Nonlinear interaction of small-scale Rossby waves with an intense large-scale zonal flow

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The system of Rossby waves is unstable with respect to modulations. This instability results in the generation of a large-scale zonal flow with zero mean vorticity. It is shown that in both stable and unstable situations, a finite-time singularity formation takes place in the system. Such a singularity has the form of peaks on the vorticity profile of the zonal flow. The situation is also considered when some zonal flow with nonzero mean vorticity is initially present. Solitary-wave solutions appropriate for the description of nonlinear behavior of such systems are found. In the case of weak mean vorticity of the zonal flow, the solitons break and the singularities develop. If the mean vorticity is strong, then the evolution of the system can be considered as the dynamics of a soliton gas. Soliton dynamics possesses some interesting properties, such as formation of soliton pairs and annihilation of solitons during collision.

I. INTRODUCTION

The interaction of Rossby waves with a background zonal flow is an important problem of the geophysical fluid dynamics.[1-6] Strong zonal winds are observed in the Earth's atmosphere and in the atmospheres of other planets of the solar system, such as Jupiter, Saturn, and Uranus.[7] These zonal flows may result from the triple interactions of Rossby waves or from other nonlinear mechanisms (Newell,[8] Hasegawa *et al.*,[9] Whitehead,[10] and Reznik and Soomere[11]).

Previous works on the interaction of Rossby waves with zonal flows (see Killworth and McIntyre,[4] and references therein, Fyfe and Held,[5] and Benilov *et al.*[6]) deal mostly with so-called *rigid lid approximation* under which the frequency of the waves on a beta plane is given by

$$\omega=\frac{\beta p}{\rho^2k^2}+pu, \tag{1}$$

where

$$\mathbf{k}=(p,q)$$

is the two-dimensional (2-D) wave vector, ρ is the Rossby radius, and u is the x component of the zonal flow velocity. We assume that this flow is parallel to the x axis and that its velocity depends only on the y coordinate, $\mathbf{v}=(u,0)$, $u=u(y,t)$, as shown in Fig. 1 (in the absence of Rossby waves, such flow would be stationary).

Dispersion law [Eq. (1)] corresponds to the limit $\rho^2k^2\gg 1$, $\beta\rho^2\gg u$ in a more general expression, taking into account both the β effect and the effect of the background flow $u(y,t)$ (Dyachenko *et al.*[12]):

$$\omega\equiv\omega(\mathbf{k},y,t)=\rho^2p\frac{\beta+u(y,t)k^2}{1+\rho^2k^2}. \tag{2}$$

One can see from (1) that in the rigid lid approximation the zonal flow affects Rossby waves through the ordinary Doppler shift. One of the main physical effects in this case is wave absorption in the critical layer, where the wave phase velocity is equal to the local velocity of the zonal flow. The linear problem of the Rossby wave absorption in the critical layer has been solved by Dickinson.[1] On the nonlinear stage the wave absorption is followed by an alternate reflection, absorption, reflection process at the critical layer (Stewartson,[2] Warn and Warn,[3] Killworth and McIntyre,[4] and Haynes and McIntyre[13]).

Working with the rigid lid approximation, Fyfe and Held[5] have found another important property of the system: a steady propagation of the Rossby wave is possible only if the zonal flow is decelerated by less than 2/5 of its initial (near the source of the wave) value.

It is noteworthy [see Eq. (2)] that Rossby waves possess dispersion, even in the absence of β effect, which is due to the joint action of the background flow $u(y,t)$ and finite Rossby radius ρ. For $\rho^2k^2\gg 1$ and $\beta=0$, we have, instead of Eq. (1),

$$\omega=pu\left(1-\frac{1}{\rho^2k^2}\right). \tag{3}$$

Such dispersion becomes more important than the usual one of the Rossby waves [the first term in (1)] when the zonal wind is strong, $u\gg\beta\rho^2$, which frequently takes place in the Earth's stratospheric (especially polar) regions and in the atmospheres of the giant rotating planets of the solar system. For example, stratospheric wind can reach magnitudes more than 75 m/s (McIntyre and Palmer[14]), which is greater than $\beta\rho^2$ for the latitudes $\phi\gg 48°$. In this case therefore, the finiteness of radius of deformation is important even if it is large, $\rho^2k^2\gg 1$.

The purpose of this paper is to investigate the dynamics of zonal flow nonlinearly coupled with Rossby waves. We will concentrate on the case of the *intense* zonal flows, which provide the major contribution to the wave disper-

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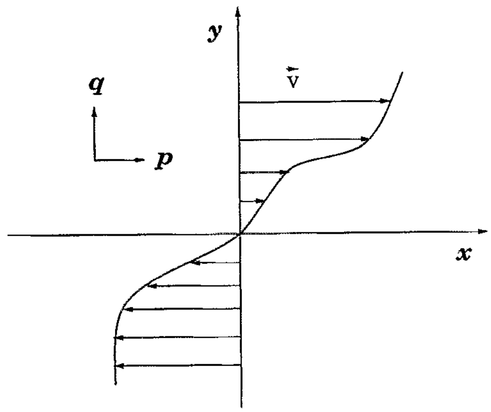

FIG. 1. Geometry of the problem.

sion (though the linear dynamics will be considered in the general case). As we will see, the zonal-flow-dominated dispersion (3) brings forth a new physics, compared to the rigid lid approximation describing the β-effect-dominated dispersive waves [see (1)].

We will assume that the characteristic length scale of the zonal flow, $L\sim(\partial/\partial y)^{-1}$, is large compared to the wavelength of small-scale Rossby waves:

$$kL\gg 1.$$

Assume also that Rossby wave amplitude is small, so that one can neglect wave/wave interactions compared with the large-scale/small-scale interactions.

Also, we will consider a one-dimensional (1-D) geometry, in the sense that all the vector and scalar fields (i.e., the zonal flow velocity and streamfunction, the spectrum of the small-scale Rossby wave turbulence; see below) depend on just one space coordinate, the "latitude" y, while the wave vector of small scales may have both components.

Of course, this is a rather idealized setup that somewhat limits the direct applicability of our results. However, this allows us to focus on essential new effects in their pure form. Issues of applicability and possible extensions of this study are discussed in the last section.

Let us define the large-scale zonal flow streamfunction $\Psi(y,t)$, according to

$$u(y,t)=\frac{\partial\Psi}{\partial y},$$

and introduce the wave-action spectrum of the small-scale turbulence:

$$n\equiv n(\mathbf{k},y,t)=\frac{\epsilon}{k^2\rho^3\beta},$$

where $\epsilon\equiv\epsilon(\mathbf{k},y,t)$ is the energy spectrum of Rossby-wave turbulence.

