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Aspects of Two-Mode Probability Density Function in Weak Wave Turbulence

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We derive the time evolution of the two-mode amplitude probability density function. Using this equation, we derive conditions for the existence of a zero flux steady-state solution. We also derive the equation for a vortex solution and show that the product of two one-mode steady-state solutions can be a two-mode steady-state solution only when an extra condition is satisfied. With this extra condition assumed, we plot the flux of probability vector on two mode's plane. It is shown that this flux lines circulate around the center (n_a, n_b), which are the mean values of the two mode's amplitude square.

KEYWORDS: three wave system, two-mode probability density function, flux of probability, zero flux solution, vortex solution
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1. Introduction

Since Zakharov has introduced Kolmogorv–Zakharov (KZ) spectra[1–3] to the wave turbulence community, many descending theories have been developed afterwards.[4–16] Wide applications of WT theory based on the KZ spectra have been found in various branches including oceanography, plasma physics, astrophysics and cosmology since then. Going far beyond the hydrodynamic description based on Navier–Stokes equation, WT theory based on the KZ spectra succeeded in explaining many aspects of turbulence phenomena. For example, the stochastic wave fields were found to be more like Kolmogorov turbulence determined by the rate at which energy cascades through scales rather than by a thermodynamic temperature describing the energy equipartition in the scale space.

However, the spectra theory, mainly utilizing the kinetic equation description, has its own limit on the qualitative analysis of turbulence. One needs to deal with the probability density function (PDF) to overcome such problems. Works along this direction, motivated by the arising interests on the intermittency phenomena, were restricted on the consideration of the nonlinear interaction potential as a coordinate function at the beginning stage.[17,18] Recently, more general description of PDF dealing with three- and four-wave Hamiltonian systems has been developed.[9–13,15,16]

Our work, in this paper, deals with a three-wave Hamiltonian system. Starting from the time evolution equation of the full-mode amplitude PDF, we derive the time evolution of two-mode amplitude PDF $P^{(2)}$ with two modes s_a and s_b as variables. The resulting time evolution equation is of the form of continuity equation as it should be. This determines the flux vector field of a given two-mode amplitude PDF. Steady state solutions of two-mode amplitude PDF can be obtained by taking this flux to be divergence-free. We investigate two types of solutions, one with zero flux and the other with non-vanishing but divergence-free flux, so called vortex solution.

It is shown that a zero flux solution is possible only when interaction coefficients satisfy a set of conditions, indicating that a generic model may not allow zero flux steady state solution for the two-mode amplitude PDF. However, a vortex solution may exist in any model. It turns out that the equation for a vortex solution is difficult to solve. So, we try with a product of two one-mode PDF as a steady state solution and obtain a condition for this PDF to be a divergence free solution. This condition is much less strict than that for zero flux solution. With this solution, we depict the shape of vector lines of the flux of probability and show that the flux lines are circulation around a point coordinated by the expectation values of the two mode variables s_a and s_b as was conjectured in ref. 11.

In the following two sections, we give reviews of the derivation of the time evolution equation of the full-mode amplitude PDF and some properties of one-mode amplitude PDF. In §4, we consider the two-mode amplitude PDF and obtain our main results. In §5, we give some concluding remarks.

2. Derivation of the Multi-Mode Probability Evolution Equation in Wave Turbulence

The physical system we deal with in this paper is a stochastic ensemble of dispersive waves interacting nonlinearly and weakly. The dynamics of a dispersive wave is described by a complex field, $c(\mathbf{x}, t)$ which represents wave actions in a d-dimensional periodic box with sides L. The time evolution of the wave is given by

$$i\dot{c}_l = \frac{\partial \mathcal{H}}{\partial \bar{c}_l}, \tag{1}$$

where c_l represents the l-th Fourier mode of $c(\mathbf{x}, t)$, and Hamiltonian $\mathcal{H}$ is the sum of the free term ($\mathcal{H}_2$) and perturbative terms ($\mathcal{H}_3, \mathcal{H}_4, \ldots$),

$$\mathcal{H} = \mathcal{H}_2 + \mathcal{H}_3 + \mathcal{H}_4 + \cdots. \tag{2}$$

Each term $\mathcal{H}_j$ can be represented by the interaction coefficients T and j amplitudes c_l,

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$$\mathcal{H}_j = \sum_{q_1,q_2,q_3,\ldots,q_n,p_1,p_2,\ldots,p_m} (T^{q_1,q_2,\ldots,q_n}_{p_1,p_2,\ldots,p_m} \bar{c}_{q_1}\bar{c}_{q_2}\cdots\bar{c}_{q_n}c_{p_1}c_{p_2}\cdots c_{p_m} + \text{c.c.}), \quad n+m=j$$

where $q_1, q_2, q_3, \ldots, q_n$ and $p_1, p_2, \ldots, p_m$ are wavevectors on a d-dimensional Fourier space lattice. The coefficients $T^{q_1,q_2,\ldots,q_n}_{p_1,p_2,\ldots,p_m}$ represent the wave-wave interactions where n waves collide to create m waves or $n \to m$.

