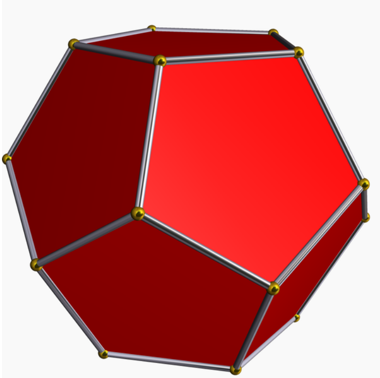
Categorification and TQFTs

Warwick Mathematics Society

Nicholas Jackson

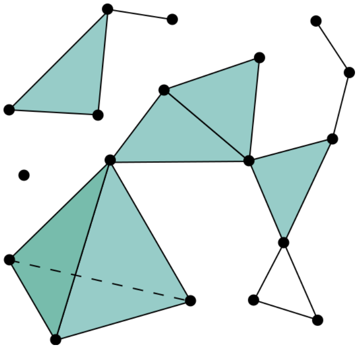
November 2013

Leonhard Euler (1707–1783)



$$\begin{aligned}\chi(\text{dodecahedron}) &= \#(\text{vertices}) - \#(\text{edges}) + \#(\text{faces}) \\ &= 20 - 30 + 12 \\ &= 2\end{aligned}$$

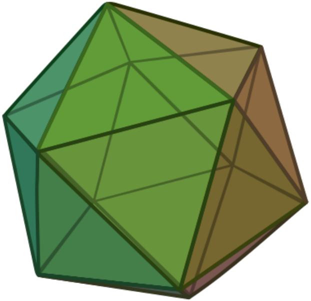
Extend this to simplicial complexes:



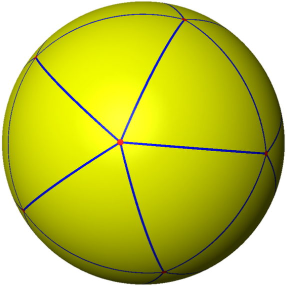
$$\chi(K) = \sum_{k=0}^{n} (-1)^k \#(k\text{–simplices})$$
$$= 18 - 23 + 8 - 1$$
$$= 2$$

Extend this to (some) topological spaces.
Triangulation: simplicial complex K and a homeomorphism

$$h\colon K \to X.$$



$\longrightarrow$



Euler characteristic: invariance

The Euler characteristic χ is a topological invariant:

$$\text{If } X \cong Y \text{ then } \chi(X) = \chi(Y).$$

Euler characteristic: invariance

The Euler characteristic χ is a topological invariant:

$$\text{If } X \cong Y \text{ then } \chi(X) = \chi(Y).$$

But. . .

$$\chi(X) = \chi(Y) \text{ doesn't necessarily mean that } X \cong Y.$$

Euler characteristic: invariance

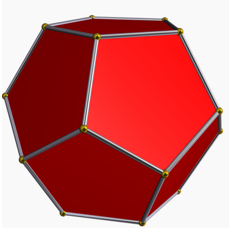
The Euler characteristic χ is a topological invariant:

$$\text{If } X \cong Y \text{ then } \chi(X) = \chi(Y).$$

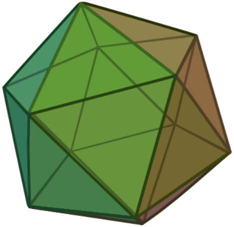
But. . .

$$\chi(X) = \chi(Y) \text{ doesn't necessarily mean that } X \cong Y.$$

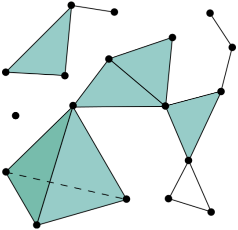
For example:



$\cong$



$\ncong$



From MA251 Algebra I:

Theorem (finitely generated abelian groups)

Let A be a finitely generated abelian group. Then

$$A \cong \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{r \text{ copies}} \oplus \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_k}.$$

(We say A has rank r.)