Then, under the assumptions made above, the equations for Ψ and Rossby-wave turbulent spectrum $n(\mathbf{k},y,t)$ can be obtained as the 1-D version of the 2-D equations of Dyachenko *et al.*:[12]

$$\frac{\partial}{\partial t}\left(\frac{\partial^2\Psi}{\partial y^2}-\rho^{-2}\Psi\right)=\frac{\partial^2 A}{\partial y^2}; \tag{4}$$

with

$$A=2\rho\beta\int\frac{pq}{k^2(1+\rho k^2)}n(\mathbf{k},y,t)d\mathbf{k} \tag{5}$$

and

$$\frac{\partial n}{\partial t}+\frac{\partial\omega}{\partial q}\frac{\partial n}{\partial y}-\frac{\partial\omega}{\partial y}\frac{\partial n}{\partial q}=0; \tag{6}$$

where the frequency ω is given by (2).

It is possible to neglect nonlinear interactions between small-scale motions if

$$u^2L^2\gg\rho\beta\frac{n}{(1+\rho^2k^2)}.$$

It is principally also possible to take into account nonlinear interactions among small scales by introducing a "collisional" term onto the rhs of (6) using, e.g., the theory of weak turbulence (Longuet-Higgins and Gill,[15] Zakharov and L'vov,[16] and Reznik and Soomere[11]). Although some attempts have been made in this direction (Benilov *et al.*[6]), the correct expression for the collisional integral is not yet known. To derive such an expression, one should keep in mind that the background shear flow modifies not only the wave dispersion, but also the normal variables, in terms of which the equation of weak turbulence is to be written. The possibility of constructing the theory of weak turbulence on the background of a large-scale flow is discussed in the last section.

Equations (4) and (6) conserve the energy:

$$E=\int\frac{u^2}{2}dy+\rho\beta\int\frac{n}{\rho^2k^2}dy\,d\mathbf{k}=\text{const.} \tag{7}$$

The rhs of Eq. (4) is the Reynolds stress averaged over the small scales; it provides a source for large-scale motions due to the small-scale turbulence. We will show that the contribution of this term is positive at large times, which means that the energy cascade is inverse in such a system.

Equation (6) for small scales has a form of phase volume conservation in two-dimensional y-q space. This means that the method of contour dynamics (Zabusky *et al.*[17]) can be applied for its numerical investigation when considering evolution of piecewise constant distributions $n(q,y)$.

We will use the set of Eqs. (4)–(6) with ω and A given by (2) and (5) correspondingly as the basic model.

This paper is organized as follows: In Sec. II we address the questions of stability of a steady state with uniform spatial distribution of the small-scale spectrum and with initially infinitesimal zonal flow. We demonstrate that for small enough azimuthal wave numbers of small-scale turbulence, the system is linearly unstable and generates nonzero zonal flow velocity. The stability criterion is not affected by the β effect, while finiteness of the deformation radius plays a destabilizing role.

In Sec. III we reduce Eqs. (4)–(6) for the case of a narrow-band spectrum of small-scale waves. We consider the limiting case

$$u \gg \beta\rho^2, \quad \rho^2 \gg L^2, \tag{8}$$

when the contribution to the dispersion due to the zonal flow is dominant [see (2) and (3)], and compare it with the rigid lid approximation. We demonstrate the Hamiltonian structure of the equations obtained.

In Sec. IV we study nonlinear evolution of a nearly monochromatic wave and zonal flow in the stable (according to Sec. II) situation using equations derived in Sec. III. We notice an analogy of this case with Riemann wave propagation and demonstrate finite time singularity formation having a form of tangential discontinuity on the zonal flow velocity profile.

In Sec. V we investigate the unstable case. Analytical consideration of the early time nonlinear evolution of the unstable system indicates that the solution develops nonanalyticity in a finite time. In other words, the nonlinear evolution of the instability also results in a finite time singularity formation. We show that singularity may develop even faster than the linear instability. This is due to nonlinear generation of the higher harmonics, which grow much faster (according to the linear analysis) than the lower harmonics.

In Sec. V, the evolution of profiles with nonzero mean vorticity is considered. In this situation the small-scale azimuthal wave number $q(y)$ depends almost linearly on time (so that its graph is sliding up or down almost without changing its shape), except where it is close to zero. At those latitudes y where $|q(y)|$ is small, most of the activity takes place, since the effect of the waves with large $|\mathbf{k}|$ on the zonal flow is weak [see (4) and (5)]. We construct a solution that has a form of solitary perturbation propagating on the background of a constant-vorticity zonal flow. Such *solitons* are located around the points where the azimuthal wave vector profile changes sign. They *appear in pairs* and *annihilate during collisions*. Both the formation and annihilation of the solitons take place when a local extremum of the azimuthal wave number profile $q(y)$ crosses abscissa while sliding up or down with time. If $q(y)$ is initially bounded, then it eventually becomes sign definite, after which the absolute value $|q|$ is ever increasing with time, and hence, according to (7), the energy is transferred from the Rossby wave to the zonal flow (the inverse cascade). Numerical computation of the full dynamic equations confirms soliton formation and the profiles of simulated perturbations agree well with ideal soliton solutions. Moreover, we show that there exist singular soliton solutions analogous to shock waves. They can appear when a traveling regular soliton arrives at latitudes with sufficiently small mean vorticity.

We discuss the results (in particular, the types of singularities found) and possible generalizations of the work in Sec. VII.

II. LINEAR ANALYSIS

Equation (6) describes conservation of the small-scale spectrum along trajectories in $\mathbf{k}$-y space:

$$n(\mathbf{k},y,t) = n_0(\mathbf{k}_0,y_0), \tag{9}$$

where $\mathbf{k}_0 \equiv \mathbf{k}_0(\mathbf{k},y,t), y_0 \equiv y_0(k,y,t)$ are functions inverse to the solutions of the equations for Lagrangian trajectories,

$$\dot{p} = -\frac{\partial\omega}{\partial x} = 0, \quad \text{i.e. } p = \text{const}, \tag{10}$$

$$\dot{q} = -\frac{\partial\omega}{\partial y} = -\frac{p\rho^2k^2}{1+\rho^2k^2}\frac{\partial^2\Psi}{\partial y^2}, \tag{11}$$

$$\dot{y} = \frac{\partial\omega}{\partial q}, \tag{12}$$

with initial conditions $\mathbf{k} = \mathbf{k}_0$, $y = y_0$. Here ω is the frequency of Rossby waves propagating on the background of the zonal flow; see (2).

