the Kronecker delta defined as

$$\delta_{12}^k = 1 \ \text{ if } \ \mathbf{k} = \mathbf{k}_1 + \mathbf{k}_2 \qquad \text{and} \qquad \delta_{12}^k = 0 \ \text{ if } \ \mathbf{k} \neq \mathbf{k}_1 + \mathbf{k}_2.$$

So far, we have not made any additional assumptions, i.e. (8) is equivalent to the original RMHD equations (1) and (2). This equation is identical to the dynamical equation (3) obtained in [2] for the case $k_\parallel \ll k_\perp$ (thus they implicitly re-derived RMHD). In the appendix we explain that, contrary to previous claims, the WT approach is valid in some range of parameters within the RMHD model. Now we are going to assume that the system is weakly nonlinear so that the linear terms are greater than the nonlinear ones.

3. Very weak turbulence: 2D enslaving

Let us consider the case where the nonlinearity is weak enough for the characteristic evolution time of $a^\pm$, is long enough to satisfy the condition

$$\omega_{\mathrm{nl}} \sim 1/\tau_{\mathrm{nl}} \ll \Delta\omega = \Delta k_\parallel = 2\pi/L_\parallel, \tag{10}$$

i.e. the nonlinear frequency broadening ω_{nl} is much less than the frequency spacing between adjacent modes $\Delta\omega$. Let us introduce an intermediate time T such that

$$\frac{2\pi}{\omega} \ll T \ll \tau_{\mathrm{nl}},$$

and average equation (8) over this time. The amplitudes $a^{\pm}$ are slow and are not changed by such an averaging, and the only factor to be averaged is $e^{\pm 2ik_{\|1}t}$; we have

$$\dot{a}^{\pm}(\mathbf{k}_\perp, \mathbf{k}_\|) = \epsilon \sum_{\mathbf{k}_{\perp 1}, \mathbf{k}_{\perp 2}} V_{k12} a^{\mp}(\mathbf{k}_{\perp 1}, 0) a^{\pm}(\mathbf{k}_{\perp 2}, \mathbf{k}_\|) \delta(\mathbf{k} - \mathbf{k}_{\perp 1} - \mathbf{k}_{\perp 2}). \tag{11}$$

As we see, the nonlinear transfer occurs via the pure 2D $k_\| = 0$ modes and, in fact, $k_\| \neq 0$ modes are enslaved to the 2D component. Indeed, the solution to (11) has the form

$$a^{\pm}(\mathbf{k}_\perp, \mathbf{k}_\|) = a^{\pm}_\perp(\mathbf{k}_\perp) a^{\pm}_\|(\mathbf{k}_\|),$$

where $a^{\pm}_\|(\mathbf{k}_\|)$ = constant and for $a^{\pm}_\perp(\mathbf{k}_\perp)$ we have the following equation,

$$\dot{a}^{\pm}_\perp(\mathbf{k}_\perp) = \epsilon \sum_{\mathbf{k}_{\perp 1}, \mathbf{k}_{\perp 2}} V_{k12} a^{\mp}_\perp(\mathbf{k}_{\perp 1}) a^{\pm}_\perp(\mathbf{k}_{\perp 2}) \delta(\mathbf{k} - \mathbf{k}_{\perp 1} - \mathbf{k}_{\perp 2}). \tag{12}$$

These equations are identical to the pure 2D version of RMHD equations. Now, we are in the position to estimate $\omega_{\rm nl}$ and, therefore, establish the conditions under which the 2D enslaving occurs. Estimating as $V_{k12} \sim k_\perp$, $a^{\pm}_k \sim b_k \sim \tilde{b}/(k_\perp L_\perp k_\|^{1/2} L_\|^{1/2})$ and the effective number of summations as $k_\perp^2 L_\perp^2$, we have

$$\omega_{\rm nl} \sim k_\perp a_k k_\perp^2 L_\perp^2 \sim \frac{k_\perp^2 L_\perp \tilde{b}}{k_\|^{1/2} L_\|^{1/2}}. \tag{13}$$

Thus, the slaving condition (10) becomes

$$\frac{\tilde{b}}{B_0} \ll \frac{2\pi k_\|^{1/2}}{k_\perp^2 L_\perp L_\|^{1/2}}, \tag{14}$$

where we put back B_0 which was considered to be equal to the one above for simplicity. In recent direct numerical simulations (DNS) of MHD of [14] in the strongest value of the external field was $B_0 = 10$ (presumably in such units that $\tilde{b} \sim 1$ and the typical anisotropy could be estimated as the ratio of the parallel to perpendicular resolution, $\frac{k_\|}{k_\perp} \sim 128/512 = 1/4$. Then, the rhs of (14) is $2\pi 128^{1/2}/512^2 \sim 0.0002$. Thus we see that the condition of slaving was not satisfied.