The free Hamiltonian $\mathcal{H}_2$ is diagonalized. The triple coupling Hamiltonian $\mathcal{H}_3$ and the quadruple coupling Hamiltionian $\mathcal{H}_4$ can be written as

$$\begin{aligned}
\mathcal{H}_2 &= \sum_n \omega_n |c_n|^2, \\
\mathcal{H}_3 &= \epsilon \sum_{l,m,n} V^l_{mn}\bar{c}_l c_m c_n \delta^l_{m+n} + \text{c.c.} \\
\mathcal{H}_4 &= \epsilon^2 \sum_{m,n,\mu,\nu=1}^{\infty} W^{lm}_{\mu\nu}\bar{c}_l\bar{c}_m c_\mu c_\nu,
\end{aligned} \tag{3}$$

where $\epsilon \ll 1$ is a smallness parameter of nonlinearity.

If $\mathcal{H}_3 \neq 0$, we usually neglect $\mathcal{H}_4$ and the followings because they are of the higher order in ϵ. However if all the three-wave processes in $\mathcal{H}_3$ happen to be nonresonant, a suitable redefinition of field variables c_k removes $\mathcal{H}_3$ and the leading order perturbative Hamiltonian becomes $\mathcal{H}_4$.[3] In this paper, we restrict ourselves to the case where $\mathcal{H}_4$ and the higher order Hamiltonians can be neglected. The waves corresponding to this case include water surface capillary wave, internal waves in the ocean and Rossby waves. Now our Hamiltonian is simplified as $\mathcal{H} = \mathcal{H}_2 + \mathcal{H}_3$ and eq. (1) becomes

$$\begin{aligned}
\mathcal{H} &= \sum_n \omega_n |c_n|^2 + \epsilon \sum_{l,m,n} (V^l_{mn}\bar{c}_l c_m c_n \delta^l_{m+n} + \text{c.c.}), \\
\mathrm{i}\dot{a}_l &= \epsilon \sum_{m,n} (V^l_{mn} a_m a_n \mathrm{e}^{\mathrm{i}\omega^l_{mn}t}\delta^l_{m+n} + 2\bar{V}^m_{ln}\bar{a}_n a_m \mathrm{e}^{-\mathrm{i}\omega^m_{ln}t}\delta^m_{l+n}),
\end{aligned} \tag{4}$$

where $a_j = c_j \mathrm{e}^{\mathrm{i}\omega_j t}$, $\omega^l_{mn} \equiv \omega_{k_l} - \omega_{k_m} - \omega_{k_m}$, and $\omega_l = \omega_{k_l}$ is the dispersion relation.

Let us write the complex mode a_l as $a_l = A_l\psi_l$ where A_l is a real positive amplitude and ψ_l is a phase factor which takes values on $\mathcal{S}^1$, a unit circle centered at zero in the complex plane. Let us define the PDF $\mathcal{P}\{s, \xi\}$ as the probability density function for the wave intensities A_l^2 to be s_l and for the phase factors ψ_l to be ξ_l for all l. With this PDF, we can define the expectation of f as follows:

$$\langle f\{A^2, \psi\}\rangle = \left(\prod_l \int_0^\infty \mathrm{d}s_l \oint_{\mathcal{S}^1} \frac{|\mathrm{d}\xi_l|}{2\pi}\right)\mathcal{P}\{s, \xi\} f\{s, \xi\}. \tag{5}$$

We also introduce the generating functional $Z\{\lambda, \mu\}$ as follows:

$$Z\{\lambda, \mu\} = \left\langle \prod_l \mathrm{e}^{\lambda_l A_l^2}\psi_l^{\mu_l}\right\rangle. \tag{6}$$

Here $\lambda_l \in \mathbb{R}$ and $\mu_l \in \mathcal{Z}$. We can show that[11,12]

$$\mathcal{P}\{s, \xi\} = \sum_{\mu_l \in \mathcal{Z}} \left\langle \prod_l \delta(s_l - A_l^2)\psi_l^{\mu_l}\xi_l^{-\mu_l}\right\rangle. \tag{7}$$

The relation between the generating functional and PDF can be shown to be

$$\mathcal{P}\{s, \xi\} = \hat{\mathcal{L}}_\lambda^{-1} \sum_{\{\mu\}} \left(Z\{\lambda, \mu\}\prod_l \xi_l^{-\mu_l}\right), \tag{8}$$

where $\hat{\mathcal{L}}_\lambda^{-1}$ stands for the inverse Laplace transform with respect to all λ_l and $\{\mu\}$ are the angular harmonics indices.