The simplest solution of Eqs. (4)–(6) is the stationary state in which there are no large-scale motions and the small-scale wave distribution is homogeneous in space:

$$\Psi \equiv 0, \quad n \equiv n_0(p,q), \tag{13}$$

where n_0 is an arbitrary function.

Let us study the stability of this steady state. Consider perturbations of the form

$$q = q_0 + \tilde{q}, \quad \Psi = \tilde{\Psi}, \tag{14}$$

$$\tilde{\Psi}, \tilde{q} \propto \exp(\lambda t + i\kappa y). \tag{15}$$

Note that perturbations in function y are unimportant for linear analysis because the steady state we consider does not vary in the y direction. The meridional component of the wave number p does not vary in time [see (10)].

Substituting expressions (15) into (4), we obtain

$$-(\rho^{-2}+\kappa^2)\tilde{\Psi}\lambda = -\kappa^2\tilde{A}, \tag{16}$$

where

$$\tilde{A} = 2\rho\beta\int\frac{pq}{k^2(1+\rho^2k^2)}[n_0(p,q-\tilde{q}) - n_0(p,q)]\,d\mathbf{k}$$

$$= -2\rho\beta\int\frac{pq}{k^2(1+\rho^2k^2)}\tilde{q}\frac{\partial n_0}{\partial q}d\mathbf{k}. \tag{17}$$

Equation (11) yields

$$\lambda\tilde{q} = \frac{p\rho^2k^2}{1+\rho^2k^2}\kappa^2\tilde{\Psi}. \tag{18}$$

Substituting (18) into (17) and integrating by parts, we finally obtain the following dispersion relation:

$$\lambda^2 = \frac{2\rho\beta\kappa^4}{(\rho^{-2}+\kappa^2)}\int\frac{1+\rho^2(p^2-3q^2)}{(1+\rho^2k^2)^3}n_0\,d\mathbf{k}. \tag{19}$$

Thus, we see that every wave number contributes linearly into λ^2. Contribution of a wave with wave number $\mathbf{k} = (p,q)$ is unstable when

$$\rho^{-2}+p^2-3q^2 > 0 \tag{20}$$

[that is, if the spectrum n_0 was localized at $\mathbf{k}=(p,q)$, the system would be unstable under this condition]. Note that according to instability criterion (20), the finite radius of deformation plays a destabilizing role. Also, the condition of instability is independent of κ (i.e., of the length scale of initial perturbations) and of β. This instability is aperiodic (λ is purely imaginary in the unstable case).

Expression (19) allows us to write a simple estimate for the characteristic time τ of the linear evolution in the both stable and unstable cases. For $\rho k > \rho\kappa \sim 1$:

$$\tau \sim \frac{\rho^3 k^3}{2 u_w \kappa},$$

where u_w is the mean velocity magnitude of the Rossby wave.

Reliable values of the Rossby wave parameters needed for the estimate are not available at present, due to the lack of resolution of the fine structure in stratospheric experiments. If we assume $u_w \sim 10$ m/s ($\sim 13\%$ of the velocity in the fastest parts of the stream[14]), $\rho k \sim 3\rho\kappa \sim 2$, we obtain $\tau \sim 3$ weeks.

For less intensive and shorter Rossby waves the time scale τ can be as long as several months. Although this time would be comparable with the observed times of formation of the global stratospheric zonal flow, the neglect of the restoring forces and dissipation would not be valid for such a period of time.

On the other hand, as we will show in the next two sections, the nonlinear interaction accelerates the process and leads to a finite-time singularity formation. The actual time of the singularity formation may be even much less than the characteristic time of the linear process. Also, the system evolves faster if a finite shear flow is initially present; a situation of this type will be considered in Sec. VI.

III. INTERACTION OF NEARLY MONOCHROMATIC WAVE WITH ZONAL FLOW

Hereafter, we will study the nonlinear behavior of the system in the case of a narrow-band spectrum of small-scale waves:

$$n(\mathbf{k},y,t) = N(y,t)\,\delta(p-P)\,\delta(q-Q). \tag{21}$$

In other words, the small-scale motion is nearly a monochromatic Rossby wave, characterized at each point (latitude) y by its amplitude N called wave activity,[5] and wave vector $\mathbf{K}=(P,Q)$. Spatiotemporal evolution of K is governed by (10) and (11):

$$P = P_0 = \text{const}, \tag{22}$$

$$\frac{\partial Q(y,t)}{\partial t} = \frac{\partial}{\partial y}\left(\frac{P_0 \rho^2 (uK^2+\beta)}{(1+\rho^2 K^2)}\right). \tag{23}$$

The expression in the large parentheses should be considered a function of y, not only through the velocity u, but also through $K^2 = P^2 + Q^2$. (Actually, P can be an arbitrary function of y, but we will consider only profiles with $P=$const for simplicity.)

Equations for the wave activity $N(y,t)$ and the streamfunction of the zonal flow $\Psi(y,t)$ can be obtained by substitution of (21) into Eqs. (5) and (6), and integrating over dp, dq:

$$\frac{\partial N}{\partial t} + \frac{\partial}{\partial y}\left(N \frac{\partial}{\partial Q}\frac{P_0(u\rho^2K^2+\beta)}{(1+\rho^2K^2)}\right) = 0, \tag{24}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial^2 \Psi}{\partial y^2} - \rho^{-2}\Psi\right) = \frac{\partial^2}{\partial y^2}\left(\frac{2\beta\rho P_0 Q}{K^2(1+\rho^2K^2)} N\right). \tag{25}$$

The set of Eqs. (23)–(25) is complete for the description of a nonlinearly interacting, nearly monochromatic Rossby wave and large-scale zonal flow. These equations have a divergent form corresponding to the conservation of azimuthal impulse, the total wave activity, and potential vorticity of the zonal flow:

$$\int Q\, dy = \text{const}, \tag{26}$$

$$\int N\, dy = \text{const}, \tag{27}$$

$$\int \left(\frac{\partial^2 \Psi}{\partial y^2} - \rho^{-2}\Psi\right) dy = \text{const}. \tag{28}$$

The last conservation law is equivalent to the conservation of $\int \Psi\, dy$. Obviously, the conservation laws are valid if there is no source and dissipation, and the total flux of corresponding values through the boundaries is equal to zero (in the case of bounded systems). It can be seen from (23) and (28) that the conditions of absence of the total flux of Q in the case $\rho^2 \gg L^2$ implies that the total vorticity is zero in the system. Otherwise, the azimuthal impulse is a linear function of time.