4. Stronger disturbances: classical weak turbulence theory

Now let us consider the case when the disturbances are strong enough for the nonlinear broadening $\omega_{\rm nl}$ to be much greater than the mode spacing $\Delta\omega$. Note that the estimate for $\omega_{\rm nl}$ in this regime will be different from what we obtained in (13) and it will be estimated later. On the other hand, we want the nonlinear broadening $\omega_{\rm nl}$ to remain smaller than the linear frequency ω_k, i.e. we want the nonlinearity to be small, $\epsilon \ll 1$.

4.1. Weak nonlinearity expansion

Let us introduce time which is intermediate between the fast linear period and the slow nonlinear time,

$$\frac{2\pi}{\omega_k} \ll T \ll \tau_{\rm nl} = \frac{2\pi}{\omega_{\rm nl}},$$

and let us seek for a solution at $t = T$ in terms of series in small ϵ,

$$a_k^{\pm}(T) = a_k^{\pm(0)} + \epsilon a_k^{\pm(1)} + \epsilon^2 a_k^{\pm(2)} + \cdots \tag{15}$$

The leading order corresponds to $\epsilon = 0$, i.e. to the linear approximation. In this order, the interaction representation amplitude is time-independent and equal to its initial value at $t = 0$, i.e. $a_k^{\pm(0)} = a_k^{\pm}(0)$. The next order is obtained by substituting $a_k^{\pm(0)}$ into the rhs of (8) and integrating it from zero to T,

$$a_k^{\pm(1)} = \sum_{1,2} V_{k12} \Delta^{\pm}_{1T} a_1^{\mp(0)} a_2^{\pm(0)} \delta^k_{12}, \tag{16}$$

where

$$\Delta^{\pm}_{1T} = \frac{{\rm e}^{\pm 2{\rm i}k_{\parallel 1}T} - 1}{\pm 2{\rm i}k_{\parallel 1}}.$$

In the next order we get

$$\dot{a}_k^{\pm(2)} = \sum_{1,2} V_{k12} {\rm e}^{\pm 2{\rm i}k_{\parallel 1}t} \left(a_1^{\mp(0)} a_2^{\pm(1)} + a_1^{\mp(1)} a_2^{\pm(0)} \right) \delta^k_{12}. \tag{17}$$

Substituting here from (16) and integrating, we get

$$a_k^{\pm(2)} = \sum_{1,2,3,4} V_{k12} \delta^k_{12} \left(a_1^{\mp(0)} a_3^{\mp(0)} a_4^{\pm(0)} V_{234} \delta^2_{34} E^{+}_{31} + a_2^{\pm(0)} a_3^{\pm(0)} a_4^{\mp(0)} V_{134} \delta^2_{34} E^{-}_{31} \right), \tag{18}$$

where

$$E^{\pm}_{31} = \int_0^T \Delta^{\pm}_{1t} {\rm e}^{\pm 2{\rm i}k_{\parallel 1}t}\, {\rm d}t.$$

4.2. Statistical averaging

Statistical averaging in WT is done via considering initial wavefields with particular statistical properties. Strictly speaking, evolution equations obtained this way will be valid at $t = 0$ only and extending its validity to the long nonlinear time would require proving that the initial type of randomness survives over such a long time. This programme is possible to implement by considering the evolution of joint PDF for the wavemodes as suggested in [10]. In the present paper, we will take this property for granted and restrict our consideration to one-mode statistics only.

