Physical realizability of anisotropic weak-turbulence Kolmogorov spectra

A. M. Balk and S. V. Nazarenko

Scientific Council on the Complex Problem of Cybernetics, Academy of Sciences of the USSR
(Submitted 1 August 1990)
Zh. Eksp. Teor. Fiz. **97**, 1827–1846 (June 1990)

A general approach is developed for determining whether the anisotropic weak-turbulence Kolmogorov spectra which have recently been derived by several investigators in a variety of physical problems can be realized in practice. The conditions under which these spectra are local and the condition for their stability are found. Turbulent states near Kolmogorov spectra are derived. New nonequilibrium solutions of kinetic equations for waves are derived. The turbulence of drift waves or Rossby waves is discussed as an example.

1. INTRODUCTION

Spectra of turbulence in anisotropic media have recently been derived in several problems in hydrodynamics, plasma physics, and astrophysics on the basis of Zakharov's concept[1,2] of weak-turbulence Kolmogorov spectra. A "Kolmogorov spectrum" is any spectrum which is determined by a flux of conserved quantities, e.g., energy or momentum, through a system (or, in other words, by the rate of dissipation in the system).[2] Zakharov[1] discovered a remarkable property of weak-turbulence Kolmogorov spectra: They can be derived systematically from the equations of the medium. In other words, they are exact solutions of wave kinetic equations

$$\frac{\partial n}{\partial t}=\mathrm{St}[n] \qquad (1)$$

(which are found by taking an average of the dynamic equations of the medium). Here $n=n_k=\varepsilon_k/\omega_k$ is the spectrum of the wave action, ε_k is the energy spectrum, ω_k is the dispersion law, $\mathrm{St}[n]$ is a collision integral, which in the case of a decay dispersion law ω_k takes the form[3]

$$\mathrm{St}[n]=\int (R_{012}-R_{102}-R_{210})\,dk_1\,dk_2, \qquad (2)$$

where

$$R_{012}=2\pi|V_{012}|^2\delta(k-k_1-k_2)\delta(\omega-\omega_1-\omega_2)(n_1n_2-nn_1-nn_2),$$

and $V_{012}=V(k,k_1,k_2)$ is a matrix element of the medium. Here $\omega_i=\omega_{ki}$ and $n_i=n_{ki}$ $(i=1,2)$.

Weak-turbulence Kolmogorov spectra were introduced in a pioneering paper by Kuznetsov[4] in order to describe weak turbulence in anisotropic media. Kuznetsov derived the turbulence spectra of ion acoustic waves in a magnetized plasma:

$$n_k=C_1P^{1/2}|k_x|^{-5/2}|k_\perp|^{-2},\qquad n_k=C_2Q^{1/2}|k_x|^{-5/2}|k_\perp|^{-1}, \qquad (3)$$

where the x axis is parallel to the magnetic field, and $|k_\perp|=(k_y^2+k_z^2)^{1/2}$. Here and below, P is the energy flux, Q is the flux of x momentum (through the system), and C_1 and C_2 are dimensionless constants which depend on only the properties of the medium. Kanashov[5] found the Kolmogorov spectra of a turbulence of the magnetized plasma waves (Langmuir waves) in various physical situations. In the case $\omega_p\gg\omega_H$ (ω_p and ω_H are, respectively, the plasma frequency and Larmor frequency), for example, he found

$$n_k=C_1P^{1/2}|k_x|^{-3/2}|k_\perp|^{-5/2}. \qquad (4)$$

Monin and Piterbarg[6] derived Kolmogorov spectra for divergence-free Rossby waves:

$$n_k=C_1P^{1/2}|k_x|^{-5/2},\quad n_k=C_2Q^{1/2}|k_x|^{-5/2}|k_y|^{-1} \qquad (5)$$

(the x axis is in the direction of increasing latitude). Mikhaĭlovskiĭ *et al.*[7–10] have derived anisotropic Kolmogorov spectra for numerous problems in plasma physics. They have found spectra for oblique electron drift waves, ion drift waves, wave branches in a highly inhomogeneous plasma, and electron acoustic, magnetosonic, Alfvén, and other wave modes. For example, the Kolmogorov spectra of turbulence of drift Alfvén waves in an inhomogeneous magnetized plasma take the form[8]

$$\begin{aligned} n_k&=C_1P^{1/2}|k_x|^{-3/2}|k_y|^{-3}|k_z|^{-2},\\ n_k&=C_2Q^{1/2}|k_x|^{-1/2}|k_y|^{-4}|k_z|^{-3}, \end{aligned} \qquad (6)$$

where the magnetic field is directed along the z axis, and the density gradient along the y axis. Fridman and Dolotin[11] used weak-turbulence Kolmogorov spectra to describe density-wave turbulence in gravitating astrophysical entities.

Anisotropic Kolmogorov spectra are thus widely used to describe turbulence in a variety of physical situations. However, the question of whether these spectra can be realized physically has not yet been resolved. Can these spectra be observed in practice? If so, under what conditions? What sort of agreement with experimental data should be expected? Answering these questions is the purpose of the present paper.

A first necessary condition for the realizability of Kolmogorov spectra is that the turbulence be "local" in the sense that the behavior of the turbulence is determined primarily by interactions between waves which are of approximately the same scale (as we know, the Kolmogorov-Obukhov spectrum is based on the assumption that the turbulence is local).[1] A first condition which must be satisfied if the turbulence is to be local is that the collision integral converge for theKolmogorov spectrum (Ref. 2; only in this case is the Kolmogorov spectrum a solution of the kinetic equation). This property is referred to as the "stationary localness" of Kolmogorov spectra. The stationary localness of Kolmogorov spectra is not by itself sufficient to ensure the localness of turbulence, as was found in Ref. 13. In the case of stationary localness, there may be an evolutionary nonlocalness, in which the evolution of a spectrum which initially differs by an arbitrarily small amount from a Kolmogorov spectrum is not determined exclusively by interactions between waves of

 0038-5646/90/061031-11$03.00

approximately the same scale. If some arbitrarily small perturbation of a Kolmogorov spectrum is specified, its time evolution leads to a spectrum such that the collision integral is dominated by the edge of the inertial interval.

Another question of fundamental importance to the physical realizability of Kolmogorov spectra is the stability of such spectra with respect to both initial perturbations and an external perturbation, in which case a term $\Gamma[n]$, describing a weak auxiliary source, is added to Eq. (1):

$$\frac{\partial n}{\partial t} = \mathrm{St}[n]+\Gamma[n] \tag{7}$$

[the source primarily responsible for forming the Kolmogorov spectrum does not appear explicitly in the kinetic equations (1) and (7); Ref. 2].

The questions of the localness and stability of Kolmogorov spectra also arise in the analysis of turbulence in isotropic media. Balk and Zakharov[13] have posed these questions in a rigorous mathematical way, have developed mathematical tools for studying them, and have described all possible situations in which weak-turbulence Kolmogorov spectra can be stable. They studied the questions stated above in the case of isotropic media. In particular, they proved the evolutionary localness of Kolmogorov spectra for capillary, gravity, and sound-wave turbulence, and they analyzed the stability of these spectra.

In the present paper we examine the localness and stability of weak-turbulence Kolmogorov spectra in the anisotropic case. In particular, we derive a condition for the stability of these spectra (Sec. 4). We find the conditions under which a steady-state perturbation $\Gamma[n]$ leads to the formation of a steady-state spectrum $n(k)$, and we describe the shape of this spectrum (Sec. 5).

The results of this paper apply in general to all solutions of the kinetic equation (1) which are power functions of the components of the vectors k [by analogy with Kolmogorov spectra (3)–(6); Sec. 2]. The kinetic equations (1) can have, in addition to Kolmogorov and thermodynamically equilibrium spectrum, one- or two-parameter families of power-law solutions, whose existence was pointed out in Ref. 5. In addition, we construct solutions of the kinetic equations (1) as the sums of several power functions (Sec. 6).

The general results derived here are used to analyze the turbulence described by the Charney-Hasegawa-Mima equation with scalar and vector nonlinearities. That equation arises in several important physical problems (Sec. 7).

