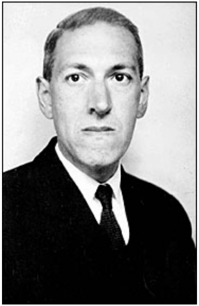
"The Geometry of the Place Was All Wrong"

Odyssey 2010

Nicholas Jackson

Easter 2010

A word from our sponsors



He talked of his dreams in a strangely poetic fashion; making me see with terrible vividness the damp Cyclopean city of slimy green stone – whose geometry, he oddly said, was *all wrong* – and hear with frightened expectancy the ceaseless, half-mental calling from underground: "Cthulhu fhtagn", "Cthulhu fhtagn."

– H P Lovecraft, *The Call of Cthulhu* (1926)

Euclid of Alexandria

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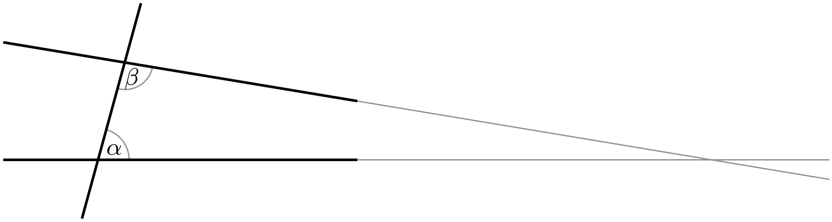
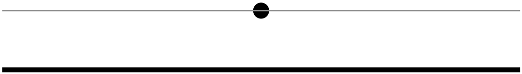
II. Any finite straight line may be extended infinitely.
III. There is a unique circle with a given centre and radius.
IV. All right angles are equal to one another.
V. The Parallel Postulate: If two lines intersect a third such that $\alpha + \beta < 180^\circ$, then the two lines must intersect each other if extended far enough.

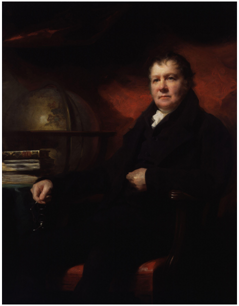


Equivalently:

PLAYFAIR'S AXIOM

At most one line can be drawn through any point not on a given line parallel to the given line.





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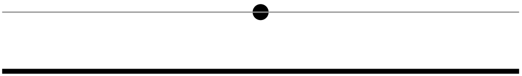

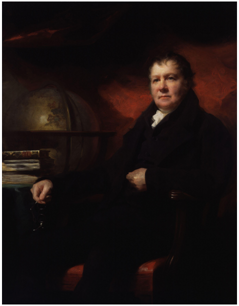

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- Due to William Ludlam (1718–1788) and Proclus Lycaeus (412–485).

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- John Wallis (1663): There exist similar triangles of different sizes.




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- Georg Simon Klügel (1763) expressed doubts that it could be proved.
- Carl Friedrich Gauss (c.1816) was convinced that it couldn't be proved.




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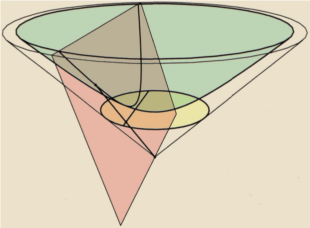
> If I commenced by saying that I am unable to praise this work, you would certainly be surprised for a moment. But I cannot say otherwise. To praise it would be to praise myself. Indeed the whole contents of the work, the path taken by your son, the results to which he is led, coincide almost entirely with my meditations, which have occupied my mind partly for the last thirty or thirty-five years.
>
> – Gauss to Farkas Bolyai

The Beltrami–Klein model



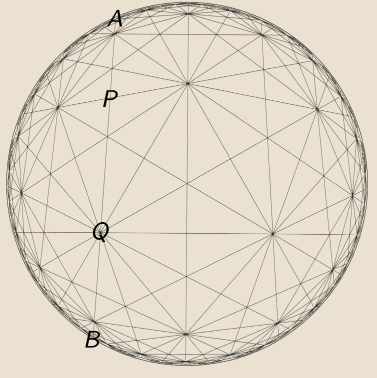


Eugenio Beltrami (1868) exhibited:

- Surfaces in Euclidean 3–space $\mathbb{E}^3$ on which the geometry is "non-Euclidean", and
- models of a consistent non-Euclidean geometry within Euclidean geometry.