First, let us write the initial values of the Fourier coefficients in the amplitude-phase representation $a_k^{\pm(0)} = A_k^{\pm} \theta_k^{\pm}$ where $A_k^{\pm} \in R^{+}$ and $\theta_k^{\pm} \in S^1$ and let us require randomness of phases and amplitudes (RPA), namely that different modes (i.e. with different **k**'s or with different superscript signs) are independent random variables.[1] Additionally, the phase factors θ_k are uniformly distributed on the unit circle in the complex plane S^1. However, RPA does not specify the shape of PDFs for amplitudes A_k because this is not necessary for the WT closure. In other words, the statistics are not fixed to be Gaussian (or close to Gaussian), and we will derive an evolution equation for the amplitude PDF which describes evolution of non-Gaussian WT states. For this, let us introduce the generating function

$$Z_k^{\pm}(\lambda, t) = \langle\langle e^{\lambda \mathcal{V}|a_k^{\pm}|^2}\rangle_\theta\rangle_A, \tag{19}$$

where the $\mathcal{V} = L_\perp^2 L_\parallel$ is the box volume, and angle brackets with subscripts θ and A mean the RPA averaging over the statistics of $\theta_k^{\pm}$ and of $A_k^{\pm}$ respectively (done separately since they are statistically independent). In terms of the generating function, the wave spectrum and the higher momenta of the wave intensity are,

$$n_k^{\pm} = \mathcal{V}\langle|a_k^{\pm}|^2\rangle = \partial_\lambda Z^{\pm}|_{\lambda=0}, \tag{20}$$

$$I_k^{(p)} = \mathcal{V}^p\langle|a_k^{\pm}|^{2p}\rangle = \partial_\lambda^p Z^{\pm}|_{\lambda=0} \quad p \in N, \tag{21}$$

and the intensity PDF is

$$P_k^{\pm}(I) = \langle\delta(I - |a_k^{\pm}|^2)\rangle = \mathcal{L}^{-1} Z_k^{\pm}(\lambda), \tag{22}$$

where $\mathcal{L}^{-1}$ denotes the inverse Laplace transform operator.

A brief explanation is about the relation and the k-space moments considered in this paper and the x-space correlators which are more common in the turbulence literature. The k-space moments are more similar to the two-point rather than one-point x-space correlators. Thus, similarly to the turbulence structure functions, the k-space moments are a sensitive measure of intermittency. In addition, the k-space moments are more natural objects in the system of weakly nonlinear waves than the x-space moments. For reference, we give the following relation between the fourth-order structure function and the k-space moments,

$$\langle[\eta(\mathbf{x}) - \eta(\mathbf{x}+\mathbf{l})]^4\rangle = \text{Gaussianpart} + 48\sum_k \left[\langle|\hat{\eta}_k|^4\rangle - 2\langle|\hat{\eta}_k|^2\rangle^2\right]\sin^4(\mathbf{k}\cdot\mathbf{l}),$$

and all higher-order structure functions could be obtained in a similar form.

To derive an equation for Z let us find its value at $t = T$ by substituting (15) into (19) and expanding in small ϵ,

$$\begin{aligned} Z_k^{\pm}(T) &= \left\langle e^{\lambda\mathcal{V}|a_k^{\pm(0)}+\epsilon a_k^{\pm(1)}+\epsilon^2 a_k^{\pm(2)}|^2}\right\rangle = Z_k^{\pm}(0) + \left\langle e^{\lambda\mathcal{V}A_k^{\pm 2}}\left[\epsilon\lambda\mathcal{V}\left\langle a_k^{\pm(1)}\bar{a}_k^{\pm(0)} + \text{cc}\right\rangle_\theta\right.\right. \\ &\quad + \epsilon^2\left\langle \left(\lambda\mathcal{V} + \lambda^2\mathcal{V}^2 A_k^{\pm 2}\right)|a_k^{\pm(1)}|^2 + \lambda\mathcal{V}\left(a_k^{\pm(2)}\bar{a}_k^{\pm(0)} + \text{cc}\right)\right. \\ &\quad \left.\left.\left. + \frac{\lambda^2\mathcal{V}}{2}\left(a_k^{\pm(1)2}\bar{a}_k^{\pm(0)2} + \text{cc}\right)\right\rangle_\theta\right]\right\rangle_A + O(\epsilon^3). \end{aligned} \tag{23}$$

[1] Importantly, not only the phases of different modes are independent, but also the amplitudes A_k are independent of each other and of the phases. This property is essential for derivation of the evolution equation for the energy spectrum, but it is often not appreciated in literature, which can be seen e.g. from the standard reading of RPA as 'Random phase approximation' which refers to the phases but not to the amplitudes.