2. POWER-LAW SOLUTIONS OF KINETIC EQUATIONS

To derive a Kolmogorov spectrum, one assumes that the medium is scale-invariant[2] in terms of the components of the vector k:

$$\omega(k)=k^{\alpha}, \qquad V(qk, qk_1, qk_2)=q^{\beta}V(k,k_1,k_2) \tag{8}$$

(q is a vector with positive components). Here and below, the multiple-index notation makes it possible to describe in a general way all the situations which arise. We introduce the dimensionality vector of the medium, d. If the medium is two-dimensional $[k=(k_x, k_y)]$ and scale-invariant with respect to k_x and k_y, we assume

$$d=(1,1), \quad \alpha=(\alpha_x,\alpha_y), \quad \beta=(\beta_x,\beta_y), \qquad k^{\alpha}=|k_x|^{\alpha_x}|k_y|^{\alpha_y}, \quad qk=(q_xk_x, q_yk_y) \text{ ,etc.}; \tag{9a}$$

If the medium is three-dimensional $[k=(k_x,k_y,k_z)]$ and scale-invariant with respect to k_x, k_y, and k_z, we assume

$$d=(1,1,1), \quad \alpha=(\alpha_x,\alpha_y,\alpha_z), \quad \beta=(\beta_x,\beta_y,\beta_z), \qquad k^{\alpha}=|k_x|^{\alpha_x}|k_y|^{\alpha_y}|k_z|^{\alpha_z}, \quad qk=(q_xk_x, q_yk_y, q_zk_z) \text{ ,etc.}; \tag{9b}$$

If the medium is three-dimensional $[k=(k_x,k_\perp)]$ and scale-invariant with respect to k_x and the absolute value of the vector $k_\perp=(k_y,k_z)$, we assume

$$d=(1,2), \quad \alpha=(\alpha_x,\alpha_\perp), \quad \beta=(\beta_x,\beta_\perp), \qquad k^{\alpha}=|k_x|^{\alpha_x}|k_\perp|^{\alpha_\perp}, \quad qk=(q_xk_x, q_\perp k_\perp) \text{ ,etc.} \tag{9c}$$

($q_\perp$ is a scalar). The sum of the components of the vector d is equal to the dimensionality of the medium. We denote by l the number of components in the vectors α, β, and d. For isotropic media, α, β, and d are scalars, and we have $l=1$.

We assume that the continuous medium has, in addition to scale invariance, the following symmetry (which is analogous to the isotropy of the medium in the case of isotropic Kolmogorov spectra[2]): In cases (9a) and (9b), the expression for $|V_{012}|^2$ is invariant under a change in sign of any component of all the vectors k, k_1, and k_2, e.g., $k_x \to -k_x$, $k_{1x}, \to -k_{1x}$, $k_{2x} \to -k_2$. In case (9c), the expression for $|V_{012}|^2$ is invariant under a change in sign of the x components of the vectors k_i $(i=0, 1, 2)$ and also under rotations of the vectors $k_{i\perp}$ $(i=0, 1, 2)$ through a common angle.

It can then be shown[2,4–7] that the kinetic equation (1) has an exact steady-state solution

$$n^0(k)=C_1P^{1/2}k^{-\nu}, \quad \nu=d+\beta \tag{10}$$

[the power is understood as in (9); cf. expressions (3)–(6)], which is a Kolmogorov spectrum with an energy flux P.

If the state of the medium is described in coordinate space by a single real function, we have

$$n(k)=n(-k), \tag{11}$$

and the phase variables of the system are the amplitudes of waves with wave vectors k from only half of k space (Ref. 14, for example). We assume that the x axis is directed in such a way that this half-space is the half with $k_x>0$. In kinetic equations (1) and (7) we should thus assume $k_x>0$ and that the integration is carried out over the half-spaces $k_{1x}>0$ and $k_{2x}>0$. In this situation, the kinetic equation (1) also has a Kolmogorov solution with an x-momentum flux Q (Refs. 4, 6, and 7):

$$n^0(k)=C_2Q^{1/2}k^{-\nu}, \quad \nu=d+\beta+(e-\alpha)/2, \quad e=(1,\underbrace{0\ldots0}_{l-1}). \tag{12}$$

The property (11) holds for the physical situations listed in Sec. 1, corresponding to Kolmogorov spectra (3)–(6), except (4). In this case we thus have only a single Kolmogorov spectrum (with an energy flux), while in each of cases (3), (5), and (6) we have two spectra. Spectra (3)–(6) are examples of Kolmogorov spectra (10) and (12) in problems differing in geometry [(9a), (9b), (9c)]. We recall that a Kolmogorov spectrum is actually a solution of Eq. (1) only if the collision integral converges for this spectrum, i.e., only if this spectrum is a local spectrum in the stationary sense. Kolmogorov spectra are exact solutions of kinetic equations along with Rayleigh-Jeans thermodynamic-equilibrium spectra

$$n(k)=\frac{T}{\omega+(v,k)} \qquad (T=\text{const},\ v=\text{const}); \tag{13}$$

the latter cause the integrand of $\mathrm{St}[n]$ to vanish identically, instead of merely causing the entire collision integral to vanish. In particular, the family of solutions (13) contains the power spectrum $n^0 = Tk^{-\alpha}$. If the property (11) holds, it also contains the power spectrum $n^0 = (T/v_x)|k_x|^{-1}$.

It was pointed out in Ref. 5 that, in addition to Kolmogorov and thermodynamic spectra, the kinetic equation (1) has some other power-law solutions:

$$n^0(k)=\mathcal{R}k^{-\nu} \quad (\mathcal{R}=\text{const}), \tag{14}$$

whose powers are determined from the equation

$$\mathrm{St}[k^{-\nu}]\equiv k^{2\beta+d-\alpha-2\nu}f(\nu)=0\,. \tag{15}$$

This equation specifies a line [in cases (9a) and (9c)] or a surface [in case (9b)] in the exponent space ν ($\dim \nu = l$).

3. BASIC EQUATIONS

To study the stability of any power-law solution (14) (in particular, a Kolmogorov or thermodynamic spectrum), we set

$$n=n^0(1+A), \quad A=A(k,t), \tag{16}$$

in Eq. (7). Assuming that the perturbation A is small, we find for it the following linearized equation:

$$\frac{\partial A}{\partial t}=\mathcal{L}(A)+\gamma. \tag{17}$$

Here $\mathcal{L}(A)$ is a linear operator whose form is determined from the expression $(1/n^0)\mathrm{St}[n^0(1+A)]$ by discarding all terms higher than first order in A:

$$\gamma=\gamma(k,t)=\frac{1}{n^0}\Gamma[n^0].$$

The quantity γ is assumed to be small (Sec. 1), so we ignore the term γA. We expand the perturbation A in an orthonormal system of eigenfunctions Y^m of the operator $\mathcal{L}$:

$$A(k,t)=\sum_m \mathcal{A}^m(k,t)Y^m. \tag{18}$$

This system of eigenfunctions is such that the expansion coefficients here depend only on the absolute values of the components of the vector k, while the dependence on the sign of the components k_x, k_y, k_z or on the direction of the vector k [in case (9c)] is reflected by the functions Y^m (cf. Ref. 13). The system of eigenfunctions Y^m evidently depends on the geometry of the problem [see (9a), (9b), and (9c)] and on satisfying condition (11). In case (9a), for example, with property (11), the expansion (18) is an expansion of the function A in a sum of odd and even parts (in terms of the variable k_y):

$$A(k,t)=\mathcal{A}^0(|k_x|,|k_y|,t)Y^0+\mathcal{A}^1(|k_x|,|k_y|,t)Y^1, \tag{19}$$

where

$$Y^0=\frac{1}{2^{1/2}}, \qquad Y^1=\frac{1}{2^{1/2}}\,\mathrm{sign}\,k_y. \tag{20}$$

If the property (11) does not hold, then four eigenfunctions are involved in the expansion (18) in case (9a):

$$Y^0={}^1/_2, \quad Y^1={}^1/_2\,\mathrm{sign}\,k_y,$$

$$Y^2={}^1/_2\,\mathrm{sign}\,k_x, \quad Y^3={}^1/_2\,\mathrm{sign}\,k_x\,\mathrm{sign}\,k_y.$$

The functions Y^m are chosen for case (9b) in a similar way. In case (9c), the eigenfunctions are as follows:

$$Y^m=\frac{1}{(2\pi)^{1/2}}e^{im\varphi}, \qquad m=0,\pm1,\pm2\ldots,$$

if the property (11) holds or

$$Y^{0m}=\frac{1}{2^{1/2}}\frac{1}{(2\pi)^{1/2}}e^{im\varphi},$$

$$Y^{1m}=\frac{1}{2^{1/2}}\,\mathrm{sign}\,k_x\frac{1}{(2\pi)^{1/2}}e^{im\varphi}, \quad m=0,\pm1,\pm2,\ldots$$

if it does not.