To filter out fast oscillations, we will seek for the solution at time T such that $2\pi/\omega \ll T \ll \tau_{nl}$. Here τ_{nl} is the characteristic time of nonlinear evolution, which is $\sim 1/\omega\epsilon^2$ for the three-wave systems. Solution at this time $t = T$ can be expressed as series expansion in small nonlinearity parameter ϵ,

$$a_l(T) = a_l^{(0)} + \epsilon a_l^{(1)} + \epsilon^2 a_l^{(2)}. \tag{9}$$

Substituting this in eq. (4), we get

$$\begin{aligned}
a_l^{(0)}(T) &= a_l(0) \equiv a_l, \\
a_l^{(1)}(T) &= -\mathrm{i}\sum_{m,n}[V^l_{mn}a_m a_n \Delta(\omega^l_{mn})\delta^l_{m+n} \\
&\quad + 2\bar{V}^m_{ln}a_m\bar{a}_n\bar{\Delta}(\omega^m_{ln})\delta^m_{l+n}],
\end{aligned} \tag{10}$$

and

$$\begin{aligned}
a_l^{(2)}(T) = \sum_{m,n,\mu,\nu} &[2V^l_{mn}(-V^m_{\mu\nu}a_n a_\mu a_\nu E[\omega^l_{n\mu\nu}, \omega^l_{mn}]\delta^m_{\mu+\nu} \\
&- 2\bar{V}^\mu_{m\nu}a_n a_\mu \bar{a}_\nu \bar{E}[\omega^{l\nu}_{n\mu}, \omega^l_{mn}]\delta^\mu_{m+\nu})\delta^l_{m+n} \\
&+ 2\bar{V}^m_{ln}(-V^m_{\mu\nu}\bar{a}_n a_\mu a_\nu E[\omega^{ln}_{\mu\nu}, -\omega^m_{ln}]\delta^m_{\mu+\nu} \\
&- 2\bar{V}^\mu_{m\nu}\bar{a}_n a_\mu \bar{a}_\nu E[-\omega^\mu_{n\nu l}, -\omega^m_{ln}]\delta^\mu_{m+\nu})\delta^m_{l+n} \\
&+ 2\bar{V}^m_{ln}(\bar{V}^n_{\mu\nu}a_m\bar{a}_\mu\bar{a}_\nu\delta^n_{\mu+\nu}E[-\omega^m_{l\nu\mu}, -\omega^m_{ln}] \\
&+ 2V^\mu_{n\nu}a_m\bar{a}_\mu a_\nu E[\omega^{\mu l}_{\nu m}, -\omega^m_{ln}]\delta^\mu_{n+\nu})\delta^m_{l+n}].
\end{aligned} \tag{11}$$

where $\Delta(x) = \int_0^T \mathrm{e}^{\mathrm{i}xt}\,\mathrm{d}t$ and $E(x, y) = \int_0^T \Delta(x-y)\mathrm{e}^{\mathrm{i}yt}\,\mathrm{d}t$.

We now expand $Z(T)$ in the power series of ϵ using $A_l^2(T) = a_l(T)\bar{a}_l(T)$ and $\psi_l(T) = [a_l(T)/\bar{a}_l(T)]^{1/2}$. Taking the limit $T \to \infty$ with the condition $2\pi/\omega \ll T \ll \tau_{nl}$, we obtain

$$Z(T) = Z(0) + T(\text{power series in } \epsilon).$$

It turns out that the term coupled to T begins with ϵ^2 order. Note that if $\epsilon = 0$, then $Z(T) = Z(0)$ as it should be. Taking only the leading order term, we finally get