Here, we have to substitute $a_k^{\pm(1)}$ and $a_k^{\pm(2)}$ from (16) and (18) respectively and perform averaging, firstly over θ and secondly over A. Linear in ϵ terms turns into zero upon θ-averaging because they contain products of an odd number (three) of θ's. Further, the term containing $a_k^{\pm(1)2}\bar{a}_k^{\pm(0)2}$ also turns into zero upon θ-averaging because they contain a non-equal number of θ^+'s and θ^-'s. θ-averaging of the remaining terms give

$$\left\langle |a_k^{\pm(1)}|^2 \right\rangle_\theta = \sum_{1,2,3,4} V_{k12} V_{k34} \Delta_{1T}^{\pm} \bar{\Delta}_{3T}^{\pm} A_1^{\mp} A_2^{\pm} A_3^{\mp} A_4^{\pm} \langle \theta_1^{\mp} \theta_2^{\pm} \bar{\theta}_3^{\mp} \bar{\theta}_4^{\pm} \rangle_\theta \delta_{12}^k \delta_{34}^k$$

$$= \sum_{1,2} V_{k12}^2 A_1^{\mp 2} A_2^{\pm 2} |\Delta_{1T}^{\pm}|^2 \delta_{12}^k, \tag{24}$$

and

$$\langle \bar{a}_k^{\pm(0)} a_k^{\pm(2)} \rangle_\theta = \sum_{1,2,3,4} V_{k12} V_{234} \delta_{12}^k \delta_{34}^2 A_k^{\pm} A_1^{\mp} A_3^{\mp} A_4^{\pm} \langle \bar{\theta}_k^{\pm} \theta_1^{\mp} \theta_3^{\mp} \theta_4^{\pm} \rangle_\theta E_{31}^{+}$$

$$= -\sum_{1,2} V_{k12}^2 \delta_{12}^k A_k^{\pm 2} A_1^{\mp 2} E_{-11}^{+}, \tag{25}$$

where we took into account that $\bar{a}_k = a_{-k}$ because $\bar{a}_k$ arises from the Fourier transform of a real function. Note that the second term in (18) did not contribute to (25) because it leads to a product of non-equal number of θ^+'s and θ^-'s and, therefore, it has a zero average. Substituting (24) and (25) into (23) we have

$$Z_k^{\pm}(T) - Z_k^{\pm}(0) = \epsilon^2 \Big\langle e^{\lambda \mathcal{V} A_k^{\pm 2}} \Big[(\lambda \mathcal{V} + \lambda^2 \mathcal{V}^2 A_k^{\pm 2}) \sum_{1,2} V_{k12}^2 A_1^{\mp 2} A_2^{\pm 2} |\Delta_{1T}^{\pm}|^2 \delta_{12}^k$$

$$-2\lambda \mathcal{V} \sum_{1,2} V_{k12}^2 \delta_{12}^k A_k^{\pm 2} A_1^{\mp 2} \mathcal{R} E_{-11}^{+} \Big] \Big\rangle_A, \tag{26}$$

where $\mathcal{R}$ denotes 'the real part of'. Usually in WT theory, the next step would be averaging over the independent amplitudes A_k, followed by the large-box limit, followed by $\epsilon \to 0$ ($T \to \infty$) limit. This sequence of taking limits is essential because the resulting frequency resonance should be broad enough to cover many wave modes. However, the Alfven wave frequency, and therefore the frequency resonance, depend on the parallel wavenumber only. Thus, for the resonance function $\Delta_T^{\pm}$ to cover many modes, it is enough to take limit $L_\parallel \to \infty$ leaving $L_\perp$ finite, which allows one to generalize the description to systems bounded in the transverse direction. Thus, let us perform averaging of (26) over amplitudes A_k and take limit $L_\parallel \to \infty$, which gives

$$Z_k^{\pm}(T) - Z_k^{\pm}(0) = \epsilon^2 \Big[(\lambda Z_k^{\pm} + \lambda^2 \partial_\lambda Z_k^{\pm}) L_\perp^{-2}$$