Substituting expansion (18) into Eq. (17), we find equations for the functions $\mathcal{A}^m(k,t)$ which are not coupled with each other. For the Mellin transform $G_m(s,t)$ of the functions $A^m(k,t)$ we find an equation (cf. Ref. 13)

$$G_m(s+h,t)=W_m(s)G_m(s,t)+\Psi_m(s+h,t). \tag{21}$$

Here $s = \sigma + i\omega$ is a complex vector variable ($\dim s = l$),

$$G_m(s,t)=\int A(k,t)\overset{*}{Y}{}^m k^{s-d}\,dk, \tag{22}$$

and

$$\Psi_m(s,t)=\int \gamma(k,t)\overset{*}{Y}{}^m k^{s-d}\,dk. \tag{23}$$

[The integration in (22), (23), and other equations is carried out over the same set of wave vectors as in the collision integral: Over the entire space $\mathbf{R}^l$ if the property (11) does not hold, or over the half-space $k_x > 0$ if the property (11) does hold.] In (21) we also have

$$h=\alpha-2\beta-d+\nu, \tag{24}$$

and $W_m(s)$ is the Mellin function.[13] A symmetrized explicit expression for this function can be derived with the help of the formula

$$\begin{aligned}\frac{1}{\mathcal{R}}(2\pi)^l\delta(r-s)W_m(s)&=\int k^{r+h-d}\overset{*}{Y}{}^m\mathcal{L}[(k')^{-s}Y^m]\,dk\\&=\int(2\pi)|V_{012}|^2\delta(k-k_1-k_2)\delta(\omega-\omega_1-\omega_2)\\&\quad\times k^{-\nu}k_1^{-\nu}k_2^{-\nu}\{[(k^{-s}Y^m+k_1^{-s}Y_1^m\\&\quad+k_2^{-s}Y_2^m)(k^\nu-k_1^\nu-k_2^\nu)\\&\quad-(k^{\nu-s}Y^m-k_1^{\nu-s}Y_1^m-k_2^{\nu-s}Y_2^m)][k^{\mu+s}\overset{*}{Y}{}^m\\&\quad-k_1^{\mu+s}\overset{*}{Y}{}_1^m-k_2^{\mu+s}\overset{*}{Y}{}_2^m]\}\,dk\,dk_1\,dk_2,\end{aligned} \tag{25}$$

where

$$\mu=\alpha-2\beta-2d+2\nu, \tag{26}$$

so the degree of homogeneity of the integrand in (25) is 0. At this point we assume $\mathscr{R} = 1$. In the two-dimensional case [(19), (20)], Eq. (25) leads to the following expression for the Mellin function:

$$W_m(s) = \int 2\pi |V(\mathbf{1}, k_1, k_2)|^2 \delta(\mathbf{1}-k_1-k_2)\delta(1-\omega_1-\omega_2) k_1^{-\nu} k_2^{-\nu}$$
$$\times \{[(1+k_1^{-s}(\operatorname{sign} k_{1y})^m + k_2^{-s}(\operatorname{sign} k_{2y})^m)(1-k_1^{\nu}-k_2^{\nu})$$
$$-(1-k_1^{\nu-s}(\operatorname{sign} k_{1y})^m$$
$$-k_2^{\nu-s}(\operatorname{sign} k_{2y})^m)][1-k_1^{\mu+s}(\operatorname{sign} k_{1y})^m$$
$$-k_2^{\mu+s}(\operatorname{sign} k_{2y})^m]\} dk_1\, dk_2$$
$$(m=0, 1;\ \mathbf{1}=(1, 1)). \qquad (27)$$

In the Appendix, the integral in (27) is put in a form convenient for calculations in the case $\alpha_x = 1$, which is a case of physical importance.

Since the spectrum (14) is a solution of Eq. (1) for any constant $\mathscr{R}$, the function $A \equiv$ const must be a solution of the linearized version of Eq. (17) with $\gamma = 0$, so the Mellin function corresponding to the eigenfunction $Y^0 =$ const should vanish at the point $s = 0$:

$$W_0(0) \equiv 2\int 2\pi |V_{012}|^2 \delta(k-k_1-k_2)\delta(\omega-\omega_1-\omega_2)$$
$$\times k^{-\nu} k_1^{-\nu} k_2^{-\nu} \{[k^{\nu}-k_1^{\nu}-k_2^{\nu}][k^{\mu}-k_1^{\mu}-k_2^{\mu}]\} k^d\, dk_1\, dk_2 = 0. \qquad (28)$$

The condition (28) is an equation which determines the exponents ν of the spectra (14). In general, it can be shown that the following identity holds for all ν:

$$f(\nu) \equiv {}^1/{}_2 W_0(0) \qquad (29)$$

[the function f is defined in (15)]. The exponents of the power-law thermodynamic and Kolmogorov spectra are found by equating the expression in braces (curly brackets) in (28) to zero (the δ-function is taken into account).

Mellin functions have the following important properties (cf. Ref. 13).

I. The function $W_m(s)$ is analytic in a cylindrical region

$$T\Omega_m = \{s = \sigma + i\omega \| \sigma \in \Omega_m,\ \omega \in \mathbf{R}^l\}, \qquad (30)$$

where Ω_m is a region in $\mathbf{R}^l$. It is assumed that Ω_m is the widest of such regions, so Ω_m is a convex region.[15]

II. The function $W_m(s)$ and $1/W_m(s)$ increase no more rapidly than algebraically in the limit $\operatorname{Im} s \to \infty$.

III. The value of the function $W_m(s)$ becomes asymptotically real (negative) in the limit $\operatorname{Im} s \to \infty$; i.e.,

$$\operatorname{Im} W_m(s)/\operatorname{Re} W_m(s) \to 0, \qquad \operatorname{Re} W(s) < 0 \quad \text{as} \quad \operatorname{Im} s \to \infty.$$

The latter property arises because part of the expression in braces in (25) oscillates rapidly in the limit $\operatorname{Im} s \to \infty$, while the rest can be written in the form

$$(k^{\mu}-k_1^{\mu}-k_2^{\mu})(k^{\nu}-k_1^{\nu}-k_2^{\nu}) - (k^{\mu+\nu}+k_1^{\mu+\nu}+k_2^{\mu+\nu}).$$

Property III then follows by virtue of (28).

4. STABILITY CONDITION

Using the real, nondegenerate matrix R, we can perform a linear transformation of the variable s in Eq. (21), i.e., $s = R\zeta$ ($\zeta = \xi + i\eta$ is a new complex variable), such that the vector e [see (12)] becomes the vector h [see (24)]: $h = Re$. Introducing

$$w_m(\zeta) = W_m(s), \quad g_m(\zeta) = G_m(s),$$
$$\psi_m(\zeta) = \Psi_m(s) \text{ with } s = R\zeta,$$

we rewrite Eq. (21) as

$$\dot{g}_m(\zeta_1+1, \zeta_2, t) = w_m(\zeta_1, \zeta_2) g_m(\zeta_1, \zeta_2 t) + \psi_m(\zeta_1+1, \zeta_2, t). \qquad (31)$$

Here the complex vector ζ_2 consists of all components of the vector ζ except the first, ζ_1. The variable ζ_2 appears as a parameter in Eq. (31). This equation can be solved at a fixed ζ_2 for the function g_m, which depends on the one scalar variable ζ_1. This circumstance makes it possible to analyze Eq. (31) by the method developed in Refs. 13.

We introduce a rotation function $\varkappa_m(\xi, \eta_2)$, as the complete increment in the argument of the complex quantity $w_m(\xi_1 + i\eta_1, \zeta_2)$, divided by 2π, as η_1 varies from $-\infty$ to $+\infty$. By virtue of property III of the Mellin functions, the rotation function can take on only integer values. We define a zero-rotation set Z_m of Mellin function $W_m(s)$:

$$Z_m = \{\sigma = R\xi \in \Omega_m \| \varkappa_m(\xi, \eta_2) = 0 \quad \text{for all} \quad \eta_2\}.$$

In other words, Z_m is the set of all $\sigma \in \Omega_m$ such that the total increment in the argument of the complex quantity $W_m(\sigma + i\omega)$ as ω moves along any straight line parallel to the vector h is zero. It can be shown that Z_m is a convex region. On the basis of the results derived in Ref. 13, we find a condition for the stability of power-law spectra:

The spectrum (14) is stable against small perturbations of the form Y_m if and only if there exists a zero-rotation region Z_m and either this region contains the null point, i.e.,

$$\varkappa_m(0, \eta_2) = 0, \quad \text{for all} \quad \eta_2, \qquad (32)$$

or the null point lies on the boundary of this region [in which case spectrum (14) is neutrally stable].

It can be shown that Z_m is a zero-rotation region if and only if the Mellin function $W_m(s)$ does not vanish in the cylindrical region

$$TZ_m = \{s = \sigma + i\omega \| \sigma \in Z_m,\ \omega \in \mathbf{R}^l\}. \qquad (33)$$

The region Z_m thus does not depend on the vector h. Consequently, the stability of the spectrum (14) is also independent of the vector h (the vector h can be chosen arbitrarily in a test of the stability of this spectrum). Just how the spectrum is established and how its perturbations behave, on the other hand, depend strongly on the vector h. Using Refs. 13, we can write explicit solutions of Eq. (21) and determine the evolution of the perturbations $A(k,t)$.

Working from the stability condition formulated above, we can derive an extremely simple necessary condition for stability. A violation of this condition frequently accompanies an instability of spectrum (14).

