Counting cubic extensions with given quadratic resolvent

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Definitions and notation

- Given $k \subset K$ two number fields, we denote by $\mathfrak{d}(K/k)$ the relative discriminant ideal of the extension.
- We define

$$N_{k,n}(X) = \sharp\{K | [K : k] = n, \mathcal{N}_{k/\mathbb{Q}}\mathfrak{d}(K/k) \le X\}/ \sim$$

Two main themes in the subject :

- Asymptotics as $X \to \infty$;
- Algorithmics.

Asymptotic results

- Asymptotic for relative quadratic extensions : Wright, Cohen-Diaz y Diaz-Olivier;
- Asymptotics for cubic extensions : Davenport-Heilbronn ($k = \mathbb{Q}$) Datskovsky-Wright (k arbitrary);
- Asymptotics for $n = 4, 5$ and $k = \mathbb{Q}$: Bhargava.

Let us focus on the case $n = 3$.

We may ask some more specific questions:

1. How many Galois (cyclic) extensions ?

   Then answer is given by Cohn for the rational case and by Cohen-Diaz y Diaz-Oliver for the general case.

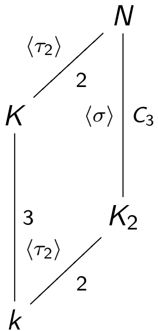
2. How many non-Galois extensions with a fixed quadratic resolvent K_2 ?

   We are going to answer to this question in this talk.

Let k be a number field, and fix K_2 a quadratic extension of k.

We define $\mathcal{F}(K_2)$ the set of cubic extensions K of k (modulo k-isomorphism) whose Galois closure N contains K_2 as quadratic subextension.

If we allow $[K_2 : k] = 1$ we can also describe cyclic cubic extensions.



Our goal

We look for an asymptotic formula for

$$N(K_2/k, X) = |\{K \in \mathcal{F}(K_2),\ \mathcal{N}_{k/\mathbb{Q}}(\mathfrak{d}(K/k)) \leq X\}| .$$

The conductor of the cyclic extension N/K_2 is of the form $\mathfrak{f}(N/K_2) = \mathfrak{f}(K/k)\mathbb{Z}_{K_2}$, where $\mathfrak{f}(K/k)$ is an ideal of k.

Since

$$\mathfrak{d}(K/k) = \mathfrak{d}(K_2/k)\mathfrak{f}(K/k)^2,$$

then we can reduce to study $M(K_2/k, X)$ where

$$M(K_2/k, X) = |\{K \in \mathcal{F}(K_2),\ \mathcal{N}_{k/\mathbb{Q}}(\mathfrak{f}(K/k)) \leq X\}|$$

Idea

We can study the fundamental Dirichlet series:

$$\Phi(s) = \frac{1}{2} + \sum_{K \in \mathcal{F}(K_2)} \frac{1}{\mathcal{N}_{k/\mathbb{Q}}(\mathfrak{f}(K/k))^s} ,$$

and thanks to a Tauberian theorem we can deduct asymptotic formulas for $M(K_2/k, X)$.

Outline of the work

1. Galois structure of the extensions (Kummer theory)

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2. Conductor of the extensions (Hecke's Theorem)
3. Study of the fundamental Dirichlet series
4. Asymptotic formulas.

Galois theory

Let $\rho = \zeta_3$ a primitive cube root of unity.
Set $L = K_2(\rho)$ and $k_z = k(\rho)$.
Let

- τ be a generator of $\mathrm{Gal}(L/K_2)$,
- τ_2 a generator of $\mathrm{Gal}(K_2/k)$
- σ be a generator of the cyclic group of order 3 $\mathrm{Gal}(N/K_2) \simeq \mathrm{Gal}(N_z/L)$, where $N_z = N(\rho)$.

We have the following relations:

$$\tau^2 = \tau_2^2 = 1 \ , \quad \tau\tau_2 = \tau_2\tau \ , \quad \tau\sigma = \sigma\tau \ .$$

Five possible situations

1. K/k cyclic and $\rho \in k \Rightarrow \tau = \tau_2 = 1$.
2. K/k cyclic and $\rho \notin k \Rightarrow \tau_2 = 1,\ \tau(\rho) = \rho^{-1}$.
3. K/k non-cyclic and $\rho \in k \Rightarrow \tau = 1$ and $\tau_2(\rho) = \rho$.
4. K/k non-cyclic and $\rho \in K_2 \setminus k \Rightarrow \tau = 1$ and $\tau_2(\rho) = \rho^{-1}$.
5. K/k non-cyclic and $\rho \notin K_2 \Rightarrow \tau(\rho) = \rho^{-1}$ and $\tau_2(\rho) = \rho$.