$$\sum_{1_\perp, 2_\perp} \int V_{k12}^2 n_1^{\mp} n_2^{\pm} |\Delta_{1T}^{\pm}|^2 \delta_{12}^{\perp k} \, \delta(k_\parallel - k_{1\parallel} - k_{2\parallel}) \, \mathrm{d}k_\parallel$$

$$-2\lambda \partial_\lambda Z_k^{\pm} L_\perp^{-2} \sum_{1_\perp, 2_\perp} \int V_{k12}^2 n_1^{\mp} \mathcal{R} E_{-11}^{+} \delta_{12}^{\perp k} \, \delta(k_\parallel - k_{1\parallel} - k_{2\parallel}) \, \mathrm{d}k_\parallel \Big], \tag{27}$$

where the summation is performed over the transverse wavenumber components and $\delta_{12}^{\perp k}$ is the Kronecker symbol with respect to the transverse wavenumber coordinates. Now let us take limit $\epsilon \to 0$ ($T \to \infty$) taking into account that

$$\lim_{T \to \infty} |\Delta_{1T}^{\pm}|^2 = \pi T \delta(k_{1\parallel}) \qquad \text{and} \qquad \lim_{T \to \infty} E_{-11}^{+} = \frac{T}{2} (\pi \delta(k_{1\parallel}) + \mathrm{i} \mathcal{P}(1/x)),$$

where $\mathcal{P}$ means the principal value part. Replacing $(Z_k^{\pm}(T) - Z_k^{\pm}(0))/T$ with $\dot{Z}_k^{\pm}$ we finally have

$$\dot{Z}_k^{\pm} = \lambda[\gamma_k^{\pm}(Z_k^{\pm} + \lambda\partial_\lambda Z_k^{\pm}) - \kappa_k^{\pm}\partial_\lambda Z_k^{\pm}], \tag{28}$$

where

$$\gamma_k^{\pm} = \pi\epsilon^2 L_\perp^{-2} \sum_{1_\perp, 2_\perp} V_{k12}^2 n_1^{\mp}(k_\perp, 0) n^{\pm}(k_{\perp 2}, k_\parallel)\delta_{12}^{\perp k}, \tag{29}$$

$$\kappa_k^{\pm} = \pi\epsilon^2 L_\perp^{-2} \sum_{1_\perp, 2_\perp} V_{k12}^2 n_1^{\mp}(k_\perp, 0)\delta_{12}^{\perp k}. \tag{30}$$

Equation (28) is the master equation of the WT theory which contains the complete information about evolution of the one-mode statistics. In particular, taking the inverse Laplace we obtain the equation for the PDF,

$$\dot{P}^{\pm}(I) + \partial_I F^{\pm} = 0, \quad \text{where} \quad F^{\pm} = -I(\kappa_k^{\pm} P^{\pm} + \gamma_k^{\pm}\partial_I P^{\pm}). \tag{31}$$

Taking the first moment, we obtain an evolution equation for the energy spectrum,

$$\dot{n}_k^{\pm} = \int \dot{I} P^{\pm}\,\mathrm{d}I = -\int I\partial_I F^{\pm}\,\mathrm{d}I = \int F^{\pm}\,\mathrm{d}I = -\kappa_k^{\pm} n_k^{\pm} + \gamma_k^{\pm}, \tag{32}$$

or, substituting from (29) and (30), we have

$$\dot{n}^{\pm}(k_\perp, k_\parallel) = \pi\epsilon^2 L_\perp^{-2} \sum_{1_\perp, 2_\perp} V_{k12}^2 n_1^{\mp}(k_\perp, 0)[n^{\pm}(k_{\perp 2}, k_\parallel) - n^{\pm}(k_\perp, k_\parallel)]\delta_{12}^{\perp k}. \tag{33}$$