Since the kinetic equation is real, we can write[1]

$$W_m(s) = \overset{*}{W}_m(s). \qquad (34)$$

If $W_m(0) > 0$, then the rotation $\varkappa_m(0,0)$ is odd, and spectrum (14) is definitely unstable. *A necessary condition for the stability of spectrum (14) against perturbations of the form Y^m [see (18)] is thus*

$$W_m(0) \leqslant 0. \qquad (35)$$

For perturbations corresponding to the eigenfunction

$Y^0 = \text{const}$ we would always have $W_0(0) = 0$, so we would have to use an expansion of the function $W_0(s)$ in a Taylor series, in which terms of up to third order are retained, in order to derive the necessary condition for stability corresponding to condition (35):

$$W_0(s) = as_x + bs_y + {}^1/_2 Ks_x^2 + {}^1/_2 Ls_y^2 + Ms_x s_y. \quad (36)$$

Here and below, in writing some of the equations we assume the two-dimensional case, (9a), for definiteness. At small values of $|\xi|$ and $|\eta_2|$, the rotation $\varkappa_0(\xi,\eta_2)$ is even if and only if we have Re $W_0(s) < 0$ and Im $W_0(s) = 0$ in a small neighborhood of the point $s = 0$. Using (36), we then find that a necessary condition for the stability of the spectrum (14) against perturbations of the form Y^0 is

$$Kb^2 + La^2 - 2Mab \geqslant 0. \quad (37)$$

If this spectrum is indeed stable, so that a zero-rotation region Z_0 exists, and its boundary passes through the null point, then the tangent to the boundary of this region at the point $\sigma = 0$ is orthogonal to the vector (a, b), and the region Z_0 itself lies with the vector (a,b) on the different sides of this tangent.

The first derivatives of Mellin function $W_0(s)$ at the point $s = 0$ coincide[2)] with the first derivatives of the function $f(\nu)$:

$$a = \left.\frac{\partial W_0}{\partial s_x}\right|_{s=0} = \frac{\partial f}{\partial \nu_x}, \quad b = \left.\frac{\partial W_0}{\partial s_y}\right|_{s=0} = \frac{\partial f}{\partial \nu_y}. \quad (38)$$

If the power spectrum is a Kolmogorov spectrum, then the second derivatives of the function $W_0(s)$ at the point $s = 0$ are (within a factor of 1/2) equal to the corresponding derivatives of the function $f(\nu)$:

$$K = \left.\frac{\partial^2 W_0}{\partial s_x^2}\right|_{s=0} = \frac{1}{2}\frac{\partial^2 f}{\partial \nu_x^2}, \quad L = \left.\frac{\partial^2 W_0}{\partial s_y^2}\right|_{s=0} = \frac{1}{2}\frac{\partial^2 f}{\partial \nu_y^2},$$

$$M = \left.\frac{\partial^2 W_0}{\partial s_x \partial s_y}\right|_{s=0} = \frac{1}{2}\frac{\partial^2 f}{\partial \nu_x \partial \nu_y} \quad (39)$$

[expressions (38) and (39) can be verified by direct calculation; see (25)–(29)].

In the case of Kolmogorov spectra, the property (39) makes it possible to offer a simple geometric interpretation of the condition (37), which is a necessary condition for stability. We denote by $\bar{\nu}$ the exponent of some Kolmogorov spectrum. Equation (15) specifies a curve of the exponents ν of the power-law solutions of Eq. (1) which passes through the point $\bar{\nu}$. According to (38) and (39), Eq. (15) can be written as follows near the point $\bar{\nu}$:

$$f(\bar{\nu}_x + \delta_x, \bar{\nu}_y + \delta_y) = a\delta_x + b\delta_y + K\delta_x^2 + L\delta_y^2 + 2M\delta_x\delta_y = 0.$$

Hence $\delta_x = \delta_x^{(1)} + \delta_x^{(2)}$, where

$$\delta_x^{(1)} = -\frac{b}{a}\delta_y, \quad \delta_x^{(2)} = -\frac{1}{a^3}(Kb^2 + La^2 - 2Mab)\delta_y^2$$

(Fig. 1). Inequality (37) is equivalent to the condition $\delta_x^{(2)} a < 0$. *A necessary condition for the stability of a Kolmogorov spectrum [(10) or (12)] with exponent $\nu = \bar{\nu}$ with respect to perturbations of the form of Y^0 is that the curve defined by Eq. (15) be convex toward positive values of the function f at the point $\bar{\nu}$* (Fig. 1).

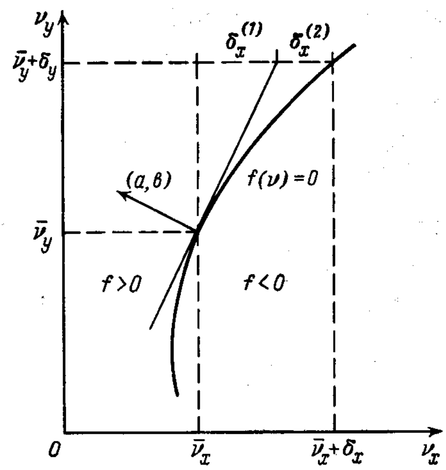

FIG. 1. Diagram used in deriving a geometric necessary condition for the stability of a Kolmogorov spectrum with respect to perturbations of the form of Y^0.

5. INTERVAL INSTABILITY AND EVOLUTIONARY LOCALITY

If there is a constant external agent $[\gamma = \gamma(k)]$, Eq. (17) can generally have steady-state solutions which are described by (18) in which we have

$$\mathscr{A}^m = \mathscr{A}^m(k) = \int_{\sigma_x - i\infty}^{\sigma_x + i\infty} \int_{\sigma_y - i\infty}^{\sigma_y + i\infty} \frac{ds_x}{2\pi i}\frac{ds_y}{2\pi i} G_m(s) k^{-s}, \quad (40)$$

where $G_m(s)$ is a steady-state solution of Eq. (21),

$$G_m(s) = \frac{\Psi_m(s+h)}{W_m(s)}. \quad (41)$$

As in Ref. 13, we assume that $\gamma(k)$ is a finite function [i.e., $\gamma(k) \equiv 0$, where any component of the vector k is close to 0 or ∞]. The function $\Psi_m(s)$ is then an entire function, and the poles of the function (41) are determined by the zeros of Mellin function $W_m(s)$. The existence of a steady-state solution of (40) requires the presence of a cylindrical region of such a nature that the function $G_m(s)$ has no poles in this region or, equivalently, the function $W_m(s)$ has no zeros in it. A steady-state solution of (40) thus exists if and only if there exists a zero-rotation region Z_m [see the assertion just before (33) in Sec. 4], and the parameter σ in (40) must belong to region Z_m. According to the results of Refs. 13, a steady-state solution of (40) of this type is thus established in the system as $t \to \infty$.

To see the asymptotic behavior of solution (40), it is convenient to transform to logarithmic variables:

$$\lambda_x = \ln|k_x|, \quad \lambda_y = \ln|k_y|; \quad \lambda = (\lambda_x, \lambda_y).$$

We can then rewrite (40) as

$$\mathscr{A}^m(\lambda) = \int_{\sigma_x - i\infty}^{\sigma_x + i\infty} \int_{\sigma_y - i\infty}^{\sigma_y + i\infty} \frac{ds_x}{2\pi i}\frac{ds_y}{2\pi i} G_m(s) e^{-(s,\lambda)}, \quad (42)$$

where $(s,\lambda) = (s_x\lambda_x, s_y\lambda_y)\sigma \in Z_m$. As λ goes off to infinity along any direction specified by the vector v, i.e., $\lambda = v\tau, \tau \to +\infty$, we have

$$\mathscr{A}^m(\lambda) = O(e^{-(\sigma,v)\tau}), \quad \tau \to +\infty \quad (\lambda = v\tau), \quad \sigma \in Z_m. \quad (43)$$

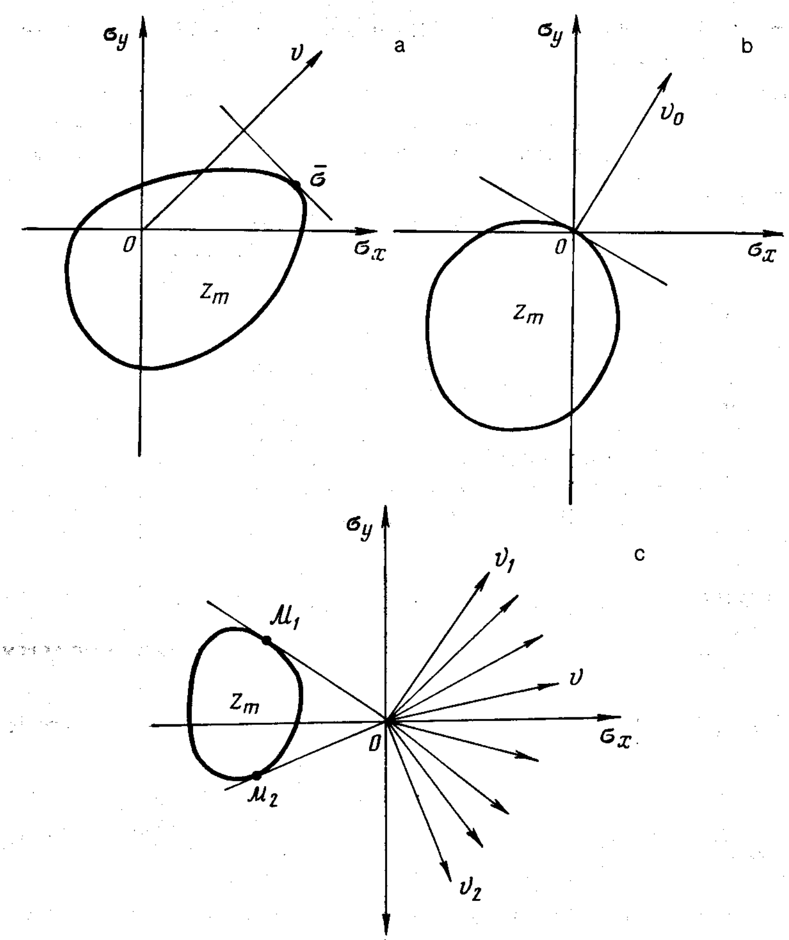

FIG. 2. Zero-rotation region Z_m. a: The point $\bar{\sigma}$ corresponds to best estimate (43) at a fixed v (the tangent to the boundary of region Z_m at the point $\bar{\sigma}$ is orthogonal to the vector v). This diagram corresponds to the case of stability. b: Case of neutral stability. Here v_0 is the unique direciton in λ space for which a relative perturbation does not decrease (the vector v_0 is orthogonal to the tangent to the boundary of region Z_m at the point $\sigma = 0$). c: Case of interval instability. The (v_1, v_2) cone of directions of v in λ space, along which the relative perturbation increases, is shown (the vectors v_1 and v_2 are orthogonal to the tangents $\mathcal{O}\mathcal{M}_1$ and $\mathcal{O}\mathcal{M}_2$, respectively, to the boundary of region Z_m).