One of these models, later refined by Felix Klein (1849–1925), is based on the disc in the Euclidean plane $\mathbb{E}^2$, but with a different distance function (or metric) and correspondingly different notions of "straight lines" (or geodesics).

The Beltrami–Klein model

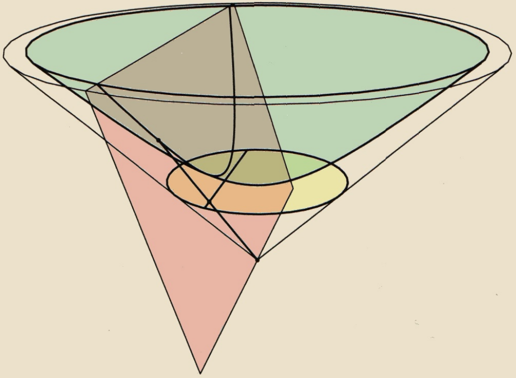


$$d(P,Q) = \frac{1}{2}\log\frac{|AQ||PB|}{|AP||QB|}$$

In particular:

- $d(P,Q) = 0$ exactly when P and Q coincide
- $d(P,Q) > 0$ otherwise

$$\{(x_0, x_1, x_2) : x_0^2 - x_1^2 - x_2^2 = 1, x_0 > 0\}$$

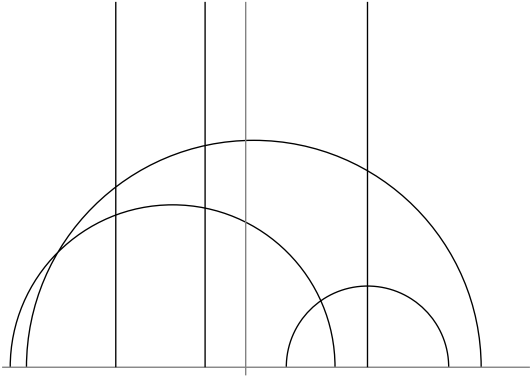


$$d(\mathbf{x}, \mathbf{y}) = \cosh^{-1}(\mathbf{x} \cdot \mathbf{y})$$

where

$$\mathbf{x} \cdot \mathbf{y} = x_0 y_0 - x_1 y_1 - x_2 y_2$$

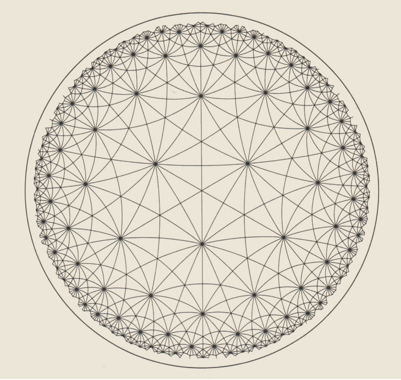
$$\mathbb{H} = \{(x, y) : y > 0\}$$



$$ds^2 = \frac{dx^2 + dy^2}{y^2}$$

Geodesics are vertical line segments, or arcs of circles centred on the horizontal axis.

$$\mathbb{D} = \{(x, y) : x^2 + y^2 < 1\}$$



$$ds^2 = \frac{2(dx^2+dy^2)}{(1-(x^2+y^2))^2}$$

Geodesics are radii and arcs which intersect the boundary circle perpendicularly.