$$\begin{aligned}
\dot{Z} = 4\pi\epsilon^2 \sum_{j,m,n} \Bigg\{ &\left(\lambda_j + \lambda_j^2 \frac{\partial}{\partial\lambda_j}\right)[|V^j_{mn}|^2\delta(\omega^j_{mn})\delta^j_{m+n} \\
&+ 2|V^m_{jn}|^2\delta(\omega^m_{jn})\delta^m_{j+n}]\frac{\partial^2 Z}{\partial\lambda_m\partial\lambda_n} \\
&+ 2\lambda_j\Bigg[-|V^j_{mn}|^2\delta(\omega^j_{mn})\delta^j_{m+n}\frac{\partial}{\partial\lambda_n} \\
&+ |V^m_{jn}|^2\delta(\omega^m_{jn})\delta^m_{j+n}\left(\frac{\partial}{\partial\lambda_m} - \frac{\partial}{\partial\lambda_n}\right)\Bigg]\frac{\partial Z}{\partial\lambda_j} \\
&+ 2\lambda_j\lambda_m[-2|V^j_{mn}|^2\delta^j_{m+n}\delta(\omega^j_{mn}) \\
&+ |V^n_{jm}|^2\delta^n_{j+m}\delta(\omega^n_{jm})]\frac{\partial^3 Z}{\partial\lambda_j\partial\lambda_n\partial\lambda_m}\Bigg\}.
\end{aligned} \tag{12}$$

Note that the derivative $\dot{Z}$ is not taken with respect to the usual time t, but T which is big enough to produce the frequency delta functions owing to the condition $2\pi/\omega \ll T \ll \tau_{nl}$. Here we notice that $\dot{Z}$ does not depend on μ at all. Therefore, up to ϵ^2 order approximation we can assume the random phase approximation and eq. (8) becomes

$$\mathcal{P}\{s\} = \hat{\mathcal{L}}_\lambda^{-1}(Z\{\lambda\}). \tag{13}$$

Applying the inverse Laplace transformation to eq. (12), we get

$$\dot{\mathcal{P}} + \sum_j \frac{\partial \mathcal{F}_j}{\partial s_j} = 0, \tag{14}$$

where $\mathcal{F}_j$ is a flux of probability in the space of the amplitude s_j,

$$\begin{aligned} -\frac{\mathcal{F}_j}{4\pi\epsilon^2 s_j} = \sum_{m,n} \Big\{ &(|V^j_{mn}|^2 \delta(\omega^j_{mn})\delta^j_{m+n} \\ &+ 2|V^n_{jm}|^2\delta(\omega^n_{jm})\delta^n_{j+m}) s_n s_m \frac{\partial \mathcal{P}}{\partial s_j} \\ &+ 2\mathcal{P}(|V^n_{jm}|^2\delta(\omega^n_{jm})\delta^n_{j+m} - |V^j_{mn}|^2\delta(\omega^j_{mn})\delta^j_{m+n}) s_m \\ &+ 2(|V^n_{jm}|^2\delta(\omega^n_{jm})\delta^n_{j+m} \\ &- 2|V^j_{mn}|^2\delta(\omega^j_{mn})\delta^j_{m+n}) s_n s_m \frac{\partial \mathcal{P}}{\partial s_m} \Big\}. \end{aligned} \tag{15}$$

Our $\dot{\mathcal{P}}$ does not depend on ξ, which indicate that if $\mathcal{P}(T=0)$ is independent of ξ, then $\mathcal{P}(T)$ is independent of ξ for all T.

Before finishing this section, we define M-mode amplitude PDF,

$$P^{(M)}_{j_1,j_2,\ldots,j_M} = \left(\prod_{l\neq j_1,j_2,\ldots,j_M} \int_{\mathcal{R}^+} \mathrm{d}s_l \right) \mathcal{P}\{s\}. \tag{16}$$

In the following sections, we consider the cases for $M = 1$ and 2.

3. Review of One-Mode Amplitude PDF

One-mode amplitude PDF $P^{(1)}(s_j)$ is obtained from $\mathcal{P}\{s\}$ by integrating over all the modes except the mode $\mathbf{k}_j$,

$$P^{(1)}(s_j) = \left(\prod_{l\neq j} \int \mathrm{d}s_l \right) \mathcal{P}\{s\}. \tag{17}$$

Note here that j stands for the momentum $\mathbf{k}_j$. The time evolution of the one-mode PDF $P^{(1)}(s_j)$ can be derived from eqs. (14) and (15) by integrating over all the other modes, and we get

$$\dot{P}^{(1)}(s_j) + \frac{\partial F(s_j)}{\partial s_j} = 0. \tag{18}$$

When we integrate the right hand side of eq. (14), all of those terms involving $\partial \mathcal{F}_l / \partial s_l$ with $l \neq j$ vanish and the remaining term is $\partial F(s_j)/\partial s_j$ with F given by

$$\begin{aligned} -\frac{F(s_j)}{4\pi\epsilon^2 s_j} = \left(\prod_{l\neq j} \int \mathrm{d}s_l \right) \sum_{m,n} \Big\{ &[|V^j_{mn}|^2 \delta(\omega^j_{mn})\delta^j_{m+n} \\ &+ 2|V^n_{jm}|^2\delta(\omega^n_{jm})\delta^n_{j+m}] s_n s_m \frac{\partial \mathcal{P}}{\partial s_j} \\ &+ 2\mathcal{P}[|V^n_{jm}|^2\delta(\omega^n_{jm})\delta^n_{j+m} - |V^j_{mn}|^2\delta(\omega^j_{mn})\delta^j_{m+n}] s_m \\ &+ 2[|V^n_{jm}|^2\delta(\omega^n_{jm})\delta^n_{j+m} \\ &- 2|V^j_{mn}|^2\delta(\omega^j_{mn})\delta^j_{m+n}] s_n s_m \frac{\partial \mathcal{P}}{\partial s_m} \Big\}. \end{aligned} \tag{19}$$