From MA251 Algebra I:

Theorem (finitely generated abelian groups)

Let A be a finitely generated abelian group. Then

$$A \cong \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{r \text{ copies}} \oplus \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_k}.$$

(We say A has rank r.)

From MA3H6 Algebraic Topology:
$H_n(X)$, the nth homology group.

- $H_n(X)$ is an abelian group.
- $H_n(X)$ "counts" the n–dimensional holes in X.
- The nth Betti number of X is $b_n = \mathrm{rank}(H_n(X))$.
- $\mathrm{rank}\, H_0(X)$ is the number of path components of X.
- $H_1(X) \cong \pi_1(X, *)^{\mathrm{ab}}$ if X is path-connected.

We can recover the Euler characteristic from the homology groups:

$$\chi(X) = \sum_{k=0}^{n} (-1)^k \operatorname{rank} H_k(X)$$

Homology and the Euler characteristic

We can recover the Euler characteristic from the homology groups:

$$\chi(X) = \sum_{k=0}^{n} (-1)^k \operatorname{rank} H_k(X)$$

Example (2–sphere, convex polyhedra)

$$H_0(S^2) = \mathbb{Z} \qquad H_1(S^2) = 0 \qquad H_2(S^2) = \mathbb{Z}$$

Hence $\chi(S^2) = \operatorname{rank}\mathbb{Z} - \operatorname{rank} 0 + \operatorname{rank}\mathbb{Z} = 1 - 0 + 1 = 2$.

Homology and the Euler characteristic

We can recover the Euler characteristic from the homology groups:

$$\chi(X) = \sum_{k=0}^{n} (-1)^k \operatorname{rank} H_k(X)$$

Example (2–sphere, convex polyhedra)

$$H_0(S^2) = \mathbb{Z} \qquad H_1(S^2) = 0 \qquad H_2(S^2) = \mathbb{Z}$$

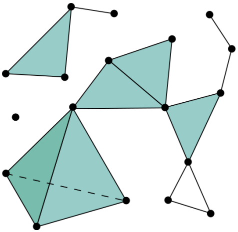
Hence $\chi(S^2) = \operatorname{rank} \mathbb{Z} - \operatorname{rank} 0 + \operatorname{rank} \mathbb{Z} = 1 - 0 + 1 = 2$.

Example (2–torus)

$$H_0(S^1 \times S^1) = \mathbb{Z} \qquad H_1(S^1 \times S^1) = \mathbb{Z} \oplus \mathbb{Z} \qquad H_2(S^1 \times S^1) = \mathbb{Z}$$

Hence $\chi(S^1 \times S^1) = \operatorname{rank} \mathbb{Z} - \operatorname{rank} \mathbb{Z} \oplus \mathbb{Z} + \operatorname{rank} \mathbb{Z} = 1 - 2 + 1 = 0$.

Homology and the Euler characteristic

$K =$ 

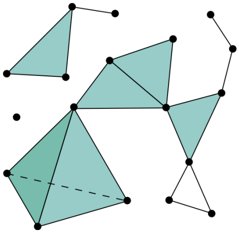
$$H_0(K) = \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z} \quad H_1(K) = \mathbb{Z} \quad H_2(K) = 0 \quad H_3(K) = 0$$

Hence

$$\chi(K) = 3 - 1 + 0 - 0 = 2 = \chi(S^2).$$

Homology and the Euler characteristic

$K =$



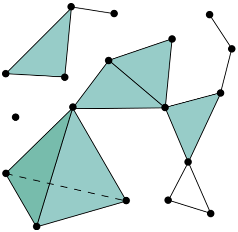
$$H_0(K) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \qquad H_1(K) = \mathbb{Z} \qquad H_2(K) = 0 \qquad H_3(K) = 0$$

Hence

$$\chi(K) = 3 - 1 + 0 - 0 = 2 = \chi(S^2).$$

But

$$H_n(K) \ncong H_n(S^2)$$

$K =$ 

$$H_0(K) = \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z} \quad H_1(K) = \mathbb{Z} \quad H_2(K) = 0 \quad H_3(K) = 0$$

Hence

$$\chi(K) = 3 - 1 + 0 - 0 = 2 = \chi(S^2).$$

But

$$H_n(K) \ncong H_n(S^2)$$

so $H_n(-)$ is a better invariant than $\chi(-)$.