It can be easily verified that Eqs. (23)–(25) conserve the total energy,

$$E = \int \left\{\frac{1}{2}\left[\left(\frac{\partial \Psi}{\partial y}\right)^2 + \rho^{-2}\Psi^2\right] + \frac{\beta\rho N}{\rho^2 K^2}\right\} dy. \tag{29}$$

One can also obtain (29) from the more general expression (7) by substituting the spectrum of form (21) and integrating over $\mathbf{k}$.

Equations corresponding to the rigid lid approximation can be obtained from (23)–(25) by taking the limit $\rho^2 \gg L^2$, $\beta\rho^2 \gg u$. Expression for the Rossby wave frequency in this case is given by (1), so that the zonal flow affects waves through the ordinary Doppler shift, and the wave dispersion is due to the β-effect only.

By combining Eqs. (24) and (25), one can see that in this case the quantity

$$u - \rho N \tag{30}$$

does not depend on time (see, e.g., Fyfe and Held[5]).

The interaction of Rossby waves with zonal wind in rigid lid approximation has been studied in detail previously,[1–6] so we will consider this case here.

As mentioned in the Introduction, often the intensity of the zonal winds is strong and their effect on the wave dispersion is comparable or even greater than the one due to the β effect. Hereafter, we will consider the case (8), when the wave dispersion is completely determined by the zonal flow, and one can disregard the β effect.

Taking the limit $\rho^2 \gg L^2$ (and therefore $\rho^2 K^2 \gg 1$), $\beta\rho^2 \gg u$ in Eqs. (23)–(25), one can see that the wave activity N does not depend on time in the leading order. For simplicity, we will assume that it does not depend on coordinate y either:

$$N=N_0.$$

Equations for the evolution of the azimuthal wave number Q and the large-scale vorticity,

$$\Omega=\frac{\partial^2\Psi}{\partial y^2}, \tag{31}$$

in the leading order are

$$\frac{\partial Q}{\partial t}=-P_0\Omega, \tag{32}$$

$$\frac{\partial\Omega}{\partial t}=\beta\rho N_0\frac{\partial^2}{\partial y^2}\left(\frac{2P_0Q}{\rho^2K^4}\right). \tag{33}$$

These equations can be also written in the Hamiltonian form:

$$\dot{Q}=-P_0\frac{\partial}{\partial y}\frac{\delta H}{\delta u}, \tag{34}$$

$$\dot{u}=P_0\frac{\partial}{\partial y}\frac{\delta H}{\delta Q}, \tag{35}$$

with the Hamiltonian function (energy):

$$H=\int\left(\frac{u^2}{2}+\frac{\beta\rho N_0}{\rho^2K^2}\right)dy\,dQ. \tag{36}$$

Note the resemblance of the Poisson brackets to those of the KdV equation.

Taking the time derivative of (32) and substituting $\partial\Omega/\partial t$ from (33)into the resulting equation, we can eliminate Ω and write, instead of (32) and (33), one second-order partial differential equation (PDE) for Q:

$$\frac{\partial^2 Q}{\partial t^2}=-\beta\rho N_0\frac{\partial^2}{\partial y^2}\left(\frac{2P_0^2Q}{\rho^2K^4}\right). \tag{37}$$

The criterion of instability (20) in this case looks like

$$P_0^2/Q_0^2>3, \tag{38}$$

where P_0 and Q_0 stand for the components of the wave vector in the stationary state. If this condition is violated, then Eq. (37) is of the hyperbolic type (strictly, the linearized equation for small perturbations will be a linear hyperbolic PDE). If condition (38) is satisfied, then Eq. (37) is of the elliptic type. Nonlinear evolution in these two cases is considered in Secs. IV and V, respectively.

IV. NONLINEAR DYNAMICS IN THE STABLE CASE

Consider a steady state $\Omega(y,t)\equiv 0$, $\mathbf{K}=(P_0,Q_0)$; $P_0,Q_0=\text{const}$. Suppose that the initial wave vector of the Rossby wave $\mathbf{K}=(P_0,Q_0)$ lies within stability bounds, i.e., condition (38) is violated.

Then the perturbations of the stationary state will be governed by a hyperbolic equation. This means that phase modulations of the Rossby wave corresponding to $\tilde{Q}$ and the perturbations of large-scale vorticity $\tilde{\Omega}$ will propagate along y at a constant phase velocity V_{ph}. This velocity can be found by substituting (21) into (19) and taking the limit $\rho^2\kappa^2,\rho^2K^2 \gg 1$:

$$V_{\text{ph}}^2=\frac{2\beta\rho N_0}{\rho^4K^6}(3Q_0^2-P_0^2). \tag{39}$$

Recall that in the situation we consider here [see (8)] the wave activity N does not depend on time, and so the perturbations of the Rossby wave amplitude do not evolve.

If the initial perturbation of the stationary state is weak, then for the description of its early evolution we can expand the governing equation (37) in small $\tilde{Q}$ and retain only the leading order of nonlinearity. As a result, we get, in the reference frame moving with velocity V_{ph}, the Riemann equation for $\tilde{Q}$:

$$\frac{\partial\tilde{Q}}{\partial t}=C\frac{\partial\tilde{Q}^2}{\partial y}, \tag{40}$$

where

$$C=\frac{4Q_0}{V_{\text{ph}}}\frac{(5Q_0^2+3P_0^2)}{(P_0^2+Q_0^2)^4}.$$

The well-known breaking of the Riemann wave corresponds here to discontinuity formation on the profile of Q, and, according to (32), formation of a δ-function-like peak on the vorticity profile $\Omega(y)$. The latter results from the fact that the time derivative in the case of weakly nonlinear wave motion is proportional to the spatial derivative. Note also that the large-scale velocity profile actually follows the profile of $\tilde{Q}$.

Characteristic time of the nonlinear evolution described by Eq.(40) is close to zero for $P_0^2\approx 3Q_0^2$. This means that the process is much faster in this case than predicted by the linear theory.

To complement this and further considerations, we performed numerical integration of the governing equation (37). A central difference scheme was used with typically 300 points per computational domain and the periodic boundary conditions; the Adams–Bashforth scheme in time was implemented.

The result of the numerical integration of Eq. (37) in the stable case is presented in Fig. 2. The sine-like initial profile has been used, corresponding to a wave traveling in the positive y direction. Figure 2 clearly demonstrates the steepening wave of Q and the formation of a sharp peak on the profile of Ω.

V. NONLINEAR DEVELOPMENT OF THE INSTABILITY

Now, let us consider a situation with condition (38) satisfied. At the linear stage perturbations exponentially grow, which correspond to a large-scale vorticity generation by the small-scale Rossby wave.