When $M = 1$, there is only one component for the flux of probability and we suppress the index j on F.

After integration, we obtain

$$F(s_j) = -\gamma_j s_j P^{(1)} - \eta_j s_j \frac{\partial P^{(1)}}{\partial s_j}, \tag{20}$$

with

$$\begin{aligned} \gamma_j &= 8\pi\epsilon^2 \sum_{m,n} \{[|V^j_{mn}|^2\delta(\omega^j_{mn})\delta^j_{m+n} n_m \\ &\quad + |V^n_{jm}|^2\delta(\omega^n_{jm})\delta^n_{j+m}](n_m - n_n)\} \\ \eta_j &= 4\pi\epsilon^2 \sum_{n,m} n_m n_n \{[|V^j_{mn}|^2\delta(\omega^j_{mn})\delta^j_{m+n} \\ &\quad + 2|V^n_{jm}|^2\delta(\omega^n_{jm})\delta^n_{j+m}]\}. \end{aligned} \tag{21}$$

In the derivation, we have used the fact that $V^l_{ln} = 0$ and n_m is the mean value of s_m. This result in eqs. (20) and (21) is exactly the same as the formula derived from the moments of the j-th mode in refs. 9, 12, 13.

A steady state solution can be obtained from the divergence free condition;

$$\frac{\partial F(s_j)}{\partial s_j} = 0. \tag{22}$$

A special solution, the zero flux solution, can be obtained by equating $F(s_j)$ to zero

$$\gamma_j P^{(1)} + \eta_j \frac{\partial P^{(1)}}{\partial s_j} = 0, \tag{23}$$

and we get

$$P^{(1)}(s_j) = \frac{1}{n_j} \exp\left(-\frac{s_j}{n_j} \right), \tag{24}$$

where $n_j = \eta_j/\gamma_j$. Note here that for a steady state solution γ_j, η_j and n_j are all constants. Further analysis on one mode steady solution can be found in refs. 10–13, 15.

4. Consideration of Two-Mode Amplitude PDF

When $M = 2$, the corresponding PDF is the two-mode amplitude PDF $P^{(2)}(s_a, s_b)$ which is obtained from $\mathcal{P}\{s\}$ by integrating over all the modes except the two modes $\mathbf{k}_a$ and $\mathbf{k}_b$

$$P^{(2)}(s_a, s_b) = \left(\prod_{l\neq a,b} \int \mathrm{d}s_l \right) \mathcal{P}\{s\}. \tag{25}$$

The time evolution of $P^{(2)}(s_a, s_b)$ can be obtained similarly as the case $M = 1$, and we get

$$\dot{P}^{(2)}(s_a, s_b) + \frac{\partial F_a}{\partial s_a} + \frac{\partial F_b}{\partial s_b} = 0. \tag{26}$$

with

$$\begin{aligned} -\frac{F_a}{4\pi\epsilon^2 s_a} &= (A_a + B_a s_b) P^{(2)} + (C_a + D_a s_b) \frac{\partial P^{(2)}}{\partial s_a} \\ &\quad + E_a s_b \frac{\partial P^{(2)}}{\partial s_b}, \end{aligned} \tag{27}$$

$$\begin{aligned} -\frac{F_b}{4\pi\epsilon^2 s_b} &= (A_b + B_b s_a) P^{(2)} + (C_b + D_b s_a) \frac{\partial P^{(2)}}{\partial s_b} \\ &\quad + E_b s_a \frac{\partial P^{(2)}}{\partial s_a}, \end{aligned} \tag{28}$$

where

$$A_a = 2{\sum_{m,n}}' v^n_{a,m}(n_m - n_n) + 2{\sum_m}' v^b_{a,m} n_m$$

$$+ 2\sum_{m,n}{}' v^a_{m,n} n_n - 2\sum_{m}{}' v^a_{m,b} n_m, \tag{29}$$

$$B_a = 2\left(\sum_{m}{}' v^m_{a,b} + \sum_{m}{}' v^a_{m,b} - \sum_{m}{}' v^b_{a,m}\right), \tag{30}$$