This is an example of categorification: roughly speaking, replace simple objects with more sophisticated ones.

This is an example of categorification: roughly speaking, replace simple objects with more sophisticated ones.

A more precise partial definition:

A categorification of a set S is a category $\mathcal{C}$ and a function

$$p\colon (\mathrm{obj}\,\mathcal{C}/\cong) \longrightarrow S.$$

Categorification

This is an example of categorification: roughly speaking, replace simple objects with more sophisticated ones.
A more precise partial definition:

A categorification of a set S is a category $\mathcal{C}$ and a function

$$p \colon (\operatorname{obj}\mathcal{C}/\cong) \longrightarrow S.$$

Example (The natural numbers)

$$S = \mathbb{N} \qquad \mathcal{C} = \mathsf{FinSet} \qquad p = |-|$$

This is an example of categorification: roughly speaking, replace simple objects with more sophisticated ones.
A more precise partial definition:

A categorification of a set S is a category $\mathcal{C}$ and a function

$$p\colon (\mathrm{obj}\,\mathcal{C}/\cong) \longrightarrow S.$$

Example (The natural numbers)

$S = \mathbb{N}$ $\qquad$ $\mathcal{C} = \mathsf{FinSet}$ $\qquad$ $p = |-|$

Example (Betti numbers)

$S = \mathbb{N}$ $\qquad$ $\mathcal{C} = \mathsf{FGAb}$ $\qquad$ $p = \mathrm{rank}$

The Jones polynomial

From MA3F2 Knot Theory: $J_K(t) \in \mathbb{Z}[t^{\pm 1/2}]$
Can be defined via the Kauffman bracket $\langle K \rangle \in \mathbb{Z}[A^{\pm 1}]$

$$\langle \bigcirc \rangle = 1$$
$$\langle K \sqcup \bigcirc \rangle = (-A^2 - A^{-2})\langle K \rangle$$
$$\langle \times \rangle = A\langle \,)(\, \rangle + A^{-1}\langle \asymp \rangle$$

This is R_2– and R_3–invariant.

The Jones polynomial

From MA3F2 Knot Theory: $J_K(t) \in \mathbb{Z}[t^{\pm 1/2}]$
Can be defined via the Kauffman bracket $\langle K \rangle \in \mathbb{Z}[A^{\pm 1}]$

$$\langle \bigcirc \rangle = 1$$
$$\langle K \sqcup \bigcirc \rangle = (-A^2 - A^{-2})\langle K \rangle$$
$$\langle \times \rangle = A\langle)(\rangle + A^{-1}\langle \asymp \rangle$$

This is R_2– and R_3–invariant.
Now work out the writhe

$$\begin{aligned} w(K) &= \#(\text{positive crossings}) - \#(\text{negative crossings}) \\ &= n_+(K) - n_-(K) \end{aligned}$$

From MA3F2 Knot Theory: $J_K(t) \in \mathbb{Z}[t^{\pm 1/2}]$
Can be defined via the Kauffman bracket $\langle K \rangle \in \mathbb{Z}[A^{\pm 1}]$

$$\begin{aligned} \langle \bigcirc \rangle &= 1 \\ \langle K \sqcup \bigcirc \rangle &= (-A^2 - A^{-2}) \langle K \rangle \\ \langle \times \rangle &= A \langle)(\rangle + A^{-1} \langle \asymp \rangle \end{aligned}$$

This is R_2– and R_3–invariant.
Now work out the writhe

$$\begin{aligned} w(K) &= \#(\text{positive crossings}) - \#(\text{negative crossings}) \\ &= n_+(K) - n_-(K) \end{aligned}$$

Define

$$X_K(A) = (-A)^{3w(K)} \langle K \rangle.$$

(This is R_1–invariant as well.)