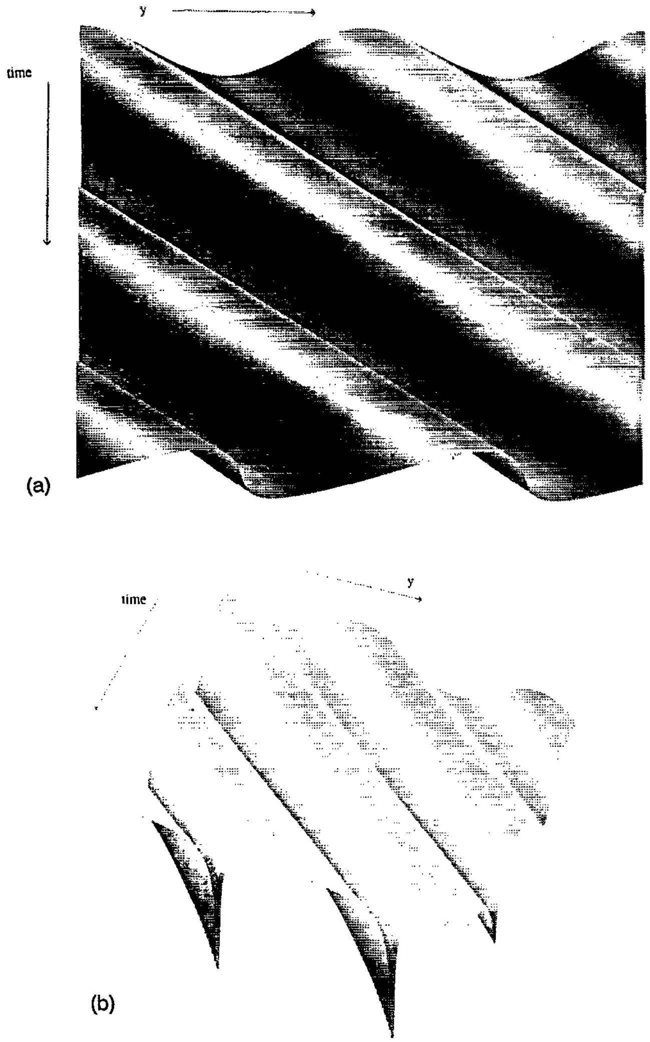

FIG. 2. Wave breaking and singularity formation in the stable case. (a) Time evolution of the azimuthal wave number profile. (b) Time evolution of the vorticity profile.

To describe the weakly nonlinear state, we expand Eq. (37) over small perturbations $\tilde{Q}$ and retain only the leading order of nonlinearity, which yields [compare with (40)]

$$\frac{\rho}{\beta N_0}\frac{\partial^2 \tilde{Q}}{\partial t^2} = -C_1 \frac{\partial^2 \tilde{Q}}{\partial y^2} + C_2 \frac{\partial^2 \tilde{Q}^2}{\partial y^2}, \tag{41}$$

where

$$C_1 = 2\frac{P_0^2 - 3Q_0^2}{(P_0^2 + Q_0^2)^3}, \tag{42}$$

$$C_2 = 4Q_0 \frac{5Q_0^2 + 3P_0^2}{(P_0^2 + Q_0^2)^4}. \tag{43}$$

We suppose that $C_1 > 0$, which corresponds to the unstable case. *A priori* the role of nonlinearity is not clear: it can either stabilize the growth of unstable disturbances or make the instability even more dramatic, perhaps leading to a finite-time singularity formation. We will show that the second possibility does actually take place.

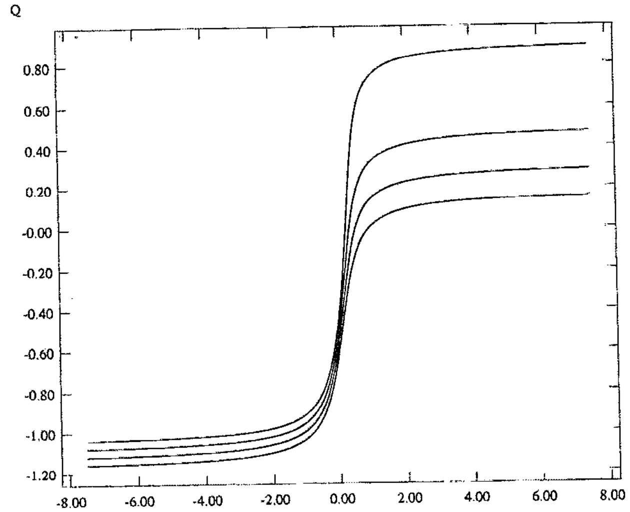

FIG. 3. Self-similar solutions in the unstable case for $C=0.157$ and $\tilde{Q}_{-\infty} = -1.04$, -1.08, -1.12, and -1.16.

We are looking for self-similar solutions of (41) in the following form:

$$\tilde{Q} \equiv \tilde{Q}(\eta); \tag{44}$$

$$\eta = \frac{\rho^{1/2}}{\beta^{1/2} N_0^{1/2}} y \Big/ (t^* - t), \tag{45}$$

where t^* is a constant.

Substituting (44) into (41) and performing one integration over η, we obtain the following equation for $\tilde{Q}(\eta)$:

$$\frac{\partial \tilde{Q}}{\partial \eta} = \frac{C}{\eta^2 + C_1 - 2C_2 \tilde{Q}}, \tag{46}$$

where C is an arbitrary constant.

According to (46), $\partial\tilde{Q}/\partial\eta$ tends to zero for $\eta \to \pm\infty$. The value of $\tilde{Q}$ approaches some different constants, $Q_{+\infty}$ and $Q_{-\infty}$, for $\eta \to +\infty$ and $\eta \to -\infty$ correspondingly.

The family of the self-similar solutions is two parametric; one can choose C and $\tilde{Q}_{-\infty}$ as independent parameters. Examples of profiles of $\tilde{Q}$ for $P_0=2$, $Q_0=1$, and different values of the parameters are shown in Fig. 3.

As time approaches t^*, the width of the transitional interval Δy, where $\tilde{Q}(y)$ changes its value from $Q_{-\infty}$ to $Q_{+\infty}$, turns to zero, so that a singularity having the form of a jump on the profile $\tilde{Q}(y)$ develops in finite time.

Figure 4 demonstrates the singularity formation obtained in the numerical simulation of the unstable case with $P_0^2=3$; $\tilde{Q}_0=0.5$, and $Q_0=0$. An almost perfect rectangular shape develops from the initial sinusoidal profile in the time of order unity [Fig. 4(a)].

