The Mathematics of Knots
Orbital 2008

Nicholas Jackson

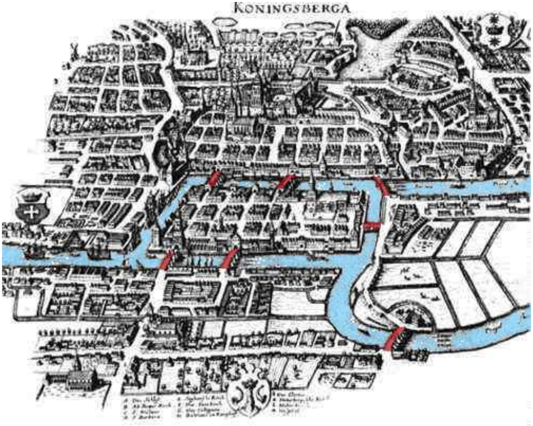
Easter 2008

What is knot theory?

Knot theory . . .

- is the study of knots from a mathematically-rigorous perspective.
- is an attempt to classify knots in 1–dimensional string (and higher-dimensional objects).
- is pure mathematics.
- is a branch of topology.

Königsberg, Prussia (now Kaliningrad, Russia) had seven bridges:



Question

Is there a route which crosses every bridge exactly once?

Answer (Leonhard Euler (1736))

No.

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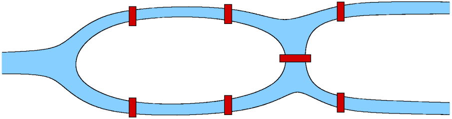
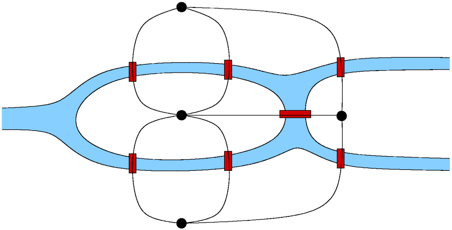
Leonhard Euler (1707–1783)

- Swiss, although spent much of his career in St Petersburg and Berlin
- Entered University of Basel in 1720 (doctorate 1726)
- One of the greatest mathematicians of the last few centuries . . .
- . . . certainly one of the most prolific: collected works run to 82 volumes (with more being edited)



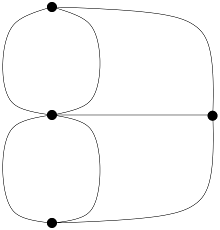
Observation 1

Exact distances and areas don't matter, only the connections, so rearrange into more convenient diagram



Observation 1

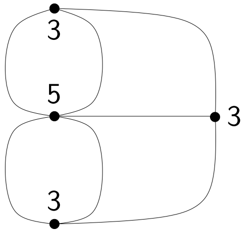
Exact distances and areas don't matter, only the connections, so rearrange into more convenient diagram, join up regions



Observation 1

Exact distances and areas don't matter, only the connections, so rearrange into more convenient diagram, join up regions, and throw away the original picture.

Topology: The bridges of Königsberg



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Each node must be visited an even number of times, except (perhaps) for the first and last (which must have the same parity).

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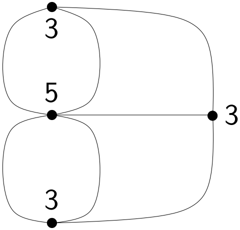