The Jones polynomial

From MA3F2 Knot Theory: $J_K(t) \in \mathbb{Z}[t^{\pm 1/2}]$
Can be defined via the Kauffman bracket $\langle K \rangle \in \mathbb{Z}[A^{\pm 1}]$

$$\begin{aligned} \langle \bigcirc \rangle &= 1 \\ \langle K \sqcup \bigcirc \rangle &= (-A^2 - A^{-2})\langle K \rangle \\ \langle \times \rangle &= A\langle \rangle\!\langle \rangle + A^{-1}\langle \asymp \rangle \end{aligned}$$

This is R_2– and R_3–invariant.
Now work out the writhe

$$\begin{aligned} w(K) &= \#(\text{positive crossings}) - \#(\text{negative crossings}) \\ &= n_+(K) - n_-(K) \end{aligned}$$

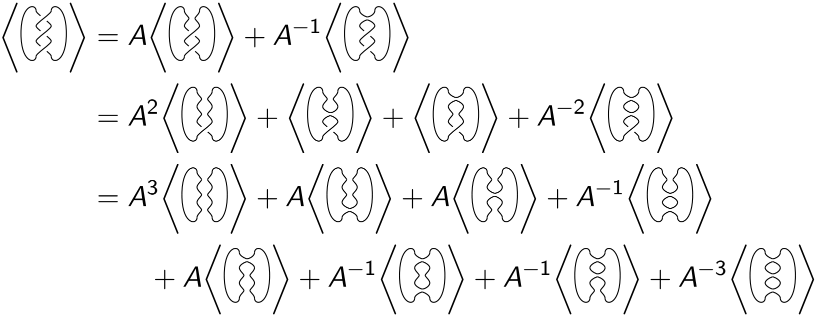
Define

$$X_K(A) = (-A)^{3w(K)}\langle K \rangle.$$

(This is R_1–invariant as well.) Finally,

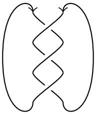
$$J_K(t) = X_K(t^{-1/4}).$$

Start by recursively calculating the Kauffman bracket:



$$
\begin{aligned}
&= A^3(-A^2 - A^{-2}) + A + A + A^{-1}(-A^2 - A^{-2}) \\
&\quad + A + A^{-1}(-A^2 - A^{-2}) + A^{-1}(-A^2 - A^{-2}) \\
&\quad + A^{-3}(-A^2 - A^{-2})^2 \\
&= -A^5 - A^{-3} + A^{-7}.
\end{aligned}
$$

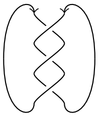
The writhe of



is $+3$, so

$$X_K(A) = (-A)^{-3\times 3}(-A^5 - A^{-3} + A^{-7}) = A^{-4} + A^{-12} - A^{-16}$$

The writhe of



is $+3$, so

$$X_K(A) = (-A)^{-3\times 3}(-A^5 - A^{-3} + A^{-7}) = A^{-4} + A^{-12} - A^{-16}$$

and hence

$$J_K(t) = X_K(t^{-1/4}) = -t^4 + t^3 + t.$$

Mikhail Khovanov introduced a slightly different formulation of the Jones polynomial.

$$\langle \bigcirc \rangle = (q + q^{-1})$$
$$\langle \times \rangle = \langle)(\rangle - q^{-1} \langle \asymp \rangle$$

Khovanov's version of the Jones polynomial

Mikhail Khovanov introduced a slightly different formulation of the Jones polynomial.

$$\langle \bigcirc \rangle = (q + q^{-1})$$
$$\langle \times \rangle = \langle)(\rangle - q^{-1} \langle \asymp \rangle$$

The unnormalised Jones polynomial

$$\hat{J}_K(q) = (-1)^{n_-} q^{n_+ - 2n_-} \langle K \rangle,$$

Mikhail Khovanov introduced a slightly different formulation of the Jones polynomial.

$$\langle \bigcirc \rangle = (q + q^{-1})$$
$$\langle \times \rangle = \langle)(\rangle - q^{-1} \langle \asymp \rangle$$