Therefore, we observe a formation of two discontinuities of the function $\tilde{Q}(y)$, and each of those can be described in terms of the discussed above self-similar solutions. On the large-scale vorticity profile these singularities manifest themselves as the development of sharp peaks [Fig. 4(b)].

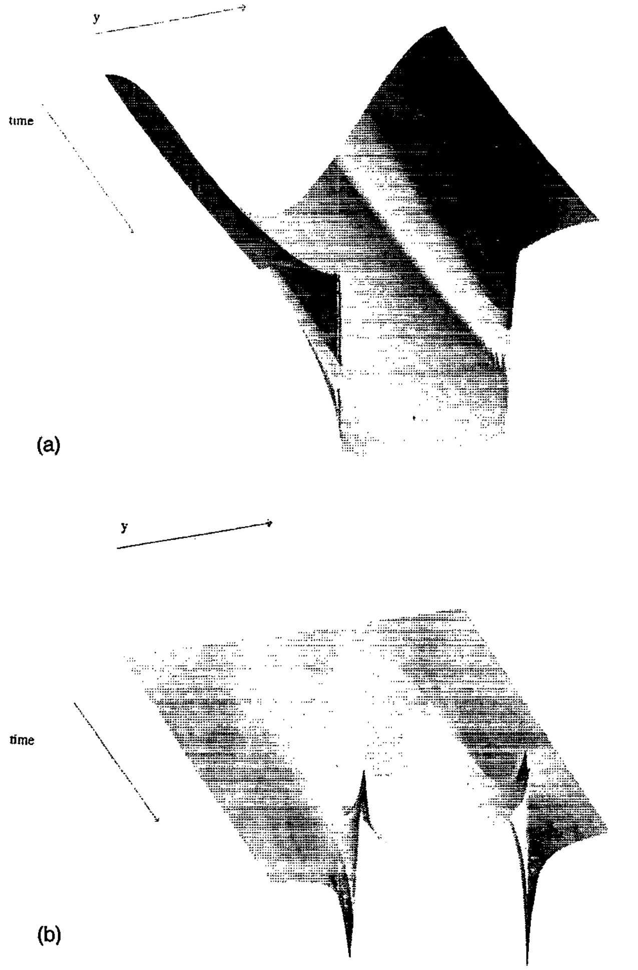

FIG. 4. Nonlinear development of the instability. (a) Formation of discontinuities on the Q profile. (b) Formation of peaks on the vorticity profile.

VI. SOLITONS PROPAGATING ON THE BACKGROUND OF A CONSTANT VORTICITY SHEAR FLOW

Up to this moment we have considered the nonlinear evolution of perturbations of a steady state with initially zero large-scale vorticity. Evolution of profiles with nonzero mean vorticity is different. If the mean vorticity of the zonal flow is not zero, then according to (32), the total y impulse of a Rossby wave $\int Q\,dy$ is a linear function of time, rather than a constant, as in the case of zero mean vorticity. For example, in the case of uniform vorticity distribution, wave number Q linearly increases or decreases (depending on the sign of the vorticity) at the same rate for all y. In other words, the graph of $Q(y)$ slides upward or downward without changing its form. According to (33), at large $|Q|$ the feedback from Rossby waves to the zonal flow is vanishing, so all the activity takes place around the points (latitudes), where the graph of $Q(y)$ intersects the abscissa. Recall that, according to (38), small $|Q|$ means instability, while large $|Q|$ corresponds to the stable situation. As the graph of Q slides vertically, the points where $Q(y)=0$ also move, and domains of active wave–flow interaction move along with them.

The basic equation (37) has a family of steadily translating analytical solutions of the form $Q=Q(y-ct)$ representing the above mechanism in its simplest form. These solutions can be written in an implicit form:

$$\eta=\frac{Q}{a}\left(c^2+\frac{2\beta N_0P_0^2}{\rho(P_0^2+Q^2)^2}\right), \tag{47}$$

where $\eta=y-ct$ is the moving coordinate, and a and c are free parameters (recall that the constant P_0 stands for the value of the x component of the wave vector; therefore, it is fixed and determined by the state of the system in the unperturbed region, $y\to\pm\infty$). Every such solution represents a linear dependence of Q on t and y at large $|\eta|$,

$$Q\sim\eta a/c^2,$$

with some nonlinear behavior when Q lies within the unstable domain around zero (according to the linear theory, see Secs. II and III). Note that by virtue of (32),

$$\Omega=\frac{c}{P_0}\frac{\partial Q}{\partial\eta},$$

for steadily translating solutions. Therefore, for such solutions large-scale vorticity is constant at $\pm\infty$, with

$$\Omega_\infty=\frac{a}{cP_0},$$

and features a localized disturbance around $\eta=0$ (however, Ω cannot change sign). The sample profiles of Q and Ω are shown in Fig. 5. We shall identify these solutions as *solitons* (which does not mean integrability here).

Obviously, for $Q(\eta)$ to be single values, the function $\eta(Q)$ should be monotonic. This implies the following regularity criterion:

$$c>c_{\rm cr},\quad \text{where } c_{\rm cr}=\frac{\beta^{1/2}N_0^{1/2}}{2P_0\rho^{1/2}},$$

or, in terms of Ω and Q,

$$\frac{\Omega_\infty}{(\partial Q/\partial\eta)_\infty}>\frac{\beta^{1/2}N_0^{1/2}}{2P_0^2\rho^{1/2}},\quad \left(\frac{\partial Q}{\partial\eta}\right)_\infty\equiv\frac{\partial Q}{\partial\eta}\quad \text{at } \eta\to\infty. \tag{48}$$

The criterion appears in a rather natural form, namely that the translation speed should be high enough, so that the instability in the domain of small wave numbers Q would not have enough time to develop singularities. When c is decreasing and approaching $c_{\rm cr}$, derivatives of Q become unbounded (as during the Riemann wave breaking), and Ω becomes unbounded too.

When condition (48) is violated, one can construct the soliton solution with a discontinuous profile of Q. The function $Q(\eta)$ defined by (47) is multivalued in this case [see Fig. 5(a)] and to obtain the profile, we have to find the position of the jump connecting the upper and the lower branches of the function $Q(\eta)$. This can be done in the same way as in the shock-wave theory, i.e., by finding the conditions on the jump using the conservation laws.

