Homological algebra for the MTV generation

Homological algebra is
'algebraic topology without the topology'

Algebraic topology. . .

Homology theory – given a topological space X, assign a series of (abelian) groups $H_n(X)$ in a well-defined way.

In particular, $X \cong Y \Rightarrow H_n(X) \cong H_n(Y)$.

Enables us to answer (some) topological questions (and ask lots more) by turning them into group-theoretic questions instead.

$H_n(X)$ (or $H_n(X; \mathbb{Z})$) somehow encodes the n-dimensional structure of X.

In particular, it ‘counts’ (with integers) the n-dimensional ‘holes’ in X.

Homology with coefficients

Use a different abelian group than $\mathbb{Z}$ to 'count' with.

For example: if X is an n-manifold, then $H_n(X; \mathbb{Z}_2)$ tells us about the orientability of X.

Cohomology

Conceptually similar, but dual ('the arrows go the other way').

We get a collection of groups $H^n(X; A)$, related in various interesting ways to the homology groups.

For example, if X is an n-manifold, the **Poincaré duality theorem** says that

$$H_i(X; A) \cong H^{n-i}(X; A)$$

. . . without the topology

Homology and cohomology provide a series of 'functors' from the category of topological spaces to the category of (abelian) groups:

$$H_*, H^* : \mathsf{Top} \to \mathsf{Ab}$$

Instead: Define similar theories

$$H_*, H^* : \mathsf{Group} \to \mathsf{Ab}$$

Why? $H_n(G; A)$ and $H^n(G; A)$ encode information about the structure of G.

For example: $\mathrm{Ext}(G, A) = H^2(G; A)$ classifies 'extensions' of G by the G-module A.

Also: corresponding theories for associative algebras, Lie algebas, commutative algebras and so on.

Is there a topological description for all this?

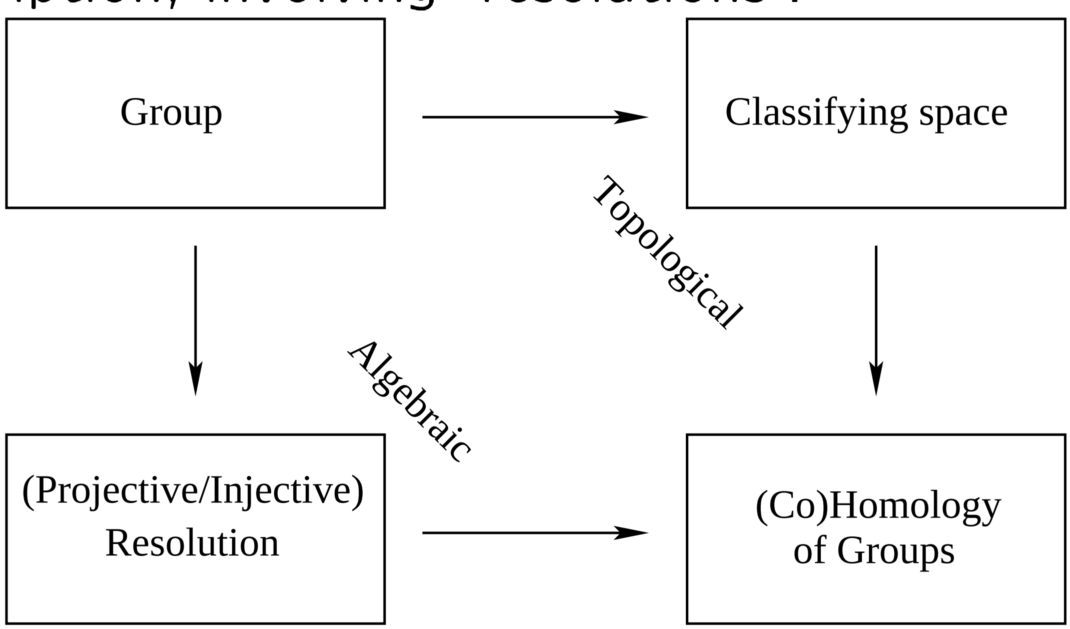
Yes: There's a well-defined way of constructing a 'classifying space' BG for G.

$$H_n(G; A) = H_n(BG; A)$$

and

$$H^n(G; A) = H^n(BG; A)$$

There's also a more abstract, algebraic description, involving 'resolutions'.



What I'm trying to do

There's a category of algebraic objects called 'racks' (and more specialised objects called 'quandles') which have applications in knot theory.

We have a nice concept of a **rack space** BX for a rack X, corresponding to the classifying space for a group.

I'm trying to devise a corresponding algebraic theory involving (some suitable generalisation of) resolutions.