Definition 1

- In cases (1) to (5) above, we set $T = \emptyset$, $\{\tau + 1\}$, $\{\tau_2 + 1\}$, $\{\tau_2 - 1\}$, $\{\tau + 1, \tau_2 + 1\}$, respectively, where T is considered as a subset of the group ring $\mathbb{Z}[\mathrm{Gal}(L/k)]$ or of $\mathbb{F}_3[\mathrm{Gal}(L/k)]$.

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- We define $\iota(\tau \pm 1) = \tau \mp 1$ and $\iota(\tau_2 \pm 1) = \tau_2 \mp 1$.
- For any group M on which T acts, we denote by $M[T]$ the subgroup of elements of M annihilated by all the elements of T.

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Lemma 2

Let M be an $\mathbb{F}_3[\mathrm{Gal}(L/k)]$-module. For any $t \in T$ we have $M[t] = \iota(t)(M)$, and conversely $M[\iota(t)] = t(M)$.

Proposition 3

1. *There exists a bijection between :*
   - *isomorphism classes of extensions K/k having quadratic resolvent isomorphic to K_2 (i. e. elements of $\mathcal{F}(K_2)$), and*
   - *elements $\overline{\alpha} \in (L^*/{L^*}^3)[T]$, $\overline{\alpha} \neq \overline{1}$, modulo the equivalence relation identifying $\overline{\alpha}$ with its inverse.*
2. *If $\alpha \in L^*$ is some representative of $\overline{\alpha}$, the extension K/k corresponding to α is the fixed field under $\mathrm{Gal}(L/k)$ of the field $N_z = L(\sqrt[3]{\alpha})$.*

Definition 4

We denote by $V_3(L)$ the group of 3-*virtual units* of L, in other words the group of $u \in L^*$ such that $u\mathbb{Z}_L = \mathfrak{q}^3$ for some ideal $\mathfrak{q}$ of L. We define the 3-*Selmer group* $S_3(L)$ of L by $S_3(L) = V_3(L)/{L^*}^3$.

Proposition 5

1. *There exists a bijection between isomorphism classes of cubic extensions K/k with given quadratic resolvent field K_2 and equivalence classes of triples $(\mathfrak{a}_0, \mathfrak{a}_1, \overline{u})$ such that*
   1. *The $\mathfrak{a}_i$ are coprime integral squarefree ideals of L such that $\overline{\mathfrak{a}_0\mathfrak{a}_1^2} \in Cl(L)^3$ and $\mathfrak{a}_0\mathfrak{a}_1^2 \in (I/I^3)[T]$, where I is the group of fractionals ideals of L.*
   2. *$\overline{u} \in S_3(L)[T]$, and $\overline{u} \neq 1$ when $\mathfrak{a}_0 = \mathfrak{a}_1 = \mathbb{Z}_L$.*

   modulo the equivalence relation $(\mathfrak{a}_0, \mathfrak{a}_1, \overline{u}) \sim (\mathfrak{a}_1, \mathfrak{a}_0, 1/\overline{u})$
2. *If $(\mathfrak{a}_0, \mathfrak{a}_1)$ is a pair of ideals satisfying (1.1) there exist an ideal $\mathfrak{q}_0$ and an element α_0 of L such that $\mathfrak{a}_0\mathfrak{a}_1^2\mathfrak{q}_0^3 = \alpha_0\mathbb{Z}_L$ with $\alpha_0 \in (L^*/{L^*}^3)[T]$. The cubic extensions K/k corresponding to such a pair $(\mathfrak{a}_0, \mathfrak{a}_1)$ are given as follows: for any $\overline{u} \in S_3(L)[T]$ the extension is the cubic subextension of $N_z = L(\sqrt[3]{\alpha_0 u})$ (for any lift u of $\overline{u}$).*

Lemma 6

1. *The condition* $\mathfrak{a}_0\mathfrak{a}_1^2 \in (I/I^3)[T]$ *is equivalent to* $\mathfrak{a}_1 = \tau(\mathfrak{a}_0)$, $\mathfrak{a}_1 = \tau_2(\mathfrak{a}_0)$, $\mathfrak{a}_0 = \tau_2(\mathfrak{a}_0)$ *and* $\mathfrak{a}_1 = \tau_2(\mathfrak{a}_1)$, *and* $\mathfrak{a}_1 = \tau(\mathfrak{a}_0) = \tau_2(\mathfrak{a}_0)$ *in cases (2), (3), (4), and (5), respectively.*