$$C_a = \sum_{m,n}{}' v^a_{m,n} n_m n_n + 2\sum_{m,n}{}' v^n_{a,m} n_m n_n, \tag{31}$$

$$D_a = 2\left(\sum_{m}{}' v^a_{b,m} n_m + \sum_{m}{}' v^b_{a,m} n_m + \sum_{m}{}' v^m_{a,b} n_m\right), \tag{32}$$

$$E_a = 2\left(\sum_{m}{}' v^m_{a,b} n_m - 2\sum_{m}{}' v^a_{b,m} n_m\right), \tag{33}$$

and A_b, B_b, C_b, D_b, E_b can be obtained by exchanging a and b. In this expression two modes a and b are excluded in the summation $\sum'$ and we use the abbreviation $v^j_{m,n} = |V^j_{mn}|^2\delta(\omega^j_{mn})\delta^j_{m+n}$ for simplicity.

Note here that γ_a and η_a in eq. (21) can be written in terms of $\sum'$ and they become

$$\gamma_a = A_a + B_a n_b - E_a,$$
$$\eta_a = C_a + D_a n_b. \tag{34}$$

These relations can also be derived directly from eq. (27). After integration of F_a over s_b we should get $F(s_a)$ in eq. (20). Comparing the coefficients of $P^{(1)}$ and $\partial P^{(1)}/\partial s_a$ we get the above relations.

4.1 Zero flux solution of two-mode amplitude PDF

It is natural to seek for a solution of two-mode amplitude PDF which gives zero flux of probability as we did in one-mode case. If there exists such a solution, then the solution will give us a steady state two-mode amplitude PDF. The corresponding equations look like

$$(A_a + B_a s_b)P^{(2)} + (C_a + D_a s_b)\frac{\partial P^{(2)}}{\partial s_a} + E_a s_b \frac{\partial P^{(2)}}{\partial s_b} = 0, \tag{35}$$

$$(A_b + B_b s_a)P^{(2)} + (C_b + D_b s_a)\frac{\partial P^{(2)}}{\partial s_b} + E_b s_a \frac{\partial P^{(2)}}{\partial s_a} = 0. \tag{36}$$

Solving these equations for $\partial P^{(2)}/\partial s_a$ and $\partial P^{(2)}/\partial s_b$, we get

$$\frac{\partial P^{(2)}}{\partial s_a} = -\frac{(C_b + D_b s_a)(A_a + B_a s_b) - (A_b + B_b s_a)E_a s_b}{(C_a + D_a s_b)(C_b + D_b s_a) - E_a E_b s_a s_b} P^{(2)}, \tag{37}$$

$$\frac{\partial P^{(2)}}{\partial s_b} = -\frac{(C_a + D_a s_b)(A_b + B_b s_a) - (A_a + B_a s_b)E_b s_a}{(C_a + D_a s_b)(C_b + D_b s_a) - E_a E_b s_a s_b} P^{(2)}. \tag{38}$$

Putting $P^{(2)} = \mathrm{const}\exp(-K)$, we get

$$\frac{\partial K}{\partial s_a} = f_a, \tag{39}$$

$$\frac{\partial K}{\partial s_b} = f_b, \tag{40}$$

where

$$f_a = \frac{(C_b + D_b s_a)(A_a + B_a s_b) - (A_b + B_b s_a)E_a s_b}{(C_a + D_a s_b)(C_b + D_b s_a) - E_a E_b s_a s_b}, \tag{41}$$

$$f_b = \frac{(C_a + D_a s_b)(A_b + B_b s_a) - (A_a + B_a s_b)E_b s_a}{(C_a + D_a s_b)(C_b + D_b s_a) - E_a E_b s_a s_b}. \tag{42}$$

The existence of a solution for K is guaranteed if the above two equations satisfy the integrability condition expressed as

$$\frac{\partial f_a}{\partial s_b} = \frac{\partial f_b}{\partial s_a}, \tag{43}$$

which, after a little work, becomes

$$g_1 + g_2 s_a + g_3 s_b + g_4 s_a^2 + g_5 s_b^2 = 0, \tag{44}$$

with

$$\begin{aligned}
g_1 &= C_a C_b(B_a C_b + A_a E_b - B_b C_a - A_b E_a) \\
&\quad + A_b D_b C_a^2 - A_a D_a C_b^2, \\
g_2 &= 2B_a C_a C_b D_b - B_b C_a C_b E_a \\
&\quad - A_b C_a D_b E_a - 2A_a C_b D_a D_b + A_a C_b E_a E_b, \\
g_3 &= -(2B_b C_a C_b D_a - B_a C_a C_b E_b \\
&\quad - A_a C_b D_a E_b - 2A_b C_a D_a D_b + A_b C_a E_a E_b), \\
g_4 &= (B_a D_b - B_b E_a)C_a D_b - A_a D_b(D_a D_b - E_a E_b), \\
g_5 &= -[(B_b D_a - B_a E_b)C_b D_a - A_b D_a(D_a D_b - E_a E_b)].
\end{aligned} \tag{45}$$