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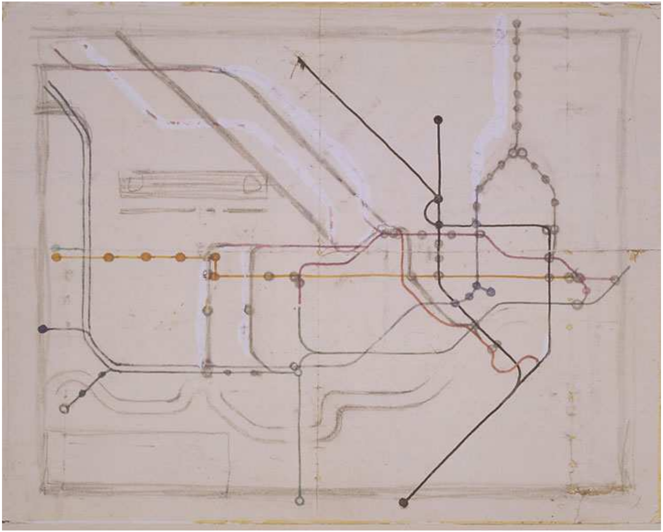
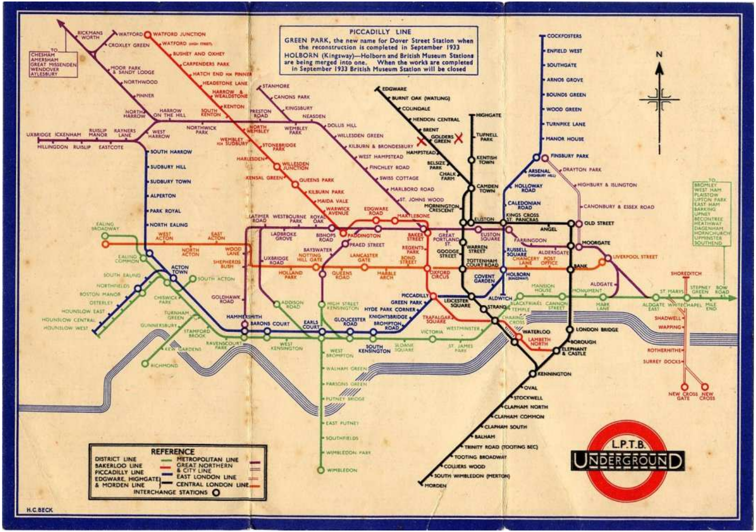
Observation 2

Each node must be visited an even number of times, except (perhaps) for the first and last (which must have the same parity). We can't do this in (18th century) Königsberg.

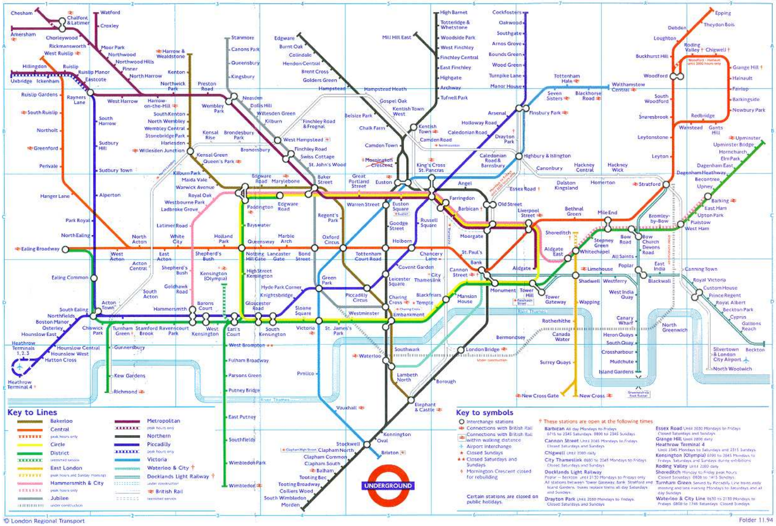


Original sketch 1931 by Harry Beck (1903–1974)



1933 version

The Great Bear



2008 version

Geographically correct version

Topology . . .

- is the study of properties of (mathematical) objects which remain invariant under continuous deformation.
- was originally known as Geometria Situs or Analysis Situs.

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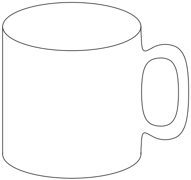

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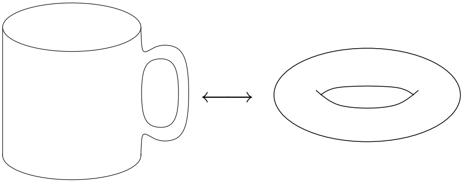

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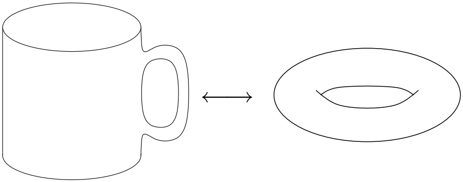
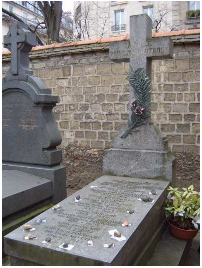


A topologist is a person who does not know the difference between a doughnut and a coffee cup. – John L Kelley

Johann Benedict Listing (1808–1882)

- Student of Carl Friedrich Gauss
- Professor of Physics, Göttingen (1839)
- Contributions to physiological optics: *Beiträge zur physiologischen Optik* (1845)
- *Vorstudien zur Topologie* (1847)
- Discovered the Möbius strip (1858)
- Nearly declared bankrupt


Jules Henri Poincaré (1854–1912)

- Mathematician, theoretical physicist and philosopher of science: "The Last Universalist".
- Work on electromagnetism, non-Euclidean geometry, number theory, dynamics (the three-body problem), relativity, . . .
- Pioneered algebraic topology (analysis situs) – use of algebraic (group theoretic) methods to solve topological problems.

