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LETTER

Sandpile behaviour in discrete water-wave turbulence

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Abstract. I construct a sandpile model for the evolution of the energy spectrum of water waves in finite basins. This model takes into account loss of resonant wave interactions in discrete Fourier space and restoration of these interactions at larger nonlinearity levels. For weak forcing, the wave action spectrum takes a critical ω^{-10} shape where the nonlinear resonance broadening overcomes the effect of the Fourier grid spacing. The energy cascade in this case takes the form of rare weak avalanches on the critical slope background. For larger forcing, this regime is replaced by a continuous cascade and a Zakharov–Filonenko ω^{-8} wave action spectrum. For intermediate forcing levels, both scalings will be relevant, ω^{-10} at small frequencies and ω^{-8} at large frequencies, with a transitional region in between, characterized by strong avalanches.

Keywords: hydrodynamic waves, intermittency, turbulence, sandpile models (theory)

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1. Introduction

The importance of basin finiteness for the statistical evolution of the free water surface was recently argued for in [1, 2] and [3]. Later in the present work, we will derive an estimate (15) according to which water surface waves of steepness α are sensitive to the basin size L if their wavelength λ is of the order of or greater than $L\alpha^4$. This means that waves with steepness $\alpha \sim 0.1$ and 1 m wavelength will 'feel' the boundaries for lakes or gulfs up to 10 km wide.

Recent direct numerical simulations of the free water surface in a finite basin in the presence of gravity revealed a bursty character of the nonlinear energy cascade from small to large wavenumbers [2]. Namely, cascade strengths were measured as functions of time at two different wavenumbers within the inertial range. Intermittent bursts were observed on these graphs, initially arising at the lower wavenumber and then propagating to the higher wavenumber. This behaviour reminds us of sandpile avalanches moving from low to high wavenumbers and their mechanism can be understood as a cycle:

- A cascade arrest due to the wavenumber discreteness leads to accumulation of energy near the forcing scale.
- This leads to widening of the nonlinear resonance.
- Sufficient resonance widening triggers the cascade thereby draining the turbulence levels and returning the system to the beginning of the cycle.

Note that the presence of forcing is essential for this scenario and it should not be expected in freely decaying fields, e.g. as in [3]. Below, we present and study a simple model for such a sandpile-like evolution of the water-wave spectrum.

2. Differential approximations for waves on an infinite surface

The evolution of random weakly nonlinear gravity surface waves in basins of infinite size and depth is well described by the Hasselmann kinetic equation [4]. However, for many purposes one can use a simpler differential equation model which preserves many

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properties of the Hasselmann equation [5]–[7]:

$$\dot{n} = \frac{C_1}{g^{3/2}\omega^3}\frac{\partial^2}{\partial\omega^2}n^4\omega^{26}\frac{\partial^2}{\partial\omega^2}\frac{1}{n}, \tag{1}$$

where C_1 is a dimensionless constant. This equation preserves the energy

$$E = \frac{2\pi}{g^2}\int \omega^4 n \, \mathrm{d}\omega \tag{2}$$

and the wave action

$$N = \frac{2\pi}{g^2}\int \omega^3 n \, \mathrm{d}\omega. \tag{3}$$

An even simpler differential approximation was suggested in [7]:

$$\dot{n} = \frac{C_2}{g^{3/2}\omega^3}\frac{\partial^2}{\partial\omega^2}(n^3\omega^{24}), \tag{4}$$

where C_2 is a dimensionless constant. This equation conserved both the energy and the wave action, but it does not have thermodynamic solutions corresponding to equipartition of these quantities.