The assertion (43) becomes progressively stronger as (σ,v) increases. Figure 2a shows how to find the point σ which determines the best estimate of (43) at a fixed value of the vector v. If the region Z_m contains the point $\sigma = 0$, the quantity $\mathscr{A}^m(\lambda)$ vanishes exponentially along all directions. This effect corresponds to stability of spectrum (14) (Sec. 4). If the point $\sigma = 0$ lies on the boundary of the region Z_m, then $\mathscr{A}^m(\lambda)$ tends toward zero along all directions expect the one direction v_0 (Fig. 2b), along which the quantity $|\mathscr{A}^m(\lambda)|$ approaches a nonzero constant (this case corresponds to a neutral stability). The latter situation always holds for perturbations of the form of Y^0, and in this case we have $v_0 = (\alpha,b)$ (Sec. 4). If the point $\sigma = 0$ does not belong to the region Z_m, the quantity $\mathscr{A}^m(\lambda)$ approaches zero only along directions outside a certain cone determined by the region Z_m (Fig. 2c). In this case, the spectrum (14) is unstable (in accordance with the stability condition).

As a result of this instability, a constant external source $[\gamma = \gamma(k)]$ gives rise to a steady-state spectrum $n(k)$ whose relative deviation from the power-law spectrum (14) increases with distance from the source in a certain region of k space (corresponding to this cone in λ space). Elsewhere in k space there is an approximately Kolmogorov spectrum. Such an instability is called an *interval* instability[13] and is of an *asymptotic nature*: Perturbations of the spectrum (14) are quite large if the inertial interval is large.

When there is no zero-rotation region Z_m at all, perturbations of the spectrum (14) of the form of Y_m can behave in two ways, according to Refs. 13 [see (18)].

1) Evolutionary nonlocalness. The evolution of such perturbations is not determined exclusively by the interactions between waves of approximately the same scale. Their evolution gives rise to a spectrum which is of such a nature that the collision integral is dominated by the extremities of the inertial interval (see also Sec. 1).

2) Absolute instability. Perturbations grow exponentially over the entire inertial interval, and a steady state is not approached ("secondary turbulence").

Only in the case of isotropic media has it been found possible to make a clear distinction between these two situations.[13] A sufficient condition for the evolutionary nonlocalness of the spectrum (14) is that there be no region of analyticity of $T\Omega_m$ [i.e., no region in which the integral which determines the Mellin function W_m converges; see (25) and (27)]. A sufficient condition for evolutionary localness is the existence of a zero-rotation region Z_m.

For power-law thermodynamic spectra in the case in which there is a region in which the Mellin function $W_m(s)$ is analytic, there always exists a zero-rotation region Z_m, which is symmetric with respect to the point $\sigma_0 = (\nu - \mu)/2$. This conclusion follows from the circumstance that Mellin function $W_m(s)$ has the following properties in this case:

$$W_m(\sigma_0+s)=W_m(\sigma_0-s),\ W_m(\sigma_0+i\omega)<0$$

[see (25) and (27)]. An absolute instability is thus not possible for thermodynamic spectra, while evolutionary nonlocalness with respect to perturbations of the form of Y^m can occur only if there is no region in which Mellin function W_m is analytic.

We conclude this section by pointing out a property of the steady-state nonlocalness which holds only for Kolmogorov spectra: In the case of a Kolmogorov spectrum, the collision integral cannot diverge only at the origin or only at infinity.

6. SPECTRA WHICH ARE SUMS OF POWER FUNCTIONS

We first consider steady-solutions of the kinetic equation (1) near the power spectrum (14), i.e., solutions $n(k)$ of the form (16), where the function $A(k)$ is a steady-state solution of Eq. (17) with $\gamma = 0$. The form of this solution $A(k)$ is determined, as in the case of isotropic media, by the zeros of the Mellin functions:

$$A(k) = \sum_m Y^m \int_{W_m(q)=0} B_m(q)\, k^{-q}\, dq, \qquad (44)$$

where $B_m(q)$ is an arbitrary function, and the integration is over the set of zeros of Mellin function $W_m(s)$. While these zeros are isolated points in the isotropic case, in the anisotropic case they form surfaces in a complex space of dimension l [see (9)]. Steady-state solutions of the type in (44) are not limiting solutions for the solutions $A(k,t)$ of Eq. (17) as $t \to \infty$, but they may determine the asymptotic behavior of a solution $A(k,t)$ as $t \to \infty$ and $\lambda \to \infty$.

Let us assume that spectrum (14) is a Kolmogorov or thermodynamic spectrum. Kats and Kontorovich[16] have shown, for the case of isotropic media, that among the power solutions (44) there are some which are universal (i.e., which do not depend on the particular form of the matrix element),

$$A(k) = \text{const}\, k^{-p} Y^m, \qquad (45)$$

and which specify so-called drift corrections to Kolmogorov and thermodynamic spectra (with the help of a correction of this type, one could, for example, superpose a small momentum flux on a Kolmogorov spectrum with an energy flux). Corresponding corrections exist for the case of anisotropic media. Indeed, as is easily seen from (25) and (27), if the power spectrum is a Kolmogorov or thermodynamic spectrum then the Mellin functions have universal zeros $s = p\,[W_m(s)|_{s=p} = 0]$. These universal zeros are listed in Table I for various values of the power ν and the index m. They determine universal power-law corrections of the form (45).

We can now show that in the case of anisotropic media the nonlinear kinetic equation (1) can have exact steady-state solutions in the form of sums of several power functions:

$$n(k) = \sum_{i=1}^{I} C_i k^{-\nu_i} \qquad (46)$$

(it is not being assumed here that one of these power functions dominates the others). When (46) is substituted in the collision integral (2), the result is

$$\mathrm{St}[n,n] = \sum_{i=1}^{I} C_i^2 f(\nu_i)\, k^{\Delta - 2\nu_i} + 2 \sum_{\substack{i,j=1,\\ i<j}}^{I} F(\nu_i,\nu_j)\, C_i C_j k^{\Delta-\nu_i-\nu_j}, \qquad (47)$$

where

$$\Delta = 2\beta - \alpha + d,$$
$$F(\nu_i,\nu_j) = F(\nu_j,\nu_i) = {}^1/_2 (\mathrm{St}[k^{-\nu_i}, k^{-\nu_j}] + \mathrm{St}[k^{-\nu_j}, k^{-\nu_i}])\, k^{-(\Delta-\nu_i-\nu_j)},$$
$$F(\nu_i,\nu_i) = f(\nu_i).$$

[Here we are writing the collision integral in the form $\mathrm{St}[n,n]$ to stress its quadratic dependence on the spectrum $n(k)$. To derive the function $F(\nu_i \nu_j)$ we need to replace terms of the form $n_1 n_2$ in the collision integral (2) by $(1/2)\ \frac{1}{2}(k_1^{-\nu_i} k_1^{-\nu_j} + k_2^{-\nu_i}.]$ If the expression (47) is to vanish identically, it is necessary to satisfy $I(I+1)/2$ relations in $(l+1)I - 1$ unknowns (one of the constants C_i is always arbitrary). We thus have $I \leqslant 4$ at $l = 2$ [see (9a) and (9c)] and $I \leqslant 7$ at $l = 3$ (see (9b)]. With $l = 2$ and $I = 3$, for example, the system of these inequalities may be of either of two types:

1) $F(\nu_i,\nu_j) = 0$ $(i,j = 1,2,3)$, with C_1, C_2, C_3 arbitrary;

TABLE I.