Lemma 6

1. *The condition* $\mathfrak{a}_0\mathfrak{a}_1^2 \in (I/I^3)[T]$ *is equivalent to* $\mathfrak{a}_1 = \tau(\mathfrak{a}_0)$, $\mathfrak{a}_1 = \tau_2(\mathfrak{a}_0)$, $\mathfrak{a}_0 = \tau_2(\mathfrak{a}_0)$ *and* $\mathfrak{a}_1 = \tau_2(\mathfrak{a}_1)$, *and* $\mathfrak{a}_1 = \tau(\mathfrak{a}_0) = \tau_2(\mathfrak{a}_0)$ *in cases (2), (3), (4), and (5), respectively.*
2. *The ideal* $\mathfrak{a}_0\mathfrak{a}_1$ *of* L *comes from an ideal* $\mathfrak{a}_\alpha$ *of* K_2 *(in other words* $\mathfrak{a}_0\mathfrak{a}_1 = \mathfrak{a}_\alpha\mathbb{Z}_L$*), and in cases (1), (2), and (3) it comes from an ideal of* k*, while in cases (4) and (5),* $\mathfrak{a}_\alpha$ *is an ideal of* K_2 *invariant by* τ_2*.*

Conductors

Definition 7

Let $\overline{\alpha} \in (L^*/{L^*}^3)[T]$ as above, let p be an ideal of k above 3, let $\mathfrak{p}$ be an ideal of K_2 above p, let $\mathfrak{p}_z$ be an ideal of L above $\mathfrak{p}$, and denote by C_k the congruence $x^3/\alpha \equiv 1 \pmod{{}^*\mathfrak{p}_z^k}$ in L. If this congruence is soluble for $k = 3e(\mathfrak{p}_z/3)/2$ we set $A_\alpha(p) = 3e(\mathfrak{p}_z/3)/2 + 1$. Otherwise, if $k < 3e(\mathfrak{p}_z/3)/2$ is the largest exponent for which it has a solution, we set $A_\alpha(p) = k$. In both cases we set

$$a_\alpha(p) = \frac{A_\alpha(p) - 1}{e(\mathfrak{p}_z/p)} .$$

Theorem 8

Let N correspond to α as above, write uniquely $\alpha\mathbb{Z}_L = \mathfrak{a}_0\mathfrak{a}_1^2\mathfrak{q}^3$ with $\mathfrak{a}_0$ and $\mathfrak{a}_1$ integral coprime squarefree ideals, and let $\mathfrak{a}_\alpha$ be the ideal of K_2 such that $\mathfrak{a}_0\mathfrak{a}_1 = \mathfrak{a}_\alpha\mathbb{Z}_L$. Then

$$\mathfrak{f}(N/K_2) = \frac{3\mathfrak{a}_\alpha \prod_{p\mid 3\mathbb{Z}_k}(p\mathbb{Z}_{K_2})^{e(p/3)/2} \prod_{\substack{p\mid 3\mathbb{Z}_k \\ e(\mathfrak{p}/3)\ odd}} (p\mathbb{Z}_{K_2})^{1/2}}{\prod_{\substack{p\mid 3\mathbb{Z}_k \\ p\nmid\mathfrak{a}_\alpha}}(p\mathbb{Z}_{K_2})^{\lceil a_\alpha(p)e(\mathfrak{p}/p)\rceil/e(\mathfrak{p}/p)}} .$$

Definition 9

- If $\mathfrak{a}$ is an ideal of k, we set $\mathcal{N}(\mathfrak{a}) = \mathcal{N}_{k/\mathbb{Q}}(\mathfrak{a})$,
- if $\mathfrak{a}$ is an ideal of K_2, we set $\mathcal{N}(\mathfrak{a}) = \mathcal{N}_{K_2/\mathbb{Q}}(\mathfrak{a})^{1/[K_2:k]}$.

Remark

This notation is consistent, in fact if $\mathfrak{a}$ is an ideal of k we have $\mathcal{N}(\mathfrak{a}) = \mathcal{N}(\mathfrak{a}\mathbb{Z}_{K_2})$.