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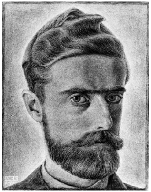
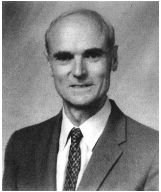
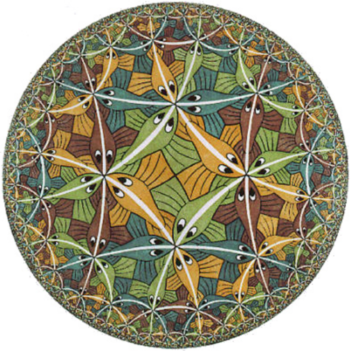
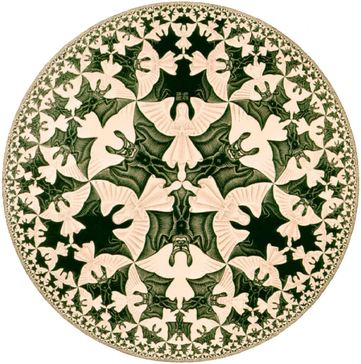
- Angles of hyperbolic triangles add up to $< 180^\circ$.
- In addition, we have spherical or elliptic geometry, already known and understood for centuries by maritime navigators. Angles of spherical triangles add up to $> 180^\circ$, and geodesics are great circles.
- In between, we have ordinary, flat, Euclidean geometry, in which the angles of a triangle add up to exactly 180°, and geodesics are ordinary straight lines.



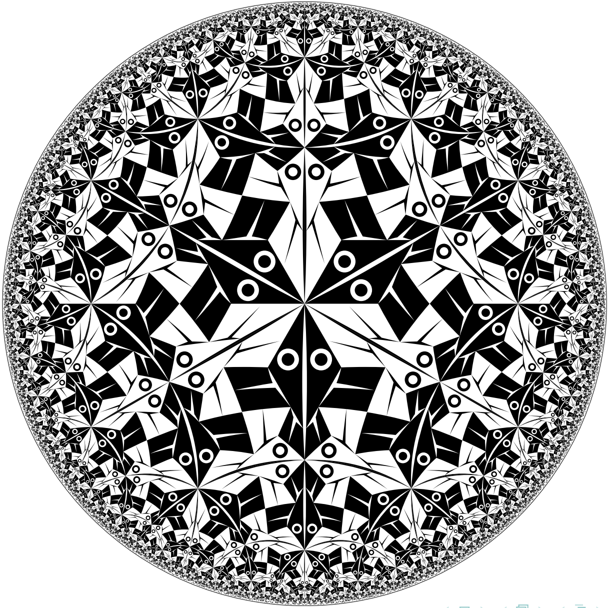
The Dutch artist Maurits Cornelis Escher (after discussion with the British/Canadian mathematician H S M Coxeter) used tessellations of the Poincaré disc model of the hyperbolic plane as the basis for his *Circle Limit* series of woodcuts.





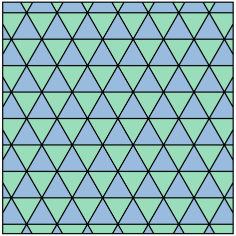


Circle Limits

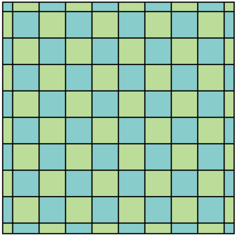


Euclidean tessellations

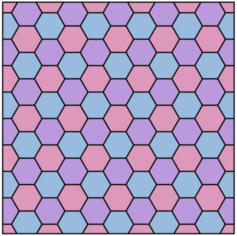
In the Euclidean plane, there are three regular tessellations:



Triangular
$\{3, 6\}$



Square
$\{4, 4\}$



Hexagonal
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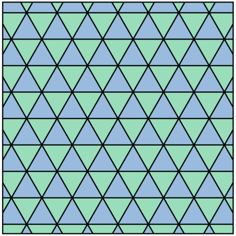

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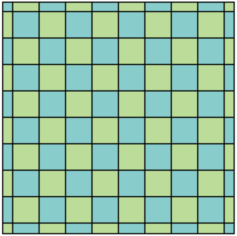

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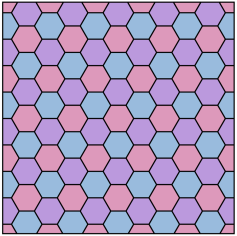

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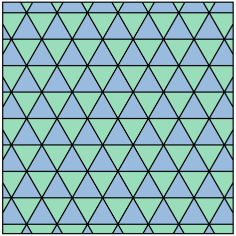