Topology

Geometric topology

- graph theory
- knot theory
- 3–manifolds, 4–manifolds, n–manifolds

Differential topology

Questions about ‘smooth’ structures and deformations.

Algebraic topology

Use algebraic (group theoretic, categorical) machinery to answer topological questions

- fundamental group $\pi_1(X)$
- higher homotopy groups $\pi_n(X)$
- homology and cohomology groups $H_n(X)$ and $H^n(X)$
- the Poincaré Conjecture

- Ancient Phrygian prophecy: whoever unties the knot will become king of Asia Minor.
- 333BC: Alexander the Great cuts the knot with his sword.
- Hellenic IV Army Corps: "Solve the knot with the sword"

One version of the legend says that the rope's ends were woven together, preventing the knot from being untied.

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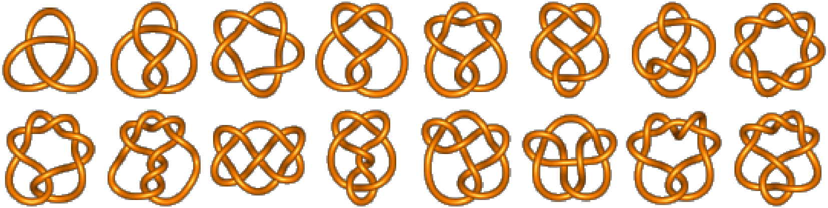
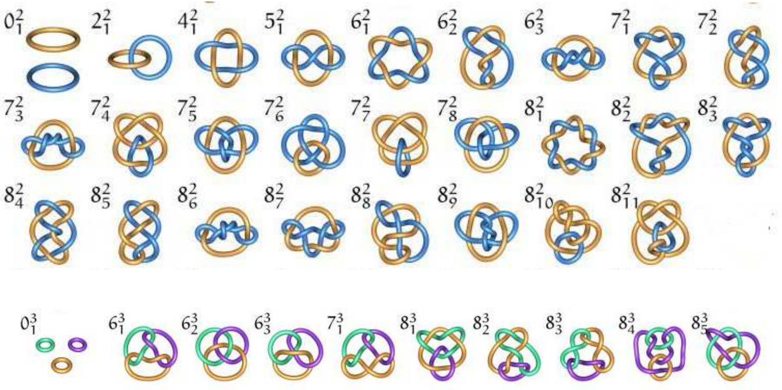
This is the sort of knot that mathematicians study.



(More precisely: "ambient isotopy classes of embeddings $S^1 \hookrightarrow S^3$")

Generalisation: consider more than one (knotted, linked) circle.



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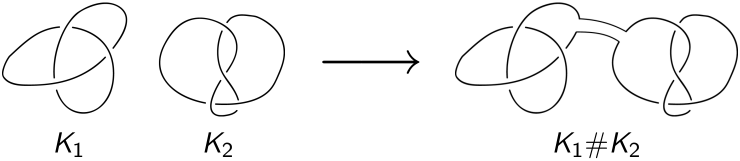

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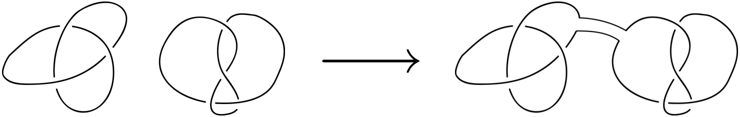
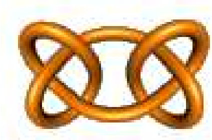
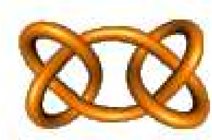


K_1 K_2 $K_1 \# K_2$

For example, the reef knot and the granny knot:



Reef knot
$3_1 \# 3_1$



Granny knot
$3_1 \# \bar{3}_1$

Conjecture (William Thomson, 1875)

Atoms are knots in the field lines of the æther.