One can see that evolution in equations (28), (31) and (33) is always mediated by interaction with $k_\parallel = 0$ mode and, as a result, $k_\parallel$ enters as an external parameter into these equations. In other words, there is no energy transfer between modes with different finite $k_\parallel$'s. For the energy spectra, this property was already found in [1] (compare also with the similar property of the inertial waves in rotating fluids [15, 16]) and here we see that the same is true for the PDFs. This allows us to separate the non-evolving $k_\parallel$ dependence and an evolving $k_\perp$ part in these equations,

$$n^{\pm}(k_\perp, k_\parallel, t) = n_\parallel^{\pm}(k_\parallel) n_\perp^{\pm}(k_\perp, t) \tag{34}$$

with $n_\parallel^{\pm}|_{k_\parallel=0} = 1$, and

$$P^{\pm}(k_\perp, k_\parallel, I, t) = P_\parallel^{\pm}(k_\parallel) P_\perp^{\pm}(k_\perp, J, t), \tag{35}$$

where $J = I/n_\parallel^{\pm}(k_\parallel)$. Then, for the evolving perpendicular part of the PDF we have the following equation,

$$\dot{P}_\perp^{\pm}(J) + \partial_J F_\perp^{\pm} = 0, \quad \text{where} \quad F_\perp^{\pm} = -J(\kappa_k^{\pm} P_\perp^{\pm} + \gamma_{\perp k}^{\pm}\partial_J P_\perp^{\pm}), \tag{36}$$

where

$$\gamma_{\perp k}^{\pm} = \pi\epsilon^2 L_\perp^{-2} \sum_{1_\perp, 2_\perp} V_{k12}^2 n_{\perp 1}^{\mp} n_{\perp 2}^{\pm}\delta_{12}^{\perp k}, \tag{37}$$

and for the spectrum we have

$$\dot{n}^{\pm}_{\perp k} = \pi\epsilon^2 L_\perp^{-2} \sum_{1_\perp,2_\perp} V^2_{k12} n^{\mp}_{\perp 1} \left[n^{\pm}_{\perp 2} - n^{\pm}_{\perp k}\right] \delta^{\perp k}_{12}. \tag{38}$$

We emphasize again that in the case of Alfven waves, taking the $L_\perp \to \infty$ limit is technically unnecessary and, therefore, WT description in this case works for systems bounded in the transverse direction (waveguides). Of course, it works for the unbounded systems too, in which case we need to take the $L_\perp \to \infty$ limit which leads to the continuous version of the above equation [1],

$$\dot{n}^{\pm}_{\perp k} = \pi\epsilon^2 \int V^2_{k12}\, n^{\mp}_{\perp 1}[n^{\pm}_{\perp 2} - n^{\pm}_{\perp k}]\delta(\mathbf{k}_\perp - \mathbf{k}_{\perp 1} - \mathbf{k}_{\perp 2})\, \mathrm{d}\mathbf{k}_{\perp 1}\, \mathrm{d}\mathbf{k}_{\perp 2}. \tag{39}$$

We can now use (38) or (39) to estimate the characteristic nonlinear frequency broadening as

$$\omega_{\mathrm{nl}} \sim \frac{k_\perp^2 \tilde{b}^2}{k_\parallel}. \tag{40}$$

For applicability of the WT approach, ω_{nl} must be greater than the k-space mode spacing $\frac{2\pi}{L_\parallel}$ and at the same time remain less than the wave frequency ω_k. This results in the following applicability condition,

$$\frac{k_\parallel}{k_\perp} \gg \frac{\tilde{b}}{B_0} \gg \frac{2\pi k_\parallel^{1/2}}{k_\perp L_\parallel^{1/2}}. \tag{41}$$

Remarkably, there is a factor $k_\perp L_\perp$ difference in the rhs of this inequality and rhs of inequality (14), which means that there is a gap between the limits of applicability of the slaved regime and the WT regime. Once again using the characteristic parameters of anisotropic simulations of [14], we see that the WT conditions (41) are not satisfied. Thus, MHD turbulence computed in this work was neither in purely slaved regime nor in a pure WT regime, but rather in an intermediate state. In the discussion section we will propose a possible interpretation of such an intermediate regime.

4.3. WT solutions: Gaussian and non-Gaussian statistics

Let us now analyse solutions of the WT evolution equations for the PDF and the spectrum obtained above. First of all, let us consider steady-state power-law spectra,

$$n^{\pm}_{\perp k} \propto k_\perp^{\nu^\pm}. \tag{42}$$

A trivial solution of this kind with $\nu^\pm = 0$ describes a thermodynamics equipartition of energy and it is valid for both discrete and continuous systems because it corresponds to expressions under the sum of (38) and in the integrand of (39) which are zero point-wise.