The integrability condition should be satisfied as an identity. In other words, eq. (44) should hold for all values of s_a and s_b. Therefore, in order to have a flux zero solution we have to require

$$g_i = 0 \tag{46}$$

for all i. It seems to be implausible for a given model with a three-wave interaction potential to satisfy all of these conditions.

If this integrability condition is satisfied, the solution for K is given by

$$K = \int_0^{s_a} f_a(s_a, s_b)\,\mathrm{d}s_a + \int^{s_b} f_b(0, s_b)\,\mathrm{d}s_b. \tag{47}$$

4.2 Comments on vortex solution

More interesting and plausible solution for the steady state two-mode amplitude PDF would be a vortex solution which is described by

$$\boldsymbol{F} = \nabla \times Q, \tag{48}$$

where $\boldsymbol{F}$ is two component vector of the flux of probability and Q is some function of s_a and s_b. Using the component notation this becomes

$$F_i = \epsilon_{ij}\frac{\partial}{\partial s_j}Q. \tag{49}$$

The equation for $P^{(2)}$ comes from $\nabla \cdot \boldsymbol{F} = 0$ and we get

$$h_1 P^{(2)} + h_2\frac{\partial P^{(2)}}{\partial s_a} + h_3\frac{\partial P^{(2)}}{\partial s_b} + h_4\frac{\partial^2 P^{(2)}}{\partial {s_a}^2} + h_5\frac{\partial^2 P^{(2)}}{\partial {s_b}^2} + h_6\frac{\partial^2 P^{(2)}}{\partial s_a \partial s_b} = 0, \tag{50}$$

with

$$\begin{aligned}
h_1 &= A_a + B_a s_b + A_b + B_b s_a, \\
h_2 &= C_a + D_a s_b + E_b S_a + A_a s_a + B_a s_a s_b, \\
h_3 &= E_a s_b + C_b + D_b s_a + A_b s_b + B_b s_a s_b, \\
h_4 &= C_a s_a + D_a s_a s_b,
\end{aligned}$$

$$h_5 = -C_b s_b + D_b s_a s_b,$$
$$h_6 = (E_a + E_b) s_a s_b. \tag{51}$$

As we did in zero flux case, we put $P^{(2)} = \text{const} \exp(-K)$, and we get

$$h_4\left[\frac{\partial^2 K}{\partial {s_a}^2} - \left(\frac{\partial K}{\partial s_a}\right)^2\right] + h_5\left[\frac{\partial^2 K}{\partial {s_b}^2} - \left(\frac{\partial K}{\partial s_b}\right)^2\right] + h_6\left(\frac{\partial^2 K}{\partial s_a \partial s_b} - \frac{\partial K}{\partial s_a}\frac{\partial K}{\partial s_b}\right) = h_1 - h_2\frac{\partial K}{\partial s_a} - h_3\frac{\partial K}{\partial s_b}. \tag{52}$$

This equation is a nonlinear second order differential equation and very hard to handle.

Instead of solving this equation explicitly, we investigate the pattern of the flux vector $\boldsymbol{F}$. For that matter we take our two-mode amplitude PDF $P^{(2)}(s_a, s_b)$ to be $\tilde{P}^{(2)}$, the product of two one-mode amplitude PDF's $P^{(1)}(s_a)$ and $P^{(1)}(s_b)$ defined as

$$\tilde{P}^{(2)}(s_a, s_b) = P^{(1)}(s_a) P^{(1)}(s_b). \tag{53}$$

With this ansatz, the divergence of flux becomes

$$\nabla \cdot \boldsymbol{F} = -4\pi\epsilon^2\left(A_a + A_b - \frac{C_a}{n_a} - \frac{C_b}{n_b}\right) \times \left(1 - \frac{s_a}{n_a}\right)\left(1 - \frac{s_b}{n_b}\right)\tilde{P}^{(2)}. \tag{54}$$

To derive this result we have used eq. (34). If we take the integral of the above equation over any one of the two modes s_a and s_b, we get zero because

$$\int_0^\infty s_i \tilde{P}^{(2)}(s_i, s_j)\, \mathrm{d}s_i = n_i P^{(1)}(s_j).$$

This is what we expect, because the remaining PDF $P^{(1)}(s_j)$ is the one-mode steady state amplitude PDF which gives divergence free flux.