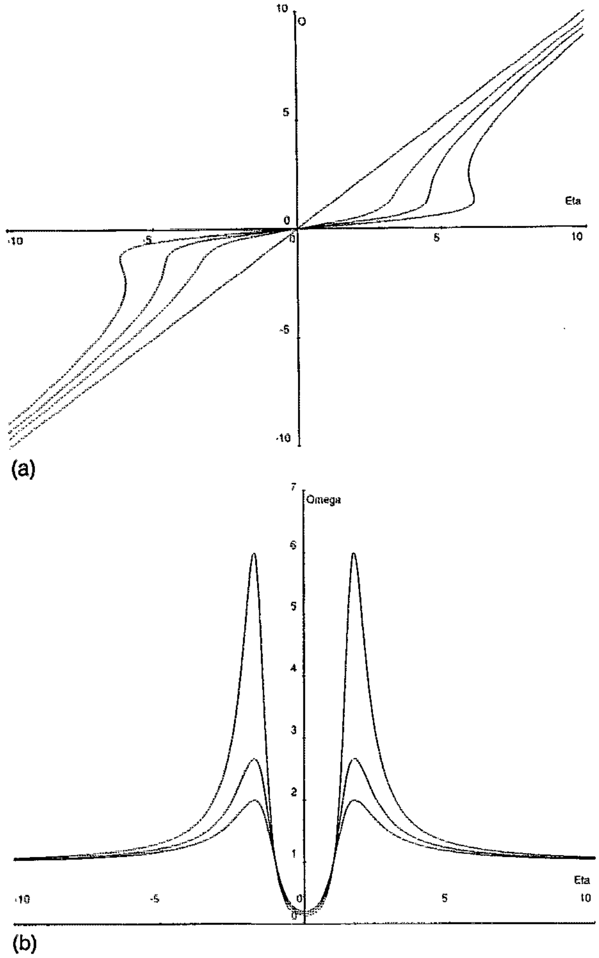

FIG. 5. Solitary wave on the background of the uniform shear flow; (a) profiles of Q. (b) Profiles of the large-scale vorticity.

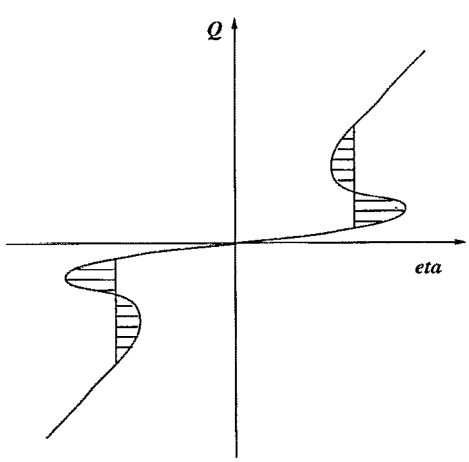

FIG. 6. The Q profile for the singular soliton.

Namely, the conservation of impulse $\int Q\,dy$ (more precisely, its linear dependence on time in the present case) implies that the areas of the segments cut by the jump on the graph of $Q(\eta)$ (see Fig. 6) must be equal.

Solitary wave solutions are of great importance for the dynamics of our system, as they develop naturally at the points where $Q(y)$ changes sign. To demonstrate this we take the initial condition $\Omega=-20$, $q=-14-10\cos 2\pi y$ at $t=0$ and integrate Eq. (37) numerically. The solution is shown in Fig. 7. One can see that the overall evolution is governed by $Q(y,t)=Q(y,0)-\Omega t$, except when Q is close to zero (the domain of instability). Strong traveling disturbances of vorticity develop at these locations. In Fig. 8 we compare instantaneous profiles of $Q(y)$ and $\Omega(y)$ at $-\Omega t=24$, with the solition solutions corresponding to the same translation velocity and slope $\partial Q/\partial\eta$ [that is, $a/c=\Omega_\infty=-20$, $a/c^2=(\partial Q/\partial\eta)_\infty=20\pi$]. It can be seen that the numerical solution follows ideal soliton profiles rather closely. Discrepancies are better seen in Ω, and they are due to both nonstationarity of the numerical solution and a weak smoothing introduced in the numerical scheme to suppress very high-wave number numerical instability.

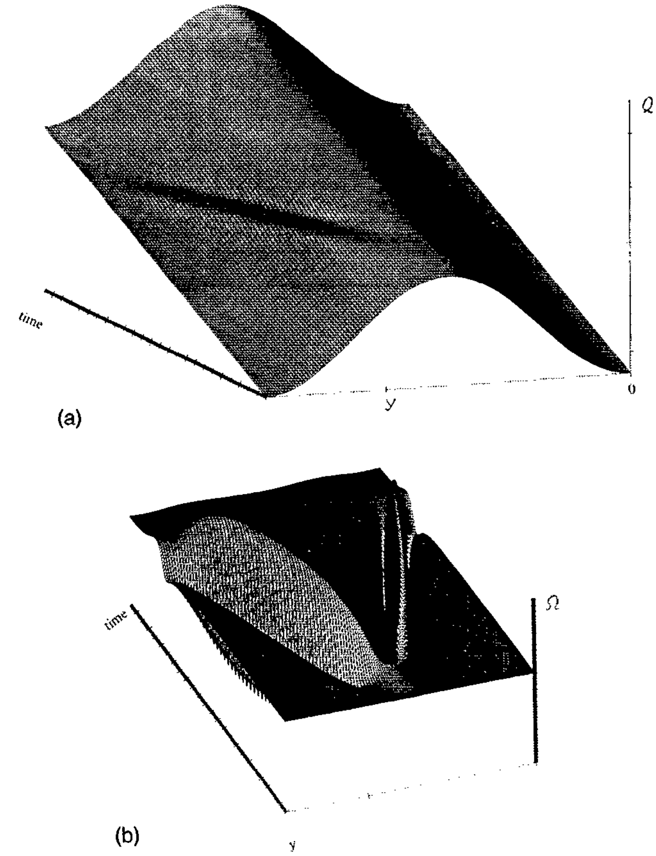

FIG. 7. Results of the numerical simulation of evolution equation (37). Creation, growth, and annihilation of the solitons. (a) Evolution of the Q profile. (b) Evolution of the vorticity profile.

As is clearly seen in Fig. 7, a pair of solitons appear at the moment when local maximum of the Q profile reaches zero, and they "annihilate" at the moment when the minimal value of Q is zero (it should be kept in mind when considering Fig. 7 that the boundary conditions are periodic in our computation). During its life cycle, the amplitude of the soliton reaches its maximum value, which corresponds to the slowest soliton speed, at the moment when $Q(y)$ intersects the abscissa at the steepest slope.