Power of spectrum	$m = 0$ (even perturbations)	$m = 1$ (odd perturbations)
$\nu = \alpha$ $(\mu = 3\alpha - 2\beta - 2d)$	$p = \nu - \alpha = 0$ $p = \nu - (1,0) = \alpha - (1,0)$ $p = \alpha - \mu = 2(\beta + d) - 2\alpha$ $p = (1,0) - \mu = 2(\beta + d) - 3\alpha + (1,0)$	$p = \nu - (0,1) = \alpha - (0,1)$ $p = (0,1) - \mu = 2(\beta + d) - 3\alpha + (0,1)$
$\nu = (1,0)$ $(\mu = \alpha - 2\beta - 2d + 2(1,0))$	$p = \nu - \alpha = (0,1) - \alpha$ $p = \nu - (1,0) = 0$ $p = \alpha - \mu = 2(\beta + d) - (2,0)$ $p = (1,0) - \mu = 2(\beta + d) - (1,0) - \alpha$	$p = \nu - (0,1) = (1, -1)$ $p = (0,1) - \mu = 2(\beta + d) - \alpha + (-2,1)$
$\nu = d + \beta$ $(\mu = \alpha)$	$p = \alpha - \mu = 0$ $p = (1,0) - \mu = (1,0) - \alpha$	$p = (0,1) - \mu = (0,1) - \alpha$
$\nu = d + \beta + {}^1/_2[(1,0) - \alpha]$ $(\mu = (1,0))$	$p = \alpha - \mu = \alpha - (1,0)$ $p = (1,0) - \mu = 0$	$p = (0,1) - \mu = (-1,1)$

$$2)\nu_1+\nu_2=2\nu_3, f(\nu_1)=f(\nu_2)=F(\nu_1,\nu_3)=F(\nu_2,\nu_3)=0,\ 2C_1C_2F(\nu_1,\nu_2)+C_3^2 f(\nu_3)=0.$$

For $l\geqslant 3$ the powers ν_i and also the constants C_i may be complex, but they must appear in complex conjugate pairs in (46). [The sum in (46) is positive only in a certain part of k space, which must include the inertial interval.] An even larger family of solutions of Eq. (1) can be found by adding several terms of the type $C_j' k^{-\nu_j'}\operatorname{sign} k_y$ to the sum in (46).

7. DRIFT-WAVE TURBULENCE

We consider a Charney-Hasegawa-Mima equation with scalar and vector nonlinearities:

$$\frac{\partial}{\partial t}(\Delta\Psi-\Psi)-\frac{\partial\Psi}{\partial x}-\mathscr{A}\left(\frac{\partial\Psi}{\partial x}\frac{\partial\Delta\Psi}{\partial y}-\frac{\partial\Psi}{\partial y}\frac{\partial\Delta\Psi}{\partial x}\right)+\mathscr{B}\frac{\partial\Psi^2}{\partial x}=0. \quad (48)$$

This equation describes Rossby-wave turbulence,[6,14,17] drift-wave turbulence in plasmas,[7-9,18] and density-wave turbulence in galactic gaseous disks.[11] In this case the dispersion law is

$$\omega=\frac{k_x}{1+k^2},\quad k=(k_x,k_y); \quad (49)$$

the matrix elements of the scalar and vector nonlinearities are of the form[9,14]

$$V_{\mathscr{B}}=\text{const}\,|k_xk_{1x}k_{2x}|^{1/2}, \quad (50)$$

$$V_{\mathscr{A}}=\text{const}\,|k_xk_{1x}k_{2x}|^{1/2}\left(\frac{k_y}{1+k^2}-\frac{k_{1y}}{1+k_1^2}-\frac{k_{2y}}{1+k_2^2}\right), \quad (51)$$

respectively; the x momentum is also called the "enstrophy." A scale invariance of the form (8) holds for motions which are approximately zonal flows with $|k_y|\gg k_x$, for either large or small values of k [$d=(1,1)$; see (9a)]. Kolmogorov spectra were derived for short waves by Monin and Piterbarg[3)] (Ref. 6); for long waves, they were derived by Mikhaĭlovskiĭ *et al.*[4)] (Refs. 7–9).

We first consider the case of a vector nonlinearity ($\mathscr{B}=0$). For short-wavelength turbulence ($|k|\gg 1, |k_y|\gg k_x$) we have

$$\omega=k_x/k_y^2,\quad \alpha=(1,-2),$$

$$V_{\mathscr{A}}=\text{const}\,|k_xk_{1x}k_{2x}|^{1/2}\left(\frac{1}{k_y}-\frac{1}{k_{1y}}-\frac{1}{k_{2y}}\right),\quad \beta=(3/2,-1).$$

The corresponding Kolmogorov spectra are given in (5). The collision integral in (2) converges for only those power-law spectra (14) (in addition to the thermodynamic spectra) for which the conditions $\nu_y=0$, $7/3<\nu_x<8/3$ hold, including Kolmogorov spectra as in (10) with exponent $\nu=(5/2,0)$. For a Kolmogorov spectrum as in (12) with $\nu=(5/2,1)$, the collision integral diverges at both large and small wave numbers. Regardless of the value of ν, Mellin function $W_m(s)$ has no region of analyticity for either $m=0$ or $m=1$. All the power spectra are thus nonlocal from the evolutionary standpoint against both even and odd perturbations [see (19) and (20)].

For long-wavelength turbulence ($|k|\ll 1, |k_y|\gg k_x$) we have

$$\omega=k_x(1-k_y^2),\quad \alpha=(1,2),$$

$$V_{\mathscr{A}}=\text{const}\,|k_xk_{1x}k_{2x}|^{1/2}(k_y^3-k_{1y}^3-k_{2y}^3),\quad \beta=(3/2,3) \quad (52)$$

[dispersion law (52) can be regarded as scale-invariant because the collision integral contains δ-functions of the wave vectors]. Figure 3a shows regions of ν for which the Mellin functions have a region of analyticity. These regions are shown by the lines, which specify families of the power solutions (14); the exponents of the thermodynamic and Kolmogorov spectra are specified (by the points).

A Kolmogorov spectrum with an energy flux [see (10)]

$$n_k=C_1P^{1/2}k_x^{-5/2}k_y^{-4} \quad (53)$$

is local in the stationary sense but nonlocal in the evolutionary sense, with respect to both even and odd perturbations (there are no regions in which Mellin functions W_0 and W_1 are analytic). Interestingly, while the collision integral con-

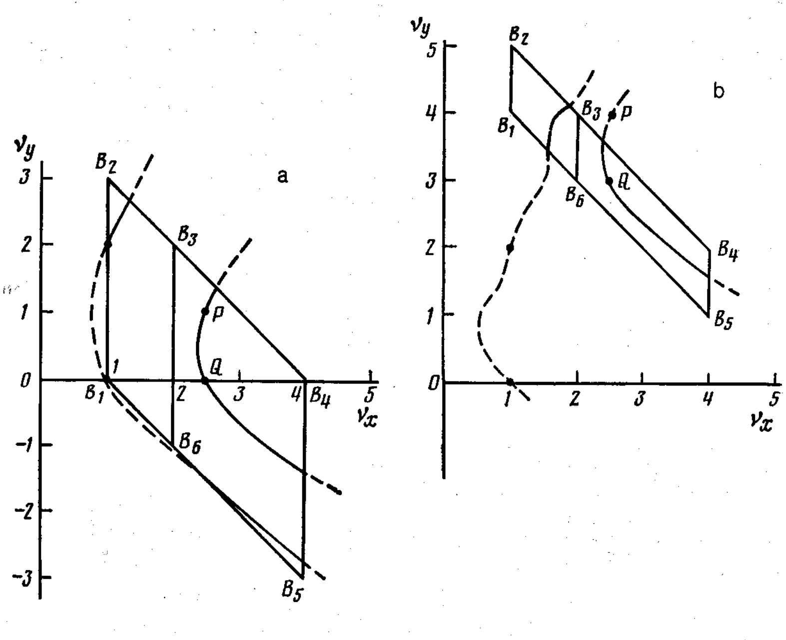

FIG. 3. Power spectra $n^0\propto k^{-\nu}$. The region of values of ν for which Mellin function W_m has an analyticity region is the parallelogram $B_1B_2B_4B_5$ in the case $m=0$ or the parallelogram $B_1B_2B_3B_6$ in the case $m=1$. The curved lines show a family of power-law solutions (14) of kinetic equation (1). These curves were found through a numerical solution of Eq. (28). The dashed lines, which are continuations of these curves outside the region in which the collision integral converges, correspond to solutions of Eq. (28) in which the integral is replaced by an integral sum, used in a numerical calculation. One of these curves passes through power-law thermodynamic spectra [$\nu=(1,2)$ and $\nu=(1,0)$], while the other passes through Kolmogorov spectra, shown by points P and Q. a—Vector nonlinearity; b—scalar nonlinearity.