Unfortunately, in a generic situation our ansatz, eq. (53), does not guarantee a divergence free flux. Consequently, $\tilde{P}^{(2)}$ is a steady state PDF only when the following condition is satisfied

$$A_a + A_b - \frac{C_a}{n_a} - \frac{C_b}{n_b} = 0. \tag{55}$$

When this condition holds, the divergence vanishes and we get a vortex solution with the corresponding scalar function Q given by

$$Q = -4\pi\epsilon^2\left(A_a - \frac{C_a}{n_a}\right) s_a s_b \tilde{P}^{(2)}.$$

We now assume that the condition in eq. (55) is satisfied for any choice of two modes a and b. Putting $A_b - C_b/n_b = \alpha$, the two components of the flux in eqs. (27) and (28) become

$$F_a = 4\pi\epsilon^2 \alpha s_a \left(1 - \frac{s_b}{n_b}\right)\tilde{P}^{(2)},$$
$$F_b = -4\pi\epsilon^2 \alpha s_b \left(1 - \frac{s_a}{n_a}\right)\tilde{P}^{(2)}. \tag{56}$$

To visualize the pattern of the flux vector, we assume that α is positive. Then, for $s_a > n_a$ F_b is positive, and for $s_a < n_a$, F_b is negative. Similarly, for $s_b > n_b$, F_a is negative, and for $s_b < n_b$, F_b is positive. Therefore, the flux lines circulate around the point (n_a, n_b) counterclockwise when we take s_a-axis to be horizontal (Fig. 1). This pattern of flux lines has already been conjectured in ref. 11.

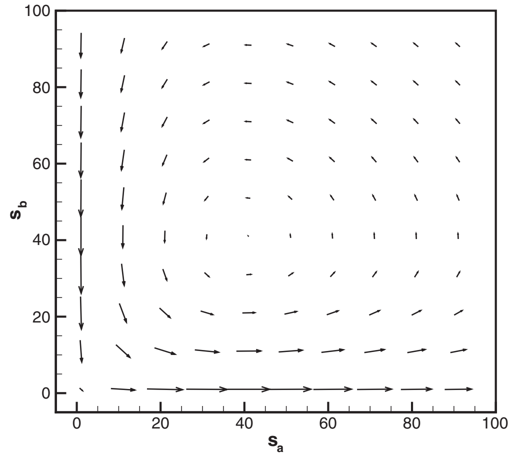

Fig. 1. Two-mode probability flux circulation. This figure shows flow patterns of the two-mode probability flux, i.e., vortex solution. We assume that $4\pi\epsilon^2\alpha$ is about 0.01, $(n_a, n_b) = (40, 40)$ and the total grid size of the frequency space is 100×100.

5. Conclusions and Discussion

In this paper we have derived the time evolution of the two-mode amplitude PDF from the time evolution equation of the full-mode amplitude PDF. We have also derived conditions to have a zero flux solution and an equation for a vortex solution, both of which define steady state two-mode amplitude PDF's.

The conditions for a zero flux solution are given in eqs. (44)–(46). When these conditions are satisfied, we can have $P^{(2)}(s_a, s_b) = \text{constant} \times \mathrm{e}^{-K}$ with K in eq. (47).

If the condition in eq. (55) is satisfied, we can have a steady state vortex solution

$$P^{(2)}(s_a, s_b) = \text{const} \times \exp\left[-\left(\frac{s_a}{n_a} + \frac{s_b}{n_b}\right)\right]. \tag{57}$$

The corresponding flux components are given in eq. (56) and plotted in Fig. 1. This solution is the solution conjectured in ref. 11. Non-vanishing flux indicates that the values of (s_a, s_b) change continuously in such a way that the two-mode PDF is unaltered. Integrating over either s_a or s_b, we get the known zero-flux one-mode PDF. The corresponding flux vanishes because the average of s is n. Note that the condition in eq. (55) for any pair of modes is a big assumption. This assumption may not apply to a generic system. However, the argument given in the previous section applies to a generic system where the condition does not hold for all pairs of modes. In this case, we use eq. (55) to select a pair of modes a and b. Considering the fact that the total number of modes is infinite, we can expect that for a given mode a there exists another mode b which together with the mode a satisfies the condition. For this selected pair of modes, our ansatz in eq. (53) defines a steady state PDF with a non-vanishing flux.

More general vortex solutions, which cannot be expressed as the product of two one-mode PDF's, may exist as solutions of eqs. (50)–(52). However, the equation is nonlinear and very hard to handle analytically. It requires numerical analysis which we are working on.

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