If the profile of Q is too steep, or the value of Ω_∞ is too small, then, according to the criterion (48), the solitons

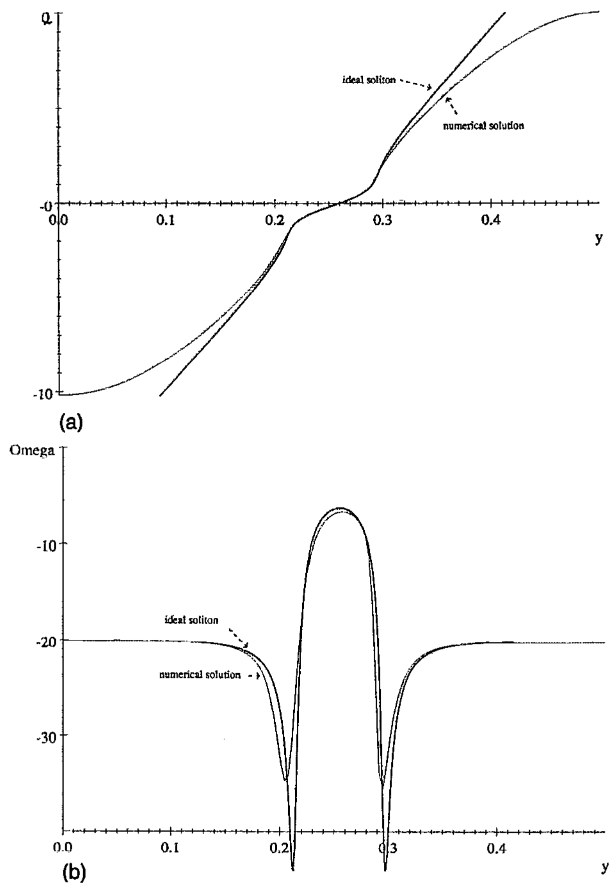

FIG. 8. Comparison of the shapes of the perturbations obtained numerically (see Fig. 6), with the analytical soliton solution. (a) The Q profile. (b) The vorticity profile.

with smooth profiles do not exist. In our computations we observe the singularity formation in this case. We can see this tendency up to the moment when the solution becomes unstable due to the high-wave number instability. The type and location of these singularities are in quantitative agreement with the singular soliton solution. Namely, the singularities have a form of two symmetrically located "walls" on the Q profile, corresponding to two peaks on the vorticity profile. Therefore, the criterion (48) and the singular solition solution can be used to predict singularity formation and its type.

VII. CONCLUSION

Summarizing the results, we would like to emphasize basic physical effects resulting from nonlinear interaction of small-scale Rossby waves with zonal flow in our 1-D formulation.

First, the system of Rossby waves can be modulationally unstable. The smaller Rossby radius, the larger is the domain of instability in k space [see the dispersion relation (19) and the criterion (20)]. This instability generates large-scale zonal flow with zero mean vorticity.

Nonlinear evolution in both stable and unstable cases leads to singularity formation. The singularity has a form of jump(s) on the azimuthal wave number profile and corresponding peak(s) on the large-scale vorticity profile. The singularity formation has been explained in terms of the Riemann wave breaking and self-similar solutions in the stable and unstable cases, respectively.

Second, if some flow with nonzero mean vorticity is initially present in the system, then the nonlinear dynamics of this system can be described in terms of soliton dynamics. The solitons move along the y axis, their amplitudes, and velocities may change, they appear in pairs and annihilate upon collision. At certain moments of time a pair of such solitons moving in opposite directions can be created in the process of evolution.

If the background vorticity is small and/or the profile of azimuthal wave number is steep, then solitons may increase in amplitude beyond the regularity limit, and then break down because of singularity formation. The condition of breaking is the opposite of (48). The type of singularity in this case is again the jump on the azimuthal wave number profile and the peak on the profile of large-scale vorticity.

Formation of such singularities qualitatively agrees with the observed steep gradient regions in the velocity profile of the Jovian zonal flows (see, e.g., Nezlin[18]). Latitude variation of the vorticity profile in the Earth's stratosphere also exhibits the regions of sharp transition (McIntyre and Palmer[14]). To make quantitative comparisons with observations other physically important factors, such as dissipation and restoring forces, should be incorporated into the model. The rest of this section is devoted to the discussion of applicability limits of our theory, and possible ways of its further generalization.

We neglected the β effect when studying the nonlinear dynamics. The possibility of zonal flow breaking in the rigid lid approximation (where the wave dispersion is due to β effect rather than the zonal flow) has been reported by Benilov *et al.*[6] Hence, the singularity formation seems to be a rather robust effect for the system of Rossby waves nonlinearly coupled with the zonal flows. It is possible to investigate a more general system, incorporating the effects of both finite β and the zonal flow on the Rossby-wave dispersion using Eqs. (23)–(25).

It should be mentioned that due to the vorticity profile steepening, the zonal flow may become unstable,

$$\frac{\partial\Omega}{\partial y}<\beta,$$

hence to make correct predictions about the further evolution of the system it is necessary to consider the full 2-D dynamics. On the other hand, as the characteristic length scale of zonal flow decreases, the scale separation assumption will eventually become invalid as well.

In other words, the observed singularity formation in our model indicates that in the real system an amplification of velocity gradients of the zonal flow should take place at least until the moment when 2-D instability comes into effect and/or when the length of zonal flow variation becomes comparable to the Rossby wavelength.

Therefore, further investigation is needed to explore the long-time behavior of the system when the present model becomes invalid. For example, to study the dynamics of the system with an unstable zonal flow, one can use 2-D equations derived by Dyachenko *et al.*[12] and, as proposed in this paper, the numerical method exploiting the phase volume conservation of the small scales and analogous to the particle-in-cell method. In this case, the interaction of the 2-D zonal-flow perturbations via the vector nonlinearity will cause developing of a strongly turbulent state characterized formation and interaction of the 2-D vortices.

Another important problem is to take into account the nonlinear interaction of small-scale Rossby waves with each other. As was mentioned in the Introduction, there is no straightforward way to do it, even in the case of weak interactions, because the zonal flow not only affects the wave dispersion, but it also modifies normal variables of this problem. This follows from the fact that the relation between the energy and wave action (i.e., the square of the modulus of the normal variable), $\epsilon_k=\omega_k n_k$, which usually holds in the weak turbulence theory, is not generally valid when Rossby waves propagate on the background of the zonal flow [see the expression for energy (7)]. In our opinion the right way to construct the theory of weak turbulence on the background of the zonal flow is to derive it from the first principles using the diagram technique developed by Zakharov and L'vov.[16] The fact that we know the expression for the square of modulus of the normal variable, i.e., the spectrum of wave action $n(\mathbf{k},y)$, may appear to be helpful for such a derivation.

ACKNOWLEDGMENTS

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