A more interesting solution is the one that corresponds to a Kolmogorov-type cascade of energy from low to high $k_\perp$'s. Unfortunately, the discrete system is much harder to examine analytically in this case, and we will have to restrict ourselves to analysis of the continuous (infinite-box) system (38). In this case, the Kolmogorov-type solutions were obtained in [1]. There is a one-parametric family of solutions of this kind with exponents satisfying $\nu^{\mp} + \nu^{\pm} = -4$. Locality of the scale interactions is checked by finding the range of convergence of the integral in the kinetic equation and, in this case, it gives the following restriction on the exponents, $-3 < \nu^{\mp}, \nu^{\pm} < -1$. Different exponents in this one-parametric family correspond to different degrees of imbalance between the forward and backward propagating Alfven waves (with the limiting values of -3 and -1 corresponding to the zero and the infinite ratios of the wave forcings).

Let us now consider the steady solutions for the PDF. One of such solutions corresponds to $F_\perp^\pm = 0$ in equation (36). This gives

$$P_\perp^\pm = \frac{\gamma_{\perp k}^\pm}{\kappa_k^\pm} \exp\left(-\frac{\gamma_{\perp k}^\pm J}{\kappa_k^\pm}\right) = \frac{1}{n_k^\pm} \mathrm{e}^{-J/n_k^\pm} \tag{43}$$

which is the Rayleigh distribution of intensity J corresponding to the Gaussian statistics of the wavefield.

However, there is also a steady solution which corresponds to nonzero values of the amplitude-space flux $F_\perp^\pm$. This solution can be obtained in terms of the integral exponential functions, and its large J asymptotics is given by

$$P_\perp^\pm \approx \frac{-F_\perp^\pm}{J}, \quad J \gg n_k^\pm, \tag{44}$$

which corresponds to strong intermittency with anomalously high probability of strong (i.e. much stronger than mean) waves. Obviously, this result cannot be extended to the arbitrarily large intensities J because the WT approach based on small nonlinearity would break down. It is important however that for weak wavefields the WT description can be valid for intensities which are in the PDF tail, $J > \langle J \rangle = n_k$, if these J remain within the weakly nonlinear range. For larger J which correspond to strong nonlinearity the behaviour is much more complicated and hard to be treated rigorously. However, on the phenomenological level one could argue that there has to be a PDF cutoff at some limiting large J because waves of greater intensities do not exist due to a wave-breaking process. On the other hand, such a wave-breaking could be viewed as a sink of waves reaching limiting amplitudes, which correspond to nonzero amplitude-space fluxes $F^\pm$ and, therefore, intermittent PDF tails.

5. Discussion

We have established that for very small wave amplitudes satisfying condition (14) the MHD turbulence is '2D enslaved', i.e. its transverse structure is behaving identically to the purely 2D system while the parallel structure is not evolving. On the other hand if the wave intensity is stronger and satisfies (41), the MHD system will be described by the classical WT theory. Note that there is a substantial range of intensities for which neither condition (14) nor (41) are satisfied. For these intermediate amplitudes we predict a plateau behaviour such that the nonlinear frequency correction remains of the order of the mode spacing, $\omega_{\mathrm{nl}} \sim 2\pi/L_\parallel$. This is because both assumptions $\omega_{\mathrm{nl}} \ll 2\pi/L_\parallel$ or $\omega_{\mathrm{nl}} \gg 2\pi/L_\parallel$ would lead to a contradiction. Since condition $\omega_{\mathrm{nl}} \sim 2\pi/L_\parallel$ is independent of the amplitude, this suggests a linear dynamic. This could be realised, e.g. if the MHD turbulence in this regime would contain two components—a strong 2D component (condensate) and a weak 3D wave component whose evolution would be mainly due to the (linear) process of shearing by the 2D vortices. Testing this prediction (e.g. numerically) and developing an effective linear theory for such a 'mesoscopic' Alfven WT would be an interesting subject for future research. In fact, the parameters of the recent numerical simulation reported in [14] in the case of the strongest anisotropy ($\frac{\tilde{b}}{B_0} \sim 1/10$) seem to fall into the range of the mesoscopic turbulence. Thus, their result that in the case of strongest anisotropy the wave spectra are the same as in the pure 2D simulation can be viewed as an indirect confirmation of the presence of a strong 2D condensate component. However, to make a reliable conclusion further effort is needed with a special emphasis of observing whether there is a tendency of condensation at the $k_\parallel = 0$ and whether there are signatures of the two-component behaviour. It would be also very interesting to simulate MHD turbulence at very low excitation levels in order to test predictions about the 2D enslaving.