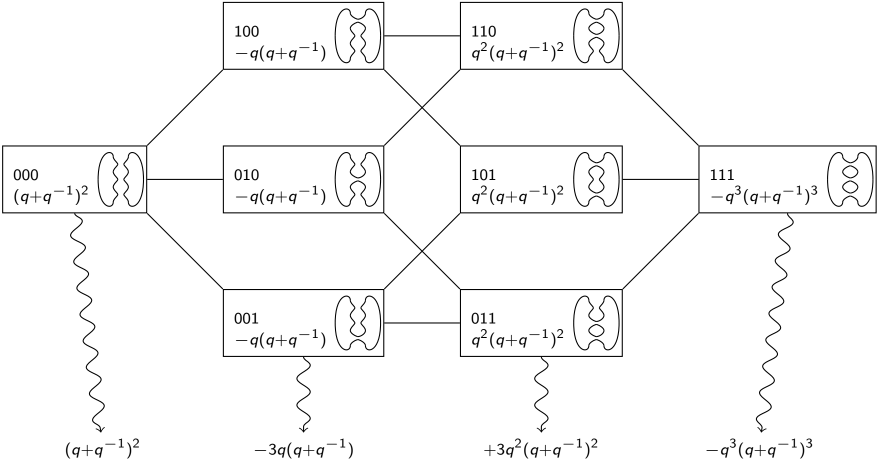
The unnormalised Jones polynomial

$$\hat{J}_K(q) = (-1)^{n_-} q^{n_+ - 2n_-} \langle K \rangle,$$

and the (original, normalised) Jones polynomial is

$$J_K(t) = \left. \frac{\hat{J}_K(q)}{q + q^{-1}} \right|_{q = -t^{1/2}}.$$

The Jones polynomial: Example



So

$$\langle K\rangle = -q^6 + q^2 + 1 + q^{-2}$$

and

$$\begin{aligned}\hat{J}_K(q) &= q^3(-q^6 + q^2 + 1 + q^{-2}) \\ &= -q^9 + q^5 + q^3 + q \\ &= (-q^8 + q^6 + q^2)(q + q^{-1})\end{aligned}$$

and hence

$$\begin{aligned}J_K(t) &= \left(-q^8 + q^6 + q^2\right)\Big|_{q=-t^{1/2}} \\ &= -t^4 + t^3 + t.\end{aligned}$$

Categorifying $J_K(t)$

How do we categorify a polynomial?

How do we categorify a polynomial?
Replace polynomials in $\mathbb{Z}[t]$ with graded vector spaces whose graded dimension is that polynomial.

How do we categorify a polynomial?
Replace polynomials in $\mathbb{Z}[t]$ with graded vector spaces whose graded dimension is that polynomial.
We want a "homology theory" for knots whose "Euler characteristic" is the Jones polynomial.

How do we categorify a polynomial?

Replace polynomials in $\mathbb{Z}[t]$ with graded vector spaces whose graded dimension is that polynomial.

We want a "homology theory" for knots whose "Euler characteristic" is the Jones polynomial.

First we need a way of turning smoothed knot diagrams into graded vector spaces.

Let nCob be the category of *n*–cobordisms:

- Objects are closed $(n-1)$–manifolds.
- Morphisms $M_1 \to M_2$ are *n*–cobordisms: *n*–manifolds W such that $\partial W = M_1 \sqcup M_2$.

Let nCob be the category of *n*–cobordisms:

- Objects are closed $(n-1)$–manifolds.
- Morphisms $M_1 \to M_2$ are *n*–cobordisms: *n*–manifolds W such that $\partial W = M_1 \sqcup M_2$.

In 2Cob:

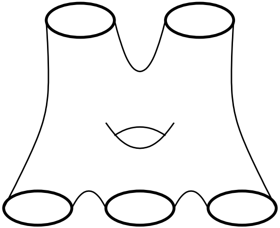
- Objects are closed 1–manifolds (disjoint unions of circles).
- Morphisms are surfaces.