If we insist that having thermodynamics equilibria is important, but ignore conservation of the wave action (which can be done at scales smaller, but not larger, than the forcing scale), then we can use another second-order differential equation,

$$\dot{n} = \frac{C}{g^{3/2}\omega^4}\frac{\partial}{\partial\omega}\left(n^2\omega^{24}\frac{\partial}{\partial\omega}(\omega n)\right), \tag{5}$$

where C is a dimensionless constant. A similar approach for Navier–Stokes turbulence is called the Leith model [8, 9]. Like in the Leith model [9], we can now find a 'warm cascade' solution, i.e. the general stationary solution of (5) which contains both finite flux and finite temperature components,

$$n = \frac{1}{\omega}\left(\frac{g^{3/2}P}{7C}\omega^{-21} + T^3\right)^{1/3}, \tag{6}$$

where P and T are (dimensional) constants measuring the energy flux and the temperature respectively. For $T = 0$ we recover the pure cascade Zakharov–Filonenko state [10],

$$n = (P/7C)^{1/3}\, g^{1/2}\omega^{-8}, \tag{7}$$

and for $P = 0$ we get the pure thermodynamic distribution,

$$n = \frac{T}{\omega}. \tag{8}$$

3. Finite basin effects

Let us now consider waves in a square basin with sides of length 2π, so that the wavenumbers take values on a discrete lattice, $\mathbf{k} \in \mathcal{Z}^2$. The main effect of the finite basin size is the loss of wavenumber resonances due to the wavenumber discreteness [1]–[3]. Indeed, let us consider the four-wave resonance conditions,

$$\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3 + \mathbf{k}_4, \tag{9}$$

$$\omega(\mathbf{k}_1) + \omega(\mathbf{k}_2) = \omega(\mathbf{k}_3) + \omega(\mathbf{k}_4). \tag{10}$$

Two different classes of such solutions were found in [2]: collinear quartets (all four wavevectors are parallel to each other) and 'tridents' (the first two wavevectors are anti-parallel and the other two are mirror symmetric with respect to the direction of the first two). Parametrization of the collinear quartets and quintets was done in [16]. Note that the collinear quartets are physically unimportant because of the zero nonlinear coefficient for such wavevectors, but the next order (the five-wave case) is nontrivial and yields an interesting kinetic equation [16]. The second class, tridents, can be parametrized as follows [2]:

$$\mathbf{k} = (a, 0), \qquad \mathbf{k}_1 = (-b, 0), \qquad \mathbf{k}_2 = (c, d), \qquad \mathbf{k}_3 = (c, -d)$$

with

$$a = (l^2 + m^2 + lm)^2, \qquad b = (l^2 + m^2 - lm)^2, \qquad c = 2lm(l^2 + m^2), \qquad d = l^4 - m^4,$$

where l and m are integers. New solutions can be obtained by further rescaling and rotating these tridents by rational angles.

Further, more resonances appear due to the nonlinear resonance broadening even when this broadening is significantly less than the wavenumber grid spacing [2]. However, the total numbers of exact and approximate resonances both remain significantly depleted with respect to the continuous case and, therefore, they are inefficient for supporting the turbulent cascade unless the nonlinear resonance broadening becomes of the order of the k-grid spacing. This allows us to formulate a simplified model for wave turbulence in finite basins as explained in the next section.

4. Wave turbulence in finite basins

First, we need to evaluate the four-wave resonance broadening which is determined by the characteristic nonlinear time $\tau_{\rm NL}$. The estimates for $\tau_{\rm NL}$ found using (1), (4), (5) and the original Hasselmann kinetic equation [4] will be the same. $\tau_{\rm NL}$ can also be found from a simple dimensional argument and the result is

$$\tau_{\rm NL} \sim g^{10}\omega^{-19}n^{-2}. \tag{11}$$

This corresponds to the resonance broadening in the k-space given by

$$\kappa_{\rm NL} = \frac{1}{(\partial\omega/\partial k)\tau_{\rm NL}} \sim g^{-11}\omega^{20}n^2. \tag{12}$$