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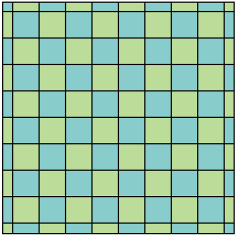

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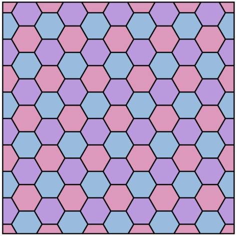

Hexagonal
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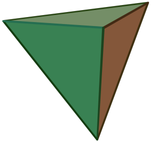
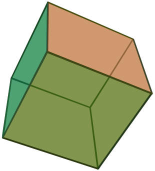
In the Euclidean case, we need $(p-2)(q-2)=4$.

Spherical tessellations

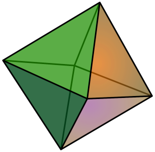
On the 2–sphere $\mathbb{S}^2$, there are five regular tessellations, each corresponding to one of the regular (Platonic) polyhedra:



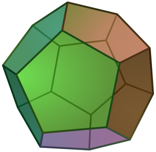
Tetrahedron
$\{3, 3\}$



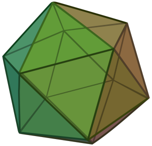
Cube
$\{4, 3\}$



Octahedron
$\{3, 4\}$



Dodecahedron
$\{5, 3\}$



Icosahedron
$\{3, 5\}$

In spherical geometry, we require that $(p - 2)(q - 2) < 4$.

There are infinitely many regular tessellations of the hyperbolic plane.
These have Schläfli symbols $\{p, q\}$ (q regular p–gons meeting at each vertex) where p and q satisfy

$$(p-2)(q-2) > 4.$$

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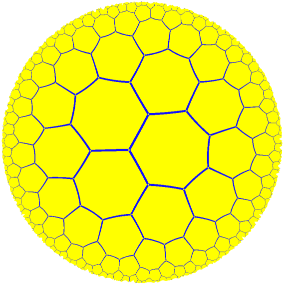
The simplest cases are $\{7, 3\}$ (three heptagons meeting at each vertex) and $\{3, 7\}$ (seven triangles meeting at each vertex).



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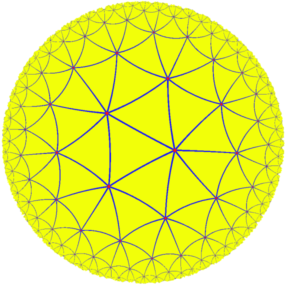

$\{3, 7\}$

Things get a bit more complicated/interesting when we move into three dimensions.

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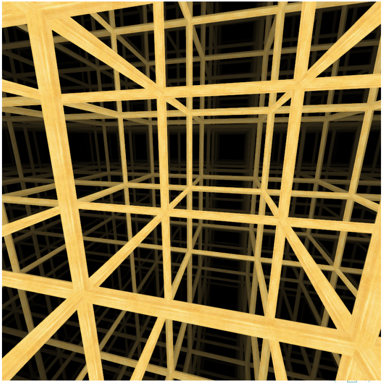
In Euclidean 3–space $\mathbb{E}^3$ there is just one regular tessellation: the cubic honeycomb $\{4, 3, 4\}$



The three-sphere

The higher-dimensional analogue of the 2–sphere is the 3–sphere $\mathbb{S}^3$. This can be constructed in a number of different ways:

- The subset of $\mathbb{E}^4$ consisting of points of unit distance from the origin: $\mathbb{S}^3 = \big\{(x, y, z, w) : x^2 + y^2 + z^2 + w^2 = 1\big\} \subset \mathbb{E}^4$

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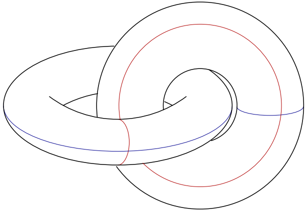
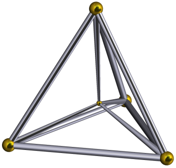
- Glue two 3–dimensional solid balls along their (2–sphere) surfaces.
- Glue two solid tori along their surfaces, gluing meridians to longitudes and vice-versa:



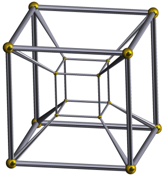
What we end up with is something which is locally 3–dimensional, but which curves back on itself in a way which $\mathbb{E}^3$ doesn't.