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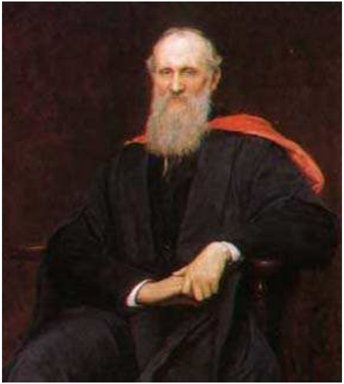


William Thomson (1824–1907)

- Irish mathematical physicist and engineer
- Studied at Glasgow (1834–1841) and Peterhouse, Cambridge (1841–1845)
- Professor of Natural Philosophy, Glasgow (1846)
- President of the Royal Society (1890–1895)
- 1st Baron Kelvin of Largs (1892)

Kelvin's theory of vortex atoms inspired serious attempts at classifying and enumerating knots.

Peter Guthrie Tait (1831–1901)

Thomas Penyngton Kirkman (1806–1895)

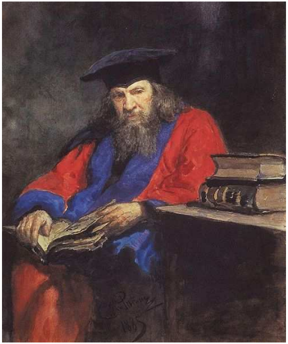
Also Charles Newton Little (1858–1923). Between them, they produced tables of all knots with up to eleven crossings.

Collapse of the Vortex Atom theory

Kelvin's theory collapsed due to the Michelson–Morley experiment (which disproved the existence of the æther) and the lack of any clear correlation between Tait, Kirkman and Little's knot tables and Mendeleev's periodic table.

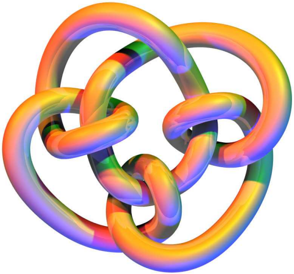
Albert Michelson (1852–1931)

Edward Morley (1838–1923)



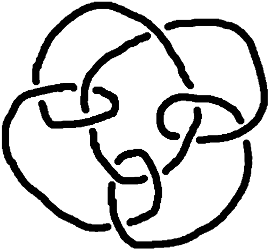
Dmitri Mendeleev (1834–1907)

3–dimensional knots aren't so easy to work with, so we represent them as 2–dimensional diagrams:



Knot (10_{75})

$\longleftrightarrow$



Diagram

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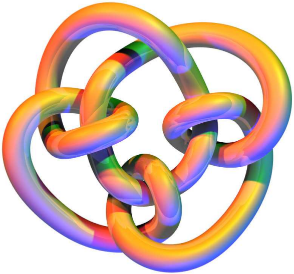

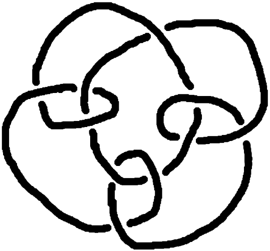

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allowed manipulations	$\longleftrightarrow$	allowed moves

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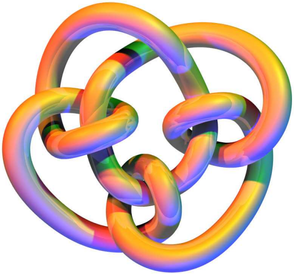
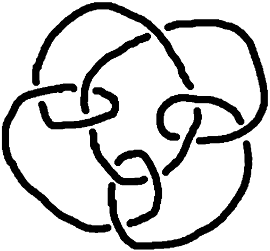


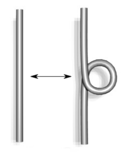


Knot (10_{75})	$\longleftrightarrow$	Diagram
allowed manipulations	$\longleftrightarrow$	allowed moves
ambient isotopy	$\longleftrightarrow$	Reidemeister moves

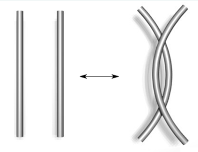
Reidemeister moves

Theorem (Reidemeister (1927))

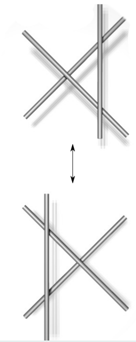
Diagrams representing equivalent knots are related by a finite sequence of moves of the following types:



type I



type II



type III

Kurt Reidemeister
(1893–1971)