Although the numerical simulations represent the most immediate application of the finite-box theory developed here, one could also extend our approach to such naturally occurring MHD waveguides as coronal loops on the Sun [17] solar wind 'spaghetti' structures [18]. Of course, these applications would require inclusion of the fluid compressibility effects and specific geometry of the bounding volume.

It would also be interesting to extend the theory presented in this paper to the case without strong anisotropy. Even though this case is generally much harder, great simplifications were shown to arise in [19] if the pseudo-mode (associated with the parallel fields) is not excited.

In conclusion, I generalized the classical WT theory of MHD turbulence by taking into account the finite box effects and by extending this theory to describing the wave PDFs. We found a new regime of low-intensity 2D-enslaved MHD turbulence which exists in the finite-box systems. Inclusion of the wave PDF into the description allowed us to find non-Gaussian solutions which correspond to strong MHD intermittency.

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Appendix. Conditions of validity of RMHD

Here, we will show that the RMHD approach does not restrict the linear time to be greater than the nonlinear one and, therefore, there is a range of amplitudes where WT approach is applicable. For this, following [2] (and slightly changing the notations), we will write the exact dynamical MHD equations in terms of the Fourier transform of the 'perturbed' Elsasser variables,

$$\hat{\mathbf{z}}_k^\pm = \hat{\mathbf{u}}_k \pm \hat{\mathbf{b}}_k.$$

We have,

$$\dot{\hat{z}}_{kj}^\pm \mp \mathrm{i} B_0 k_\parallel \hat{z}_{kj}^\pm = -\mathrm{i}\epsilon k_m P_{jn} \sum_{1,2} \hat{z}_{1m}^\mp \hat{z}_{2n}^\pm \delta_{12}^k, \tag{A.1}$$

where $P_{jn} = \delta_n^j + k_j k_n / k^2$ is the projector operator. Now, following [2], we consider the case $k_\parallel \ll k_\perp$ and simplify the above equation. In particular, we will neglect the parallel component in the incompressibility condition,

$$\hat{\mathbf{z}}^\pm \times \mathbf{k} = 0 \approx \hat{\mathbf{z}}_\perp^\pm \times \mathbf{k}_\perp .$$

Note that this approximation, in addition to $k_\parallel \ll k_\perp$, requires $\hat{\mathbf{z}}_\perp^\pm \times \mathbf{k}_\perp \gg \hat{z}_\parallel^\pm k_\parallel$, i.e. that the parallel components of velocity and magnetic field are not too big. Simplifying the rhs of (A.1) under the same assumptions we get our equation (8) with

$$a_k^\pm = \mathrm{i}\frac{(\mathbf{k}_\perp \times \mathbf{z}_k^\pm)_\parallel}{k_\perp} \mathrm{e}^{\mathrm{i}\omega^\pm t}/\epsilon. \tag{A.2}$$

On the other hand, our equation (8) is identical to equation (3) of [2] in a symmetrized form (our $a_k^\pm$ corresponds to their $k_\perp a_k^\pm / k_y$). But, as we already mentioned in the text, this equation is identical to the original RMHD equations re-written in Fourier space without any extra assumptions.

Thus we conclude, that the conditions of applicability of RMHD are

$$k_\parallel \ll k_\perp$$

and

$$b_\perp, u_\perp \gg \tilde{b}_\parallel \frac{k_\parallel}{k_\perp}, u_\parallel \frac{k_\parallel}{k_\perp},$$

Note that there is no requirement $\tilde{b}_\perp, u_\perp \gg B_0 k_\parallel / k_\perp$, i.e. the linear time is allowed to be smaller than the nonlinear time. This means that the WT approach can be used within the RMHD model for sufficiently small amplitudes.

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