Definition 10

The fundamental Dirichlet series is defined by

$$\Phi(s)=\frac{1}{2}+\sum_{K\in\mathcal{F}(K_2)}\frac{1}{\mathcal{N}(\mathfrak{f}(K/k))^s}\;.$$

By the fundamental bijection

$$\Phi(s)=\frac{1}{2}\sum_{(\mathfrak{a}_0,\mathfrak{a}_1)\in J}\quad\sum_{\overline{u}\in S_3(L)[T]}\frac{1}{\mathcal{N}(\mathfrak{f}(N/K_2))^s}\;,$$

where J is the set of pairs $(\mathfrak{a}_0,\mathfrak{a}_1)$ of coprime integral squarefree ideals of L such that $\mathfrak{a}_0\mathfrak{a}_1^2\in(I/I^3)[T]$ and $\overline{\mathfrak{a}_0\mathfrak{a}_1^2}\in Cl(L)^3$, and where $\mathfrak{f}(N/K_2)$ is the conductor of the extension N/K_2 corresponding to the triple $(\mathfrak{a}_0,\mathfrak{a}_1,\overline{u})$.

So from the theorem above,

$$\Phi(s) = \frac{1}{2 \cdot 3^{(3/2)[k:\mathbb{Q}]s} \prod_{\substack{p \mid 3\mathbb{Z}_k, \\ e(\mathfrak{p}/3) \text{ odd}}} \mathcal{N}(p)^{s/2}} \sum_{(\mathfrak{a}_0, \mathfrak{a}_1) \in J} \frac{S_{\alpha_0}(s)}{\mathcal{N}(\mathfrak{a}_\alpha)^s},$$

where

$$S_{\alpha_0}(s) = \sum_{\overline{u} \in S_3(L)[T]} \prod_{\substack{p \mid 3\mathbb{Z}_k \\ p \nmid \mathfrak{a}_\alpha}} \mathcal{N}(p)^{\lceil a_{\alpha_0 u}(p) e(\mathfrak{p}/p) \rceil s / e(\mathfrak{p}/p)},$$

and where α_0 is any element of L such that there exists an ideal $\mathfrak{q}_0$ such that $\mathfrak{a}_0 \mathfrak{a}_1^2 \mathfrak{q}_0^3 = \alpha_0 \mathbb{Z}_L$ and $\alpha_0 \in (L^*/{L^*}^3)[T]$.

Definition 11

For $\alpha_0 \in L^*$ and $\mathfrak{b}$ an ideal of K_2 we introduce the function

$$f_{\alpha_0}(\mathfrak{b}) = \left|\{\overline{u} \in S_3(L)[T], x^3/(\alpha_0 u) \equiv 1 \pmod{{}^*\mathfrak{b}\mathbb{Z}_L} \text{ soluble in } L\}\right|$$

with the convention that $f_{\alpha_0}(\mathfrak{b}) = 0$ if $\mathfrak{b}\mathbb{Z}_L \nmid 3\sqrt{-3}$.

There exist a set $\mathcal{B}'$ of ideals $\mathfrak{b} = \prod_{\mathfrak{p}_i | 3\mathbb{Z}_{K_2}} \mathfrak{p}_i^{b_i}$ of K_2 (where $b_i \in \mathbb{Z}$ or sometimes $b_i \in \mathbb{Z}/2$ if $\mathfrak{p}_i$ ramifies in L/K_2) such that

$$\sum_{(\mathfrak{a}_0,\mathfrak{a}_1)\in J} \frac{S_{\alpha_0}(s)}{\mathcal{N}(\mathfrak{a}_\alpha)^s} = \sum_{\mathfrak{b}\in\mathcal{B}'} \lceil\mathcal{N}\rceil(\mathfrak{b})^s P_{\mathfrak{b}}(s) \sum_{(\mathfrak{a}_0,\mathfrak{a}_1)\in J'} \frac{f_{\alpha_0}(\mathfrak{b})}{\mathcal{N}(\mathfrak{a}_\alpha)^s},$$

where J' is "some" subset of J, $P_{\mathfrak{b}}(s)$ is "some" (totally explicit) function with values in $\mathbb{Q}$, and
$\lceil\mathcal{N}\rceil(\mathfrak{b}) = \prod_{\mathfrak{p}_i | 3\mathbb{Z}_{K_2}} \mathcal{N}\left(\mathfrak{p}_i^{\lceil b_i \rceil}\right)$.