1921 PhD, Hamburg

1927 Professor, Königsberg

1933 ‘politically unsound’

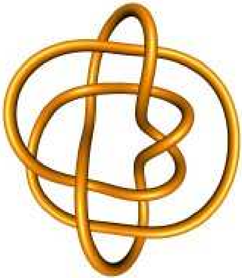
The unknot

It isn't always clear when a knot is trivial:

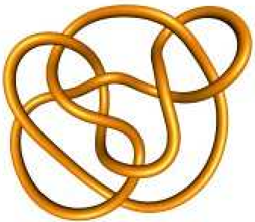
$$\simeq$$

(Possible application: a trap for vampires.)

It also isn't always easy to tell whether two knots are the same or different:



$\simeq$



- Believed distinct by Tait, Kirkman, Little, Conway, Rolfsen, ...
- 1974: Kenneth A Perko Jr showed they're the same.

We need something to do the hard work for us...

We want something that can be (relatively) easily calculated from a knot diagram, but which isn't changed by Reidemeister moves. This is called an invariant.

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More precisely:

Given two knot diagrams D_1 and D_2 (representing two knots K_1 and K_2) and an invariant f, then

$$f(D_1) = f(D_2)$$

if K_1 and K_2 are equivalent (that is, if D_1 and D_2 are related by a finite sequence of Reidemeister moves).

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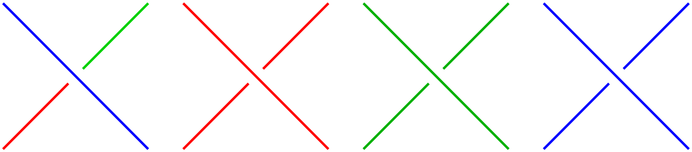
We haven't yet said what kind of object f is.
In practice, it might be a number, a polynomial (eg $z^2 - 1$ or $t^2 - 1 + t^{-2}$) or something more sophisticated (a group, a rack or quandle, a sequence of graded homology modules, ...).

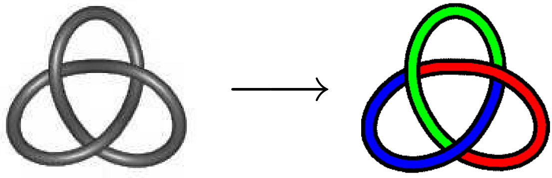
A knot diagram can be 3–coloured if we can colour each arc such that

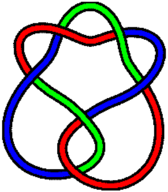
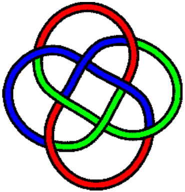
- each arc is assigned a single colour
- exactly three colours are used
- at each crossing, either all the arcs have the same colour, or arcs of all three colours meet

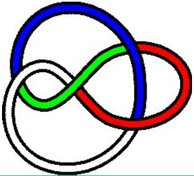


3–COLOURING THE TREFOIL

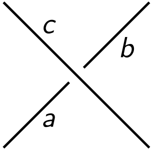
The trefoil (3_1) can be 3–coloured:



as can 6_1:  and 8_{18}: 

but the figure-eight knot (4_1) can't: 

Generalise 3–colouring:



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$$a + b \equiv 2c \pmod{n}.$$

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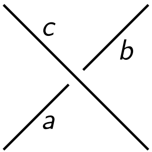

Take a knot diagram and label ('colour') each arc with a number, such that, at each crossing,

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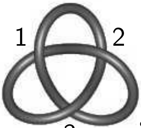
If we can consistently (and nontrivially) label the entire diagram like this, then the knot is n–colourable, or has colouring number n. Colouring numbers are invariant under Reidemeister moves.