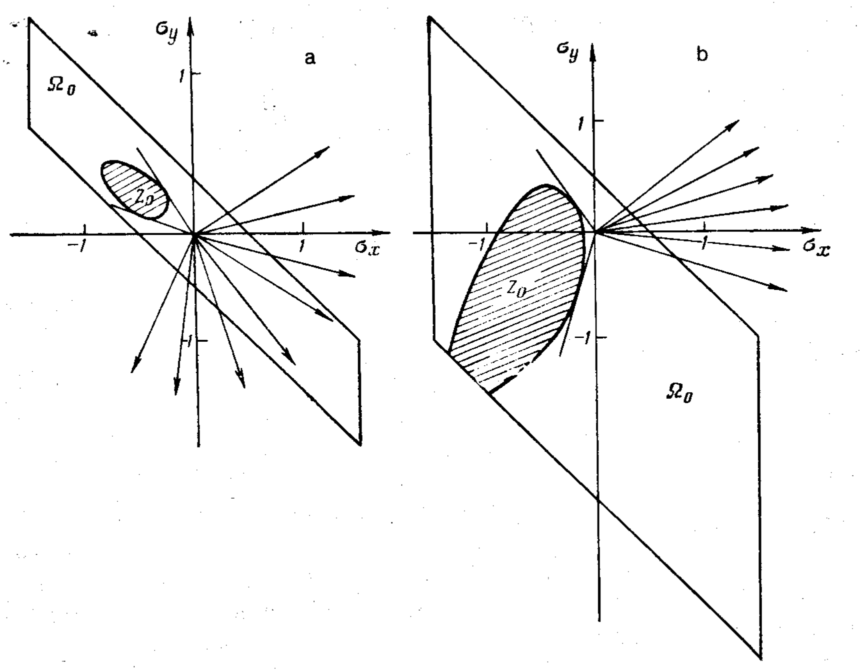

FIG. 4. Base of the region of analyticity of the function W_0 (the parallelogram Ω_0) and the zero-rotation region Z_0 for a Kolmogorov spectrum which is unstable in an interval fashion (and thus local in the evolutionary sense) with respect to even perturbations. Also shown here is the cone of directions in λ space along which a relative perturbation of a Kolmogorov spectrum grows (cf. Fig. 2c). a—Vector nonlinearity, Kolmogorov spectrum (54) with an enstrophy flux Q; b—scalar nonlinearity, Kolmogorov spectrum (55) with an energy flux P.

verges for the spectrum (53), it diverges in the long-wavelength region ($|k|\to 0$) for power-law spectra with exponents which are arbitrarily close to the Kolmogorov exponent $\nu = (5/2,4)$. A Kolmogorov spectrum with an enstrophy flux [see (12)]

$$n_k = C_2 Q^{1/3} k_x^{-5/3} k_y^{-3} \tag{54}$$

is local in the stationary sense but nonlocal in the evolutionary sense with respect to odd perturbations, since there is no region in which the Mellin function $W_1(s)$ is analytic. For both Kolmogorov spectra in (53) and (54), the expression $\mathscr{L}[(k')^{-s} \operatorname{sign} k'_y](k)|$ [$\mathscr{L}$ is a linearized collision integral; see (17)] is an integral which diverges for all s as $k' \to (0,2k_y)$.

It apparently follows that the presence of odd perturbations leads to a very nonlocal interaction between the waves and the zonal flow. It can be shown[19] that the nonlocal interaction tends to make the spectrum symmetric with respect to the k_x axis. For these or other reasons, perturbations of the Kolmogorov spectra which are odd in k_y may be forbidden, and one would study the localness of the Kolmogorov spectrum and its stability with respect to even perturbations. Numerical simulations of drift turbulence are frequently carried out with the help of "even" computation schemes (in which odd spectra are not possible). According to the dis-

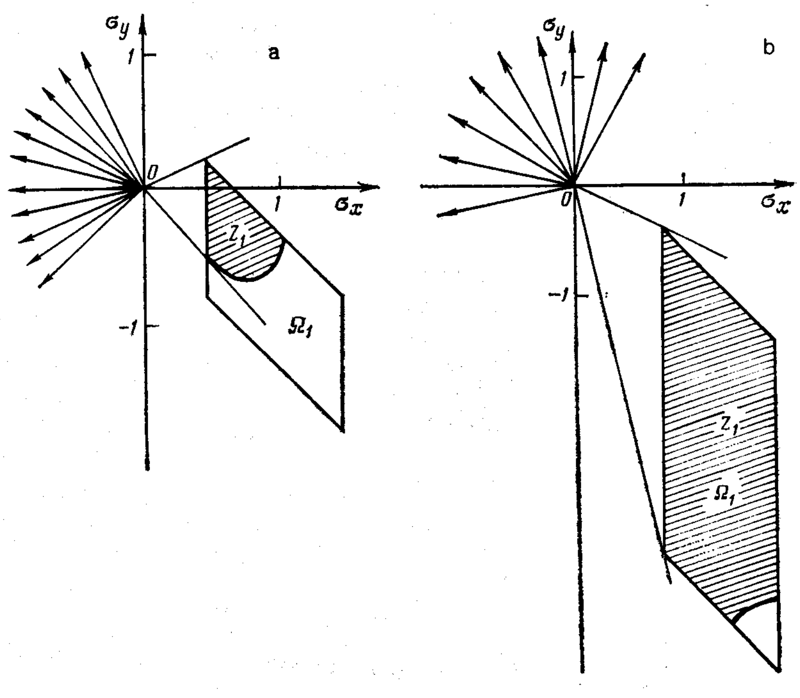

FIG. 5. Case of a power spectrum (14) for which both Mellin functions W_0 and W_1 have regions of analyticity, $T\Omega_0$ and $T\Omega_1$. This spectrum, which is stable with respect to even perturbations, is unstable in an interval fashion with respect to odd perturbations. Also shown here is the cone of directions in λ space along which a relative perturbation of a Kolmogorov spectrum grows (cf. Fig. 2c). a—Vector nonlinearity, $\nu = (3.8, 1.55)$; b—scalar nonlinearity $\nu = (1.2,2.4)$.

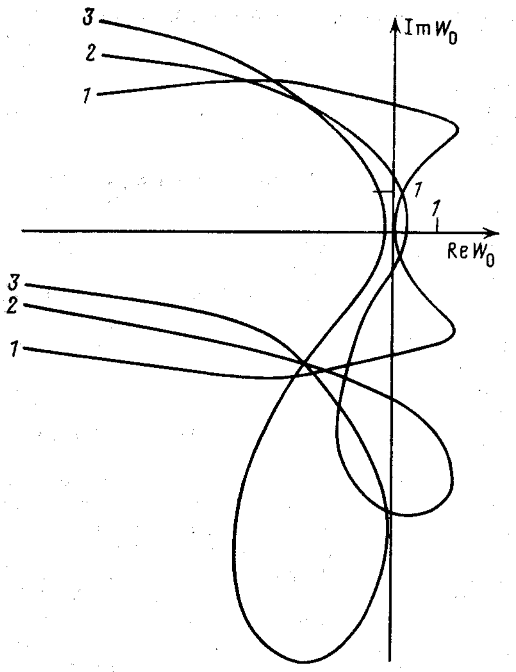

FIG. 6. Hodographs of the Mellin function $\omega_0(i\eta_1,i\eta_2) \times(-\infty<\eta_1<+\infty)$ for Kolmogorov spectrum (55) and various values of η_2 (1—$\eta_2=0$; 2—$\eta_2=0.2$; 3—$\eta_2=0.4$). For certain values of η_2, there is a rotation $\varkappa_0(0,\eta_2)=1$, so Kolmogorov spectrum (55) is unstable with respect to even perturbations.

cussion above, results calculated by such even schemes may be quite different from the results found through the use of complete computation schemes (in which the spectra n_k are not necessarily even).

Because of the geometric necessary condition for stability (see the discussion at the end of Sec. 4), the Kolmogorov spectrum (54) is unstable against even perturbations (Fig. 3a). For this spectrum, a zero-rotation region Z_0 has been found by numerical calculation (Fig. 4a). The instability of the spectrum (54) is therefore an interval instability (Sec. 5). This instability results in the establishment of a steady-state spectrum whose relative deviation from a Kolmogorov spectrum decreases in all directions (in λ space; Sec. 5) outside the direction cone shown in Fig. 4a.

It is also worthwhile to study the stability of the power spectra (14) for which both Mellin functions W_0 and W_1 have a region of analyticity (Fig. 3a). A study has been made of one such spectrum, with the exponent $\nu=(1.55, 3.8)$. It turned out to be stable with respect to even perturbations (there exists a region Z_0, and the point $\sigma=0$ lies at the boundary of this region) and to be unstable in an interval fashion with respect to odd perturbations (Fig. 5a). In this case the region of analyticity $T\Omega_1$ of Mellin function $W_1(s)$ does not contain the point $s=0$.

We now consider the case of a scalar nonlinearity [$\mathscr{A}=0$ in Eq. (48)]. Since a vector nonlinearity outweighs a scalar nonlinearity in the short-wavelength limit ($|k|\gg 1$) in most physical situations, we will consider only the long-wavelength turbulence ($|k|\ll 1$). In this case the dispersion law is as in (52), and the matrix element is given by (50). Its degree of homogeneity is $\beta=(3/2,0)$. The powers of power-law solutions (14) are shown in Fig. 3b. Both Kolmogorov spectra [see (10) and (12)],

$$n_k=C_1P^{1/2}k_x^{-5/2}k_y^{-1}, \tag{55}$$

$$n_k=C_2Q^{1/2}k_x^{-5/2} \tag{56}$$

are local in the stationary sense but nonlocal in the evolutionary sense with respect to odd perturbations. As in the case of a vector nonlinearity, the presence of odd perturbations results in a nonlocal interaction of the waves with a zonal flow.