Computation of $f_{\alpha_0}(\mathfrak{b})$

Let $\mathfrak{b} \in \mathcal{B}'$, $\mathfrak{b}_z = \mathfrak{b}\mathbb{Z}_L$ and $Cl_{\mathfrak{b}}(L)$ the ray class group.

Definition 12

$S_{\mathfrak{b}}(L)[T] = \{\overline{u} \in S_3(L)[T],\ x^3 \equiv u \pmod{^*\mathfrak{b}_z} \text{ soluble}\}$, where u is any lift of $\overline{u}$ coprime to $\mathfrak{b}_z$, and the congruence is in L.

Lemma 13

Let $\mathfrak{a}_0, \mathfrak{a}_1$ as in condition (1) of Proposition 5. Then

$$f_{\alpha_0}(\mathfrak{b}) = \begin{cases} |S_{\mathfrak{b}}(L)[T]| & \text{if } \overline{\mathfrak{a}_0\mathfrak{a}_1^2} \in Cl_{\mathfrak{b}}(L)^3 \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 14

Set $Z_{\mathfrak{b}} = (\mathbb{Z}_L/\mathfrak{b}_z)^*$. *Then*

$$|S_{\mathfrak{b}}(L)[T]| = \frac{|(U(L)/U(L)^3)[T]||(Cl_{\mathfrak{b}}(L)/Cl_{\mathfrak{b}}(L)^3)[T]|}{|(Z_{\mathfrak{b}}/Z_{\mathfrak{b}}^3)[T]|}.$$

Remark 15

We can compute explicitly $|(U(L)/U(L)^3)[T]|$ *and* $|(Z_{\mathfrak{b}}/Z_{\mathfrak{b}}^3)[T]|$*, but we can't compute* $|(Cl_{\mathfrak{b}}(L)/Cl_{\mathfrak{b}}(L)^3)[T]|$*. Luckily it is not necessary, because this term will disappear in subsequent computations.*

Final form of the Dirichlet series

Theorem 16

We have

$$\Phi(s) = \frac{|(U(L)/U(L)^3)[T]|}{2 \cdot 3^{(3/2)[k:\mathbb{Q}]s} \prod_{\substack{p|3\mathbb{Z}_k, \\ e(\mathfrak{p}/3) \text{ odd}}} \mathcal{N}(p)^{s/2}} \cdot$$

$$\cdot \sum_{\mathfrak{b} \in \mathcal{B}'} \left(\frac{\lceil \mathcal{N} \rceil (\mathfrak{b})}{\mathcal{N}(\mathfrak{r}^e(\mathfrak{b}))} \right)^s \frac{P_{\mathfrak{b}}(s)}{|(Z_{\mathfrak{b}}/Z_{\mathfrak{b}}^3)[T]|} \sum_{\chi \in \widehat{G_{\mathfrak{b}}}} F(\mathfrak{b}, \chi, s) \, .$$

Where

$$G_{\mathfrak{b}} = (Cl_{\mathfrak{b}}(L)/Cl_{\mathfrak{b}}(L)^3)[T], \quad \mathfrak{r}^e(\mathfrak{b}) = \prod_{\substack{\mathfrak{p}|3\mathbb{Z}_{K_2}, \mathfrak{p} \nmid \mathfrak{b} \\ e(\mathfrak{p}/3) \text{ even}}} \mathfrak{p} .$$

Thanks to the theorem, we can now expand $\Phi(s)$ around the pole $s = 1$:

1. In cases (1) and (4) we get

$$\Phi(s) = \frac{C(K_2/k)}{(s-1)^2} + \frac{C(K_2/k)D(K_2/k)}{(s-1)} + O(1),$$

2. while in the other cases we get

$$\Phi(s) = \frac{C(K_2/k)}{(s-1)} + O(1),$$

where the formulas for $C(K_2/k)$ and $D(K_2/k)$ are totally explicit.