Related to the colouring numbers is the determinant: Label each arc and crossing:



Construct a matrix

	a	b	c
1	-1	-1	2
2	2	-1	-1
3	-1	2	-1

$$\longrightarrow A_+ = \begin{bmatrix} -1 & -1 & 2 \\ 2 & -1 & -1 \\ -1 & 2 & -1 \end{bmatrix}$$

and delete one column and one row (it doesn't matter which):

$$A = \begin{bmatrix} -1 & -1 \\ 2 & -1 \end{bmatrix}$$

Then for any knot K, we define $\det(K) := |\det(A)|$.
So

$$\det(3_1) = \left|\begin{smallmatrix} -1 & -1 \\ 2 & -1 \end{smallmatrix}\right| = |(-1) \times (-1) - (-1)(2)| = 3.$$

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Theorem

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The determinant

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Theorem

$$\det(K_1 \# K_2) = \det(K_1)\det(K_2)$$

James Waddell Alexander II (1888–1971)

- Pioneer of algebraic topology
- Princeton, Institute for Advanced Study
- Keen mountaineer
- 'Politically unsound'

The Alexander polynomial $\Delta_K(t)$ of a knot K is defined up to multiplication by $\pm t^n$, and is invariant under Reidemeister moves.

EXAMPLE

$$\Delta_{3_1}(t) = t - 1 + t^{-1}$$

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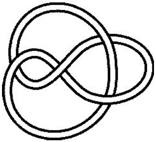
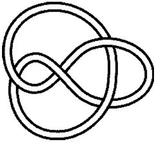
This is an extension of the determinant:

THEOREM

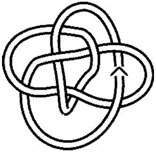
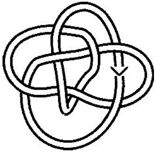
$$|\Delta_K(-1)| = \det(K)$$

Reflection and reversion

Some knots K are equivalent to their mirror image $\overline{K}$ – we call these amphicheiral or achiral.

$4_1 =$  $\simeq$  $= \overline{4_1}$

Some (oriented) knots K are equivalent to their orientation inverse $-K$ – we call these invertible or reversible.

$8_{17} =$  $\not\simeq$  $= -8_{17}$

Theorem

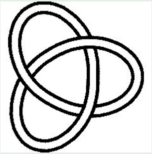
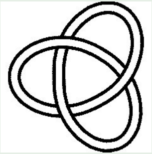
$$\Delta_K(t) = \Delta_{\overline{K}}(t^{-1}) = \Delta_{-K}(t^{-1})$$

So, the Alexander polynomial can't always tell the difference between a knot and its mirror image or orientation inverse:

Example

Δ can't distinguish the right- and left-handed trefoils:

$$\Delta_{3_1}(t) = t - 1 + t^{-1} = \Delta_{\overline{3_1}}(t^{-1})$$

$3_1 =$  $\neq$  $= \overline{3_1}$

John Conway (1937–)

- Cambridge, Princeton
- Number theory, group theory, knot theory, cellular automata ('Life'), game theory, . . .
- Monstrous Moonshine

The Conway polynomial

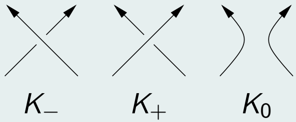
John Conway (1937–)

- Cambridge, Princeton
- Number theory, group theory, knot theory, cellular automata (‘Life’), game theory, . . .
- Monstrous Moonshine

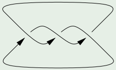
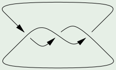
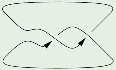
Skein relation

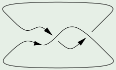
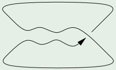
$$\nabla_{K_+} - \nabla_{K_-} = z\nabla_{K_0}$$

where $\nabla_{\bigcirc} = 1$, ($\Longrightarrow$ $\nabla_{\bigcirc\bigcirc} = 0$), and



Example

∇  $= \nabla$  $+ z\nabla$ 

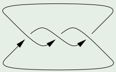
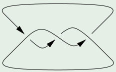
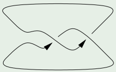
$= 1 + z \Big($ ∇  $+ z\nabla$  $\Big)$

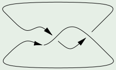
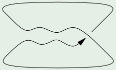
$$= 1 + z(0 + z)$$

$$= 1 + z^2$$

The Conway polynomial

Example

∇  $= \nabla$  $+ z\nabla$ 

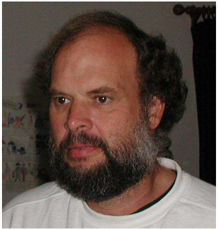
$= 1 + z \Big(\nabla$  $+ z\nabla$  $\Big)$

$= 1 + z(0 + z)$

$= 1 + z^2$

Theorem

$$\Delta_K(t^2) = \nabla_K(t - t^{-1})$$



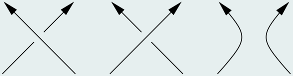
Vaughan F R Jones (1952–)