In our model, we will postulate that the wave spectrum will not evolve at ω if the resonance broadening $\kappa_{\rm NL}$ is less that the k-grid spacing κ, and it will evolve as in the continuous case for $\kappa_{\rm NL} > \kappa$. Such a 'frozen turbulence' state was first observed in numerical simulations of the capillary waves [11] and it was later discussed in [2, 12]. One has to be careful, however, not to interpret the absence of the spectrum evolution at small amplitudes literally, because a small number of exact resonances do survive for the gravity waver waves (see the previous section) and further resonances may re-appear at resonance broadening which is much less than the k-grid spacing [2]. However, the number of such resonant modes is too small for evolving the spectrum efficiently and in our simple model we just put the Heaviside step function $H(\kappa_{\rm NL} - \kappa)$ as a pre-factor to the equation (5) in order to get a model for the spectrum evolution in discrete k-space:

$$\dot{n} = \frac{H(\kappa_{\rm NL} - \kappa)}{g^{3/2}\omega^4} \frac{\partial}{\partial \omega} \left(n^2 \omega^{24} \frac{\partial}{\partial \omega}(\omega n) \right) + \gamma(\omega) n. \tag{13}$$

Here, we have added a function $\gamma(\omega)$ which models forcing at low ω's (e.g. by wind) and dissipation at high ω values (by wave breaking) and which, in principle, can be a function of n. We will assume that in between the forcing and dissipation scales there exists an inertial range where $\gamma \approx 0$. From now on, we ignore the dimensionless order-1 pre-factor C since our resonance broadening is given by an order-of-magnitude estimate.

5. Behaviour predicted by the model

Equation (13) with $\kappa_{\rm NL}$ given by (12) will be our master model for the water-wave turbulence spectrum in a finite basin. Let us consider the consequences of this model qualitatively. Let us assume that initially there are no waves in the basin and let us start forcing the system at low frequencies. Then, there will be no transfer over scales and the spectrum will grow with the growth rate γ until it reaches the critical value where $\kappa_{\rm NL} = \kappa$. After that the nonlinear transfer will become activated and the energy will spill into the adjacent range of frequencies. If the forcing is so weak that its characteristic $\tau_{\rm F} = 1/\max\{\gamma(\omega)\}$ is much longer than the characteristic nonlinear time $\tau_{\rm NL}$, then the level of turbulence will never greatly exceed its critical value and the critical spectrum with $\kappa_{\rm NL} \approx \kappa$ will gradually occupy the entire inertial range. The condition $\kappa_{\rm NL} \approx \kappa$ gives for the critical spectrum

$$n_{\rm c} \sim g^{11/2} \kappa^{1/2} \omega^{-10}. \tag{14}$$

It is useful to rewrite this relation in terms of the water surface angle α characterizing the wave steepness,

$$\alpha_{\rm c} \sim (\lambda/L)^{1/4}, \tag{15}$$

where L is the box size and λ is the wavelength. One can interpret this relation as an expression for the minimal steepness for which the finite box effect can be ignored. For example, for a box containing 10^4 wavelengths the finite box effects can be ignored only for $\alpha > 0.1$.

If the forcing is stochastic then the subsequent evolution will consist of small avalanches going down the critical slope with time intervals Δt greater than the time $\sim\tau_{\rm NL}$ needed (according to (13)) for the avalanche to travel from the forcing to the dissipation scale. Note that this is a classical condition for sandpile models. To be specific, let us consider a type of forcing such that after each interval Δt we add an

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increment $(\Delta a)e^{i\phi}$ where phase ϕ is random and uniform in $(0, 2\pi]$ and a is a small positive value, $\Delta a \ll \sqrt{n_c}$. If at some moment of time $n = n_c$ at the forcing scales $k \in (k_F, k_F + \Delta k)$, then with probability 1/2 the spectrum will get greater than critical in the forcing range after time interval Δt and, approximately, $n = n_c + \Delta a\sqrt{n_c}$. For $\Delta t \gg \tau_{NL}$, such a disturbance will have enough time to travel/diffuse away before the next spectrum disturbance might appear at k_F after another Δt interval. Thus, the evolution of each supercritical disturbance can be treated separately. Because each such disturbance is small, one can use the linearized version of the evolution equation (13),

$$\dot{n} = \kappa g^{19/2}\omega^{-4}\frac{\partial}{\partial\omega}\left(\omega^4\frac{\partial}{\partial\omega}(\omega n)\right), \tag{16}$$

with the initial condition

$$n|_{t=0} = \Delta a\sqrt{n_c(\omega_F)} \qquad \text{for } \omega \in \omega_F + \Delta\omega, \qquad \text{and} \quad n = 0 \quad \text{otherwise.} \tag{17}$$