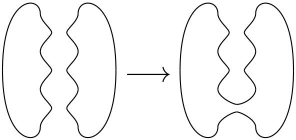
Let nCob be the category of n–cobordisms:

- Objects are closed $(n-1)$–manifolds.
- Morphisms $M_1 \to M_2$ are n–cobordisms: n–manifolds W such that $\partial W = M_1 \sqcup M_2$.

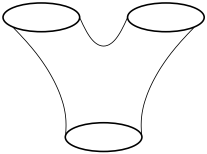
In 2Cob:

- Objects are closed 1–manifolds (disjoint unions of circles).
- Morphisms are surfaces.

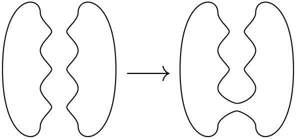




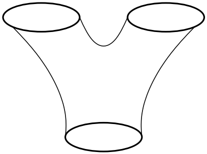
is really just



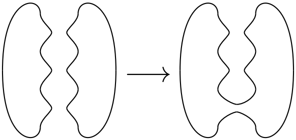
So Khovanov's Kauffman bracket gives us an n–cube of cobordisms.



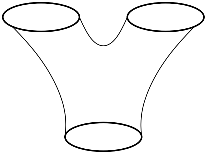
is really just



So Khovanov's Kauffman bracket gives us an n–cube of cobordisms. We want to turn this into an n–cube of (graded) linear maps between (graded) vector spaces.



is really just



So Khovanov's Kauffman bracket gives us an n–cube of cobordisms.
We want to turn this into an n–cube of (graded) linear maps between (graded) vector spaces.
We need a functor from 2Cob to $\mathrm{GrVect}_{\mathbb{C}}$.

In quantum mechanics,

- the state of a system is represented by an element of a vector space (usually a Hilbert space),

In quantum mechanics,

- the state of a system is represented by an element of a vector space (usually a Hilbert space),
- observables correspond to operators on this space,

In quantum mechanics,

- the state of a system is represented by an element of a vector space (usually a Hilbert space),
- observables correspond to operators on this space, and
- possible values of observations correspond to eigenvalues of these operators, with the subsequent state given by the corresponding eigenvector (or eigenstate).

A QFT is a general framework for describing fundamental processes or forces in physics.

- QED describes electromagnetism
- QCD describes the strong force
- The Standard Model describes EM, Weak, Strong and Higgs.

A QFT is a general framework for describing fundamental processes or forces in physics.

- QED describes electromagnetism
- QCD describes the strong force
- The Standard Model describes EM, Weak, Strong and Higgs.

General Relativity

space	$(n-1)$–manifold
spacetime	n–cobordism

composition of cobordisms

identity cobordism

Quantum mechanics

states	Hilbert space
process	linear operator

composition of operators

identity operator

Atiyah et al axiomatised QFT as a functor

$$F\colon \text{nCob} \to \text{Hilb or Vect}_k$$

What this means is that

- to each $(n-1)$–manifold we assign a vector space, and
- to each n–cobordism we assign a linear map, such that composition works and identity cobordisms correspond to identity maps.

This is a topological quantum field theory or TQFT.

TQFTs

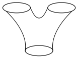
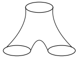
Atiyah et al axiomatised QFT as a functor

$$F\colon \mathrm{nCob} \to \mathrm{Hilb} \text{ or } \mathrm{Vect}_k$$

What this means is that

- to each $(n-1)$–manifold we assign a vector space, and
- to each n–cobordism we assign a linear map, such that composition works and identity cobordisms correspond to identity maps.

This is a topological quantum field theory or TQFT.
To define a 2–dimensional TQFT, we need to decide:

- What vector space the circle  corresponds to. (Disjoint unions $\sqcup$ correspond to tensor products $\otimes$.)
- What linear maps the cobordisms , ,  and  correspond to.