Asymptotic formulas

Thanks to a Tauberian theorem we get

1. In cases (1) and (4) :

$$M(K_2/k, X) = C(K_2/k)X(\log(X) + D(K_2/k) - 1) + O(X^{1/2+\varepsilon}).$$

2. In the other cases :

$$M(K_2/k, X) = C(K_2/k)X + O(X^{1/2+\varepsilon}).$$

Examples

Cyclic cubic extensions of $\mathbb{Q}$

In this case we obtain

$$\sum_{K/\mathbb{Q} \text{ cyclic cubic}} \frac{1}{f(K)^s} = -\frac{1}{2}+\frac{1}{2}\left(1+\frac{2}{3^{2s}}\right)\prod_{p\equiv 1 \pmod 3}\left(1+\frac{2}{p^s}\right),$$

and so

$$M(\mathbb{Q},X) = C(\mathbb{Q})X + O(X^{1/2+\varepsilon}),$$

with

$$C(\mathbb{Q}) = \frac{11\sqrt{3}}{36\pi}\prod_{p\equiv 1 \pmod 3}\left(1-\frac{2}{p(p+1)}\right)$$
$$= 0.1585282583961420602835078203575\ldots$$

Pure cubic fields over $\mathbb{Q}$

Case (4) : $K_2 = \mathbb{Q}(\rho)$ and $L = K_2$, so $K/\mathbb{Q}$ is a *pure cubic field* i. e. $K = \mathbb{Q}(\sqrt[3]{m})$. We get

$$\sum_{K/\mathbb{Q} \text{ pure cubic}} \frac{1}{f(K)^s} = -\frac{1}{2} + \frac{1}{6}\left(1 + \frac{2}{3^s} + \frac{6}{3^{2s}}\right) \prod_{p \neq 3} \left(1 + \frac{2}{p^s}\right) + \frac{1}{3} \prod_{p \equiv \pm 1 \pmod 9} \left(1 + \frac{2}{p^s}\right) \prod_{p \not\equiv \pm 1 \pmod 9} \left(1 - \frac{1}{p^s}\right).$$

and so

$$M(\mathbb{Q}(\sqrt{-3}), X) = C \cdot X \cdot (\log(X) + D - 1) + O(X^{1/2+\varepsilon}),$$

where

$$C = C(\mathbb{Q}(\sqrt{-3})) = \frac{7}{30} \prod_p \left(1 - \frac{3}{p^2} + \frac{2}{p^3}\right)$$
$$= 0.066907733301378371291841632984295\dots$$
$$D = D(\mathbb{Q}(\sqrt{-3})) = 2\gamma - \frac{16}{35}\log(3) + 6\sum_p \frac{\log(p)}{p^2+p-2}$$
$$= 3.4502227978305919627907119196711 10\dots,$$

and γ is Euler's constant.

Case (5) : $K_2 = \mathbb{Q}(\sqrt{D})$ **with** $D \neq -3$

There exists a function $\phi_D(s)$ holomorphic for $\mathrm{Re}(s) > 1/2$ such that

$$\sum_{K \in \mathcal{F}(K_2)} \frac{1}{f(K)^s} = \phi_D(s) + \frac{3^{r_2(D)}}{6} L_3(s) \prod_{\left(\frac{-3D}{p}\right)=1} \left(1 + \frac{2}{p^s}\right),$$

where

$$L_3(s) = \begin{cases} 1 + 2/3^{2s} & \text{if } 3 \nmid D, \\ 1 + 2/3^s & \text{if } D \equiv 3 \pmod{9}, \\ 1 + 2/3^s + 6/3^{2s} & \text{if } D \equiv 6 \pmod{9}. \end{cases}$$

$r_2(D) = 1$ for $D < 0$, $r_2(D) = 0$ otherwise.

Set $D' = -3D$ if $3 \nmid D$ and $D' = -D/3$ if $3 \mid D$, and denote as usual by $\chi_{D'}$ the character $\left(\frac{D'}{.}\right)$. Then if $D \neq -3$ is a fundamental discriminant, for all $\varepsilon > 0$ we have

$$M(\mathbb{Q}(\sqrt{D}), X) = C(\mathbb{Q}(\sqrt{D}))X + O(X^{1/2+\varepsilon}),$$

with

$$C(\mathbb{Q}(\sqrt{D})) = \frac{3^{r_2(D)}\ell_3 L(\chi_{D'}, 1)}{\pi^2} \prod_{p \mid D'} \left(1 - \frac{1}{p+1}\right) \cdot \prod_{\left(\frac{D'}{p}\right)=1} \left(1 - \frac{2}{p(p+1)}\right), \quad \text{where}$$

$$\ell_3 = \begin{cases} 11/9 & \text{if } 3 \nmid D, \\ 5/3 & \text{if } D \equiv 3 \pmod 9, \\ 7/5 & \text{if } D \equiv 6 \pmod 9. \end{cases}$$