Tessellations of $\mathbb{S}^3$

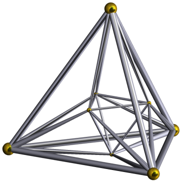
As with $\mathbb{S}^2$, these correspond to the 4–dimensional regular polytopes or polychora. There are six of them:



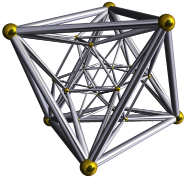
4–simplex
$\{3, 3, 3\}$



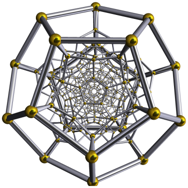
Hypercube
$\{4, 3, 3\}$



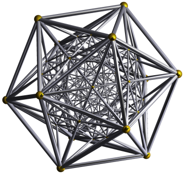
4–cross
$\{3, 3, 4\}$



24–cell
$\{3, 4, 3\}$



120-cell
$\{5, 3, 3\}$



600-cell
$\{3, 3, 5\}$

Construct 3–dimensional analogues of the 2–dimensional models to get models for $\mathbb{H}^3$. For example, the Poincaré ball model or the half-space model.

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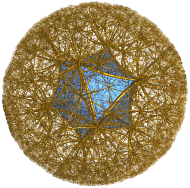
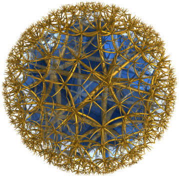
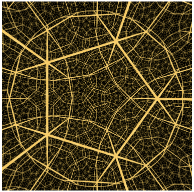
There are four regular tessellations of hyperbolic 3–space $\mathbb{H}^3$:



$\{3,5,3\}$



$\{4,3,5\}$



$\{5,3,4\}$

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More generally, we can impose geometric structures on manifolds.

A manifold is an object which is *locally* topologically equivalent (that is, modulo continuous deformations) to ordinary n–dimensional space.

Examples: surfaces, curves, knots, Poincaré dodecahedral space, lens spaces $L(p, q)$, 4–dimensional curved spacetime, etc.

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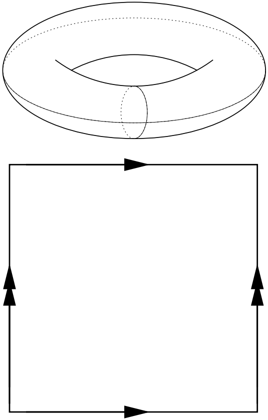
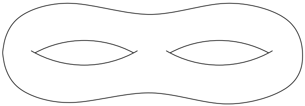
Equivalently:

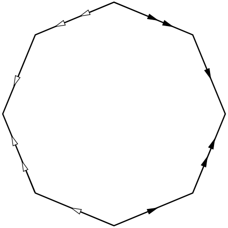
An n–dimensional manifold is Euclidean (or spherical, hyperbolic, etc) if it is locally isometric to $\mathbb{E}^n$ (or $\mathbb{S}^n$, $\mathbb{H}^n$, etc).

The only compact surfaces which admit Euclidean structures are the torus and the Klein bottle:



Many surfaces admit hyperbolic structures, starting with the double-torus:





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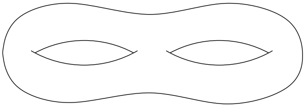

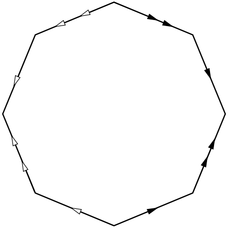

Theorem

There are exactly three two-dimensional model geometries: Euclidean $\mathbb{E}^2$, spherical $\mathbb{S}^2$ and hyperbolic $\mathbb{H}^2$.