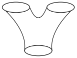
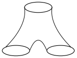
Atiyah et al axiomatised QFT as a functor

$$F\colon \text{nCob} \to \text{Hilb or Vect}_k$$

What this means is that

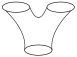
- to each $(n-1)$–manifold we assign a vector space, and
- to each n–cobordism we assign a linear map, such that composition works and identity cobordisms correspond to identity maps.

This is a topological quantum field theory or TQFT.
To define a 2–dimensional TQFT, we need to decide:

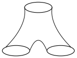
- What vector space the circle  corresponds to. (Disjoint unions $\sqcup$ correspond to tensor products $\otimes$.)
- What linear maps the cobordisms , ,  and  correspond to.

Khovanov's TQFT

Khovanov's TQFT maps

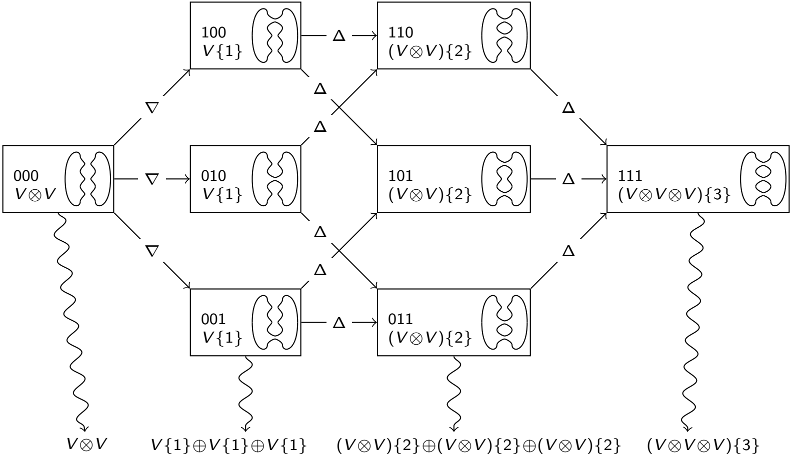
-  to a graded vector space V with one basis vector v_+ in degree $+1$ and one basis vector v_- in degree -1.
-  to $\nabla\colon V \otimes V \to V$ such that

$$v_+ \otimes v_+ \mapsto v_+, \quad v_+ \otimes v_- \mapsto v_-, \quad v_- \otimes v_+ \mapsto v_-, \quad v_- \otimes v_- \mapsto 0.$$

-  to $\Delta\colon V \to V \otimes V$ such that

$$v_+ \mapsto v_+ \otimes v_- + v_- \otimes v_+, \qquad v_- \mapsto v_- \otimes v_-.$$

This TQFT enables us to turn our cube of cobordisms into a cube of (graded) vector spaces and (graded) linear maps.



This gives a chain complex

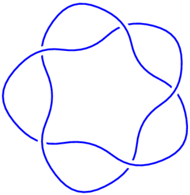
$$V_0 \xrightarrow{d_0} V_1 \xrightarrow{d_1} V_2 \xrightarrow{d_2} \cdots$$

whose homology modules $H_n(K) = \ker(d_n)/\operatorname{im}(d_{n-1})$ are Reidemeister-invariant, and whose graded Euler characteristic is J_K.

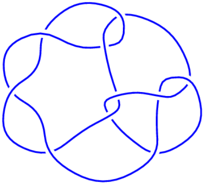
This gives a chain complex

$$V_0 \xrightarrow{d_0} V_1 \xrightarrow{d_1} V_2 \xrightarrow{d_2} \cdots$$

whose homology modules $H_n(K) = \ker(d_n)/\operatorname{im}(d_{n-1})$ are Reidemeister-invariant, and whose graded Euler characteristic is J_K. But. . .



and



both have $J_K(t) = t^2 + t^4 - t^5 + t^6 - t^7$, but different Khovanov homology modules.

Further reading

- **Joachim Kock**, *Frobenius Algebras and 2D Topological Quantum Field Theories*, LMS Student Texts 59, Cambridge University Press (2003)
- **Dror Bar-Natan**, *On Khovanov's categorification of the Jones polynomial*, Algebraic and Geometric Topology 2 (2002) 337–370 arXiv:math/0201043