Because of the geometric necessary condition for stability, both Kolmogorov spectra (55) and (56) are unstable with respect to even perturbations (Fig. 3b). We have observed an instability of these spectra, and also of the spectrum (54), through a direct application of the stability condition [this approach is illustrated for the case of the spectrum (55) in Fig. 6].

The Kolmogorov spectrum (55) is interval-unstable against even perturbations (Fig. 4b) and is thus local in the evolutionary sense with respect to these perturbations. The Kolmogorov spectrum (56) is nonlocal in the evolutionary sense or absolutely unstable (Sec. 5) against even perturbations, since there is no zero-rotation region Z_0 for it.

The stability of the spectrum with exponent $\nu=(1.20; 2.40)$, for which both Mellin functions W_0 and W_1 have regions of analyticity, was also studied (Fig. 3b). Like the corresponding spectrum in the case of a vector nonlinearity, this spectrum is stable against even perturbations and interval-unstable against odd perturbations (Fig. 5b).

In all these cases, there is only a single local Kats-Kontorovich correction (the zero of the Mellin function which corresponds to this correction falls in its region of analyticity). This correction occurs in the case of a scalar nonlinearity for the Kolmogorov spectrum (55) and is given by the value $p=(0,2)$ Table I). It describes a small enstrophy flux against the background of a Kolmogorov spectrum with an energy flux.

The general results derived in this study can also be used to analyze weak turbulence in many other situations, e.g., those mentioned in Sec. 1. The results can easily be extended to nondecay situations and to various other kinetic equations in anisotropic media.

This analysis of the stability and locality of Kolmogorov spectra makes it possible to draw some important conclusions about the structure of the turbulence and the nature of the transfer of integrals of motion from scale to scale.

We are deeply indebted to V. E. Zakharov for valuable ideas, for useful discussions, and for stimulating interest in this study.

APPENDIX

In order to calculate the Mellin functions we need to transform the integral which determines them from an integral over a resonant manifold into an integral over a simple region. We will do this in the case of two-dimensional media, (9a), (19), (20), (25), with $\alpha_x=1$ (this value holds in most physically interesting situations[6-9,11]). Since the integral (25) is symmetric with respect to k_1 and k_2, we assume $|k_{2y}|>|k_{1y}|$, inserting a factor of 2 in front of the integral. We introduce $q_i=|k_{iy}|$ ($i=1,2$). We now change variables $(k_{1x},k_{2x})\to(\theta_1,\theta_2)$:

$$\theta_1=k_{1x}+k_{2x},\quad \theta_2=k_{1x}q_1^{\alpha_y}+k_{2x}q_2^{\alpha_y},$$

$$\frac{\partial(\theta_1,\theta_2)}{\partial(k_{1x},k_{2x})}=|q_2^{\alpha_\nu}-q_1^{\alpha_\nu}|\neq 0.$$

By virtue of the three δ-functions $\delta(1-\theta_1)$, $\delta(1-\theta_2)$, and $\delta(1-k_{1y}-k_{2y})$ in the integral, we can integrate over the variables θ_1, θ_2, and k_{2y}. Using $k_{1x},k_{2x}>0$, we then find the single integral

$$W_m(s)=2\int_0^1 2\pi|V(1,k_1,k_2)|^2 k_1^{-\nu}k_2^{-\nu}\frac{1}{|q_2^{\alpha_\nu}-q_1^{\alpha_\nu}|}$$

$$\times[(1+k_1^{-s}(-1)^m+k_2^{-s})(1-k_1^{\nu}-k_2^{\nu})$$

$$-(1-k_1^{\nu-s}(-1)^m+k_2^{\nu-s})]$$

$$\times[1-k_1^{\mu+s}(-1)^m-k_2^{\mu+s}]dq_1,$$

where all the variables are functions of q_1:

$$k_{1y}=-q_1,\ k_{2y}=q_2=1+q_1,$$
$$k_{1x}=\frac{q_2^{\alpha_\nu}-1}{q_2^{\alpha_\nu}-q_1^{\alpha_\nu}},\quad k_{2x}=\frac{q_1^{\alpha_\nu}-1}{q_1^{\alpha_\nu}-q_2^{\alpha_\nu}}.$$

This expression makes it possible, in particular, to determine the stationary localness of the power-law spectra and to find the regions of analyticity of the Mellin functions.

[1)] In case (9c), under condition (34), the medium must also be symmetric under mirror reflections in the $k_x=0$ plane.

[2)] Pointed out by G. E. Fal'kovich.

[3)] Expressions for the weak-turbulence Kolmogorov spectra of Rossby waves were derived previously on the basis of dimensional considerations by Pelinovskiĭ and Sazontov.[19]

[4)] Mikhaĭlovskiĭ *et al.*[7] also discuss the localness of these Kolmogorov spectra. In their paper, "localness" is understood as being nearly (but not exactly) the same as a stationary localness.

[1] V. E. Zakharov, Zh. Prikl. Mekh. Tekh. Fiz. No. 4, 35 (1965).

[2] V. E. Zakharov, in *Basic Plasma Physics* (eds. A. A. Galeev and R. Sudan), Vol. 2, North-Holland, Amsterdam, 1984 (Russ. transl., Energoatomizdat, Moscow, 1984, p. 48).

[3] V. E. Zakharov, Izv. Vyssh. Uchebn. Zaved., Radiofiz. **17**, 431 (1974).

[4] E. A. Kuznetsov, Zh. Eksp. Teor. Fiz. **62**, 584 (1972) [Sov. Phys. JETP **35**, 310 (1972)].

[5] A. A. Kanashov, Preprint No. 291, Institute of Automation and Electrometry, Siberian Branch of the Academy of Sciences of the USSR, Novosibirsk, 1982.

[6] A. S. Monin and L. I. Piterbarg, Dokl. Akad. Nauk SSSR **295**, 816 (1987) [Sov. Phys. Dokl. **32**, 622 (1987)].

[7] A. B. Mikhaĭlovskiĭ, S. V. Novakovskiĭ, V. P. Lakhin *et al.*, Zh. Eksp. Teor. Fiz. **94**(7), 159 (1988) [Sov. Phys. JETP **67**, 1386 (1988)].

[8] S. V. Novakovskii, A. V. Mikhailovskii, and O. G. Onishchenko, Phys. Lett. A **132**, 33 (1988).

[9] A. V. Mikhailovskii, S. V. Nazarenko, S. V. Novakovskii *et al.*, Phys. Lett. A **133**, 407 (1988).

[10] A. B. Mikhaĭlovskiĭ, E. A. Novakovskiĭ, V. P. Lakhin *et al.*, Zh. Eksp. Teor. Fiz. **95**, 1579 (1989) [Sov. Phys. JETP **68**, 924 (1989)].

[11] V. V. Dolotin and A. M. Fridman, in *Dynamics of Astrophysical Discs* (ed. Z. Seilwood), Cambridge University Press, Cambridge, 1989, p. 71.

[12] A. N. Kolmogorov, Dokl. Akad. Nauk SSSR **30**, 299 (1941); A. M. Obukhov, Izv. Akad. Nauk SSSR Geograf. geofiz. **5** (1941).

[13] A. M. Balk and V. E. Zakharov, Dokl. Akad. Nauk SSSR **299**, 1112 (1988) [Sov. Phys. Dokl. **33**, 270 (1988)]; A. M. Balk and V. E. Zakharov, in *Plasma Theory and Nonlinear and Turbulent Processes in Physics. Proceedings of International Workshop* (Kiev, April 1987), World Scientific, Singapore, 1988, p. 359; A. M. Balk and V. E. Zakharov, in *Integrability and Kinetic Equations for Solitons* (eds. V. G. Bar'yakhtar, V. E. Zakharov, and V. M. Chernousenko), Nauk. Dumka, Kiev, 1990 (in press).

[14] V. E. Zakharov and L. I. Piterbarg, Dokl. Akad. Nauk SSSR **295**, 86 (1987) [Sov. Phys. Dokl. **32**, 560 (1987)].

[15] V. S. Vladimirov, *Methods of the Theory of Functions of Many Complex Variables*, Nauka, Moscow, 1964, p. 411.

[16] A. V. Kats and V. M. Kontorovich, Zh. Eksp. Teor. Fiz. **64**, 153 (1973) [Sov. Phys. JETP **37**, 80 (1973)].

[17] M. S. Longuet-Higgins and A. E. Hill, *Nonlinear Theory of Wave Propagation* (Russ. transl., ed. G. I. Barenblat, Mir, Moscow, 1970, p. 161).

[18] A. Hasegawa and K. Mima, Phys. Fluids **21**, 87 (1978).

[19] E. N. Pelinovskiĭ, Okeanologiya **18**, 192 (1978); A. G. Sazontov, *Fine Structure and Synoptic Variability of Seas: Proceedings of All-Union Seminar-Symposium*, Tallin, 1980, p. 147.

Translated by D. Parsons