Equation (16) can be rewritten as

$$\dot{\epsilon} = \kappa g^{19/2}\left(4\frac{\partial\epsilon}{\partial\omega} + \omega\frac{\partial^2\epsilon}{\partial\omega^2}\right), \tag{18}$$

where $\epsilon = \omega n$ is the spectral energy density. According to this equation, the disturbance generated by forcing will propagate toward high k with speed $4\kappa g^{19/2}$ while getting diffused at an increased rate (due to moving to higher ω values). The stationary solution of (18) decays as $\epsilon \sim \omega^{-3}$ or $n \sim \omega^{-4}$, i.e. significantly slower than $n_c \sim \omega^{-10}$. Therefore, for a long enough inertial range the linear approximation will fail at some $\omega = \omega^*$ and the critical slope $n \sim \omega^{-10}$ will be replaced by the Zakharov–Filonenko slope $n \sim \omega^{-8}$ for $\omega > \omega^*$. The transitional range with $\omega \sim \omega^*$ will be characterized by strong avalanches.

For stronger forcing, transition to the Zakharov–Filonenko spectrum occurs at lower frequencies or even right at the forcing scale if $\omega_F > \omega^*$ (i.e. when α at the forcing scale is steeper than α_c at this scale). In numerical simulations, efforts are typically made to overcome 'frozen turbulence' and generate the cascade. At the present level of resolution (up to 512^2 modes) this goal can be achieved with only partial success because, according to estimate (15), the condition that turbulence is not frozen can only be marginally reconciled with the condition for the wave turbulence theory to work, $\alpha < 1$. Thus, in all existing simulations (e.g. [1]–[3], [13]–[15]), turbulence, although not frozen, was still quite sensitive to the finite box effects. This state was named mesoscopic turbulence in [3]. In the presence of forcing, it shows up via strong cascade avalanches coexisting with Zakharov–Filonenko state occupying about a decade long wavenumber interval.

6. Discussion

In this paper, we presented an evolution model for the spectrum of gravity water waves in finite basins. It has the following features:

- The model is given by a nonlinear second-order equation in Fourier space.
- The model has the cascade Zakharov–Filonenko spectrum and the thermodynamic spectrum among its solutions. It also has a general stationary solution where both the flux and the temperature effects are present.

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- The model takes into account the k-space discreteness by switching off the nonlinear evolution when the spectrum falls below a critical value. The critical spectrum is determined by the condition that the nonlinear resonance widening is equal to the k-grid spacing.

On the basis of this model, we established that for very weak forcing the spectrum takes the critical slope $n \sim \omega^{-10}$ with occasional weak 'avalanches' running down this slope. For larger forcing the system does not feel discreteness and the spectrum takes the Zakharov–Filonenko form, $n \sim \omega^{-8}$. For intermediate levels of forcing, the spectrum may have the -10 exponent at low frequencies and -8 at large frequencies within the inertial range. Such an intermediate case is characterized by strong avalanches down the mean spectral slope, a feature observed in recent numerical simulations of the free water surface [2].

It is interesting that slopes steeper than Kolomogorov ones were also previously obtained for waves with narrow band forcing [17]. The narrow band forcing also leads to a quasi-discrete character of the mode excitations. To conclude, it is worth mentioning that there are plenty of other examples of dispersive waves whose resonant interaction may be affected by the finite size box and where one could expect similar avalanche-like behaviour. Interestingly, irrespective of discreteness, the sandpile analogy has also been previously invoked in the wave turbulence context to illustrate the sudden readjustments necessary to balance the turbulence sources and sinks [18].

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