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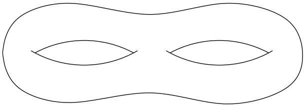

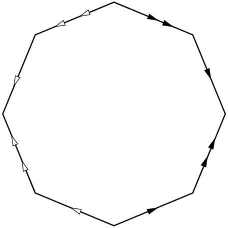

THEOREM

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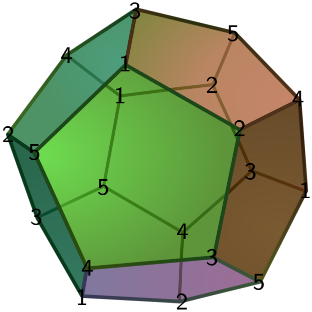
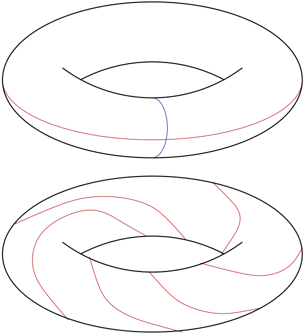
THEOREM (UNIFORMISATION THEOREM)

Every surface admits one of the above three geometric structures.

Poincaré dodecahedral space and lens spaces $L(p,q)$ admit spherical structures.





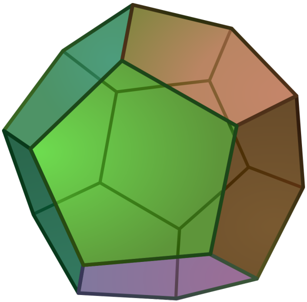
Poincaré dodecahedral space

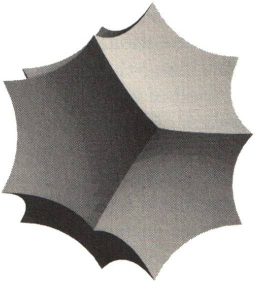
Identify opposite faces of a solid dodecahedron with a $\frac{1}{10}$ twist.

Lens spaces $L(p,q)$

Glue two solid tori along their surfaces, attaching the meridian of one to a (p,q) torus knot.

Seifert–Weber dodecahedral space admits a hyperbolic structure.



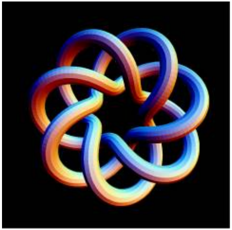


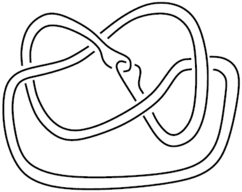
Identify opposite faces of a solid dodecahedron with a $\frac{3}{10}$ twist. The thirty edges are identified in six groups of five, and all twenty of the vertices are glued together.

Let K be a knot embedded in the 3–sphere S^3. Thicken K and remove it, so we're left with S^3 with a K–shaped tubular hole in it.

Theorem (William Thurston)

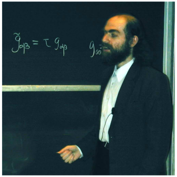
The complement $S^3 \setminus K$ admits a hyperbolic structure unless K is either a torus or a satellite knot.





Most knots are hyperbolic: of the 1 701 936 prime knots with 16 or fewer crossings, only twelve are torus, and twenty are satellites.

Three-dimensional geometries



Theorem (William Thurston)

There are eight three-dimensional model geometries: $\mathbb{E}^3$, $\mathbb{S}^3$, $\mathbb{H}^3$, $\mathbb{S}^2 \times \mathbb{E}^1$, $\mathbb{H}^2 \times \mathbb{E}^1$, $\widetilde{SL}(2, \mathbb{R})$, Nil *and* Sol.

Thurston's Geometrisation Conjecture

Every oriented, prime, closed 3–manifold can be decomposed into toroidal components, each of which admits one of these geometries.

Thurston was awarded a Fields Medal in 1982 for his work on three-dimensional geometry. The Geometrisation Conjecture was proved by Grigori Perelman in 2003; he was offered, but declined, a Fields Medal for his work.

References

H S M Coxeter, *The Beauty of Geometry*, Dover (1999)
Euclid, *Elements* (c.300BC)
Jeremy Gray, *Worlds Out of Nothing*, Springer (2007)
H P Lovecraft, *The Call of Cthulhu*, Weird Tales (1928)
William Thurston, *Three-Dimensional Geometry and Topology*, Princeton University Press (1997)