- New Zealand
- PhD, Geneva 1979
- Fields Medal, Kyoto 1990
- DCNZM 2002

SKEIN RELATION

$$(t^{1/2} - t^{-1/2})V_{K_0} = t^{-1}V_{K_+} - tV_{K_-}$$

where $V_{\bigcirc} = 1$ and



K_- K_+ K_0

The Jones polynomial

Theorem

$$V_K(t) = V_{\overline{K}}(t^{-1}), \qquad V_{K\#L} = V_K V_L$$

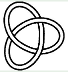
The Jones polynomial

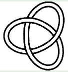
Theorem

$$V_K(t) = V_{\overline{K}}(t^{-1}), \qquad V_{K\#L} = V_K V_L$$

The Jones polynomial, unlike the Alexander polynomial (and hence the determinant) can distinguish the left- and right-handed trefoils:

Example

$$V_{3_1} = V_{\text{(knot)}}$$

$$= t^{-1} + t^{-3} - t^{-4}$$

$$V_{\overline{3_1}} = V_{\text{(knot)}}$$

$$= t^{1} + t^{3} - t^{4}$$

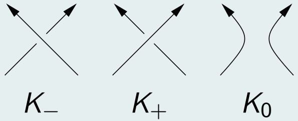
The HOMFLY(PT) polynomial

A two-variable polynomial invariant devised, independently, by Jim Hoste, Adrian Ocneanu, Kenneth Millett, Peter Freyd, Raymond Lickorish, David Yetter, Jozef Przytycki and Pawel Traczyk.

Skein relation

$$a^{-1}P_{K_+}(a,z) - aP_{K_-}(a,z) = zP_{K_0}(a,z)$$

where $P_{\bigcirc} = 1$ and



This encapsulates both the Alexander and Jones polynomials:

Theorem

$$V_K(t) = P_K(t, t^{1/2} - t^{-1/2}), \qquad \Delta_K(t) = P_K(1, t^{1/2} - t^{-1/2})$$

The Jones polynomial can be interpreted as a function $V_K : \mathbb{C} \to \mathbb{C}$:

• Colour (label) the knot (or link) diagram with the fundamental representation W of the quantised enveloping algebra $U_q(sl_2)$ of the Lie algebra sl_2.
• Read up the page, interpreting $\bigcup$ as a map $\mathbb{C} \to W \otimes W^*$, a crossing $\times$ as an isomorphism $W \otimes W \to W \otimes W$, and $\bigcap$ as a map $W^* \otimes W \to \mathbb{C}$.
• If you define these maps appropriately, the composite function $\mathbb{C} \to \mathbb{C}$ is the Jones polynomial.

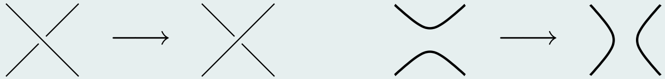
Replacing W with a representation of a different quantum group (quasitriangular Hopf algebra) yields a different knot invariant.

- A homology theory KH_* for knots and links, whose graded Euler characteristic is the Jones polynomial.
- $KH_*(K)$ contains more information than V_K – there are non-equivalent links which have the same Jones polynomials, but whose Khovanov homology is different.
- Similar theories developed for the Alexander polynomial Δ_K (Ozsváth–Szabó's knot Floer homology), the HOMFLYPT polynomial, and the sl_3 quantum invariant (Khovanov–Rozansky homology).
- General technique called categorification.

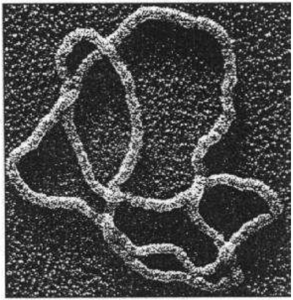
Other topics

- Finite-type (Vassiliev) invariants
- The Kontsevich integral
- Witten's QFT interpretation of the Jones polynomial
- Higher-dimensional knots (knotted spheres in 4–space)
- Racks and quandles, cocycle state-sum invariants

Two enzymes, Topoisomerase I and II, act on strands of DNA, performing crossing changes and smoothing:



This causes the DNA to become knotted, linked and/or supercoiled:



Credits

- http://www.knotplot.com/
- http://katlas.math.toronto.edu/
- http://www-groups.dcs.st-and.ac.uk/˜history/