What I Did On My Holidays

or

A Cautionary Tale For Graduate Students

by

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Sir Michael Atiyah: Algebra is a kind of Faustian pact – we sell our souls, giving up our geometric intuition for the promise of powerful algebraic machinery.

Algebraic topology: Turn a geometric or topological question into an algebraic question.

Homological algebra: Basically algebraic topology without the topology.

Homology theory

Chain complex: sequence of Abelian groups (or modules)

$$\mathbf{C} : \ldots \xrightarrow{d_{n+2}} C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \ldots$$

where $\operatorname{im} d_{n+1} \leqslant \ker d_n$ ($\Leftrightarrow d_n d_{n+1} = 0$).

Homology groups:

$$H_n(\mathbf{C}) := \ker d_n / \operatorname{im} d_{n+1}$$

Elements of $\ker d_n$ are n–**cycles**, and elements of $\operatorname{im} d_n$ are n–**boundaries**.

Exact sequence: chain complex with trivial homology. So $H_n(\mathbf{C})$ measures how much $\mathbf{C}$ fails to be exact at the nth step.

Cochain complex: 'dual' of a chain complex:

$$\mathbf{C} : \ldots \xleftarrow{d^{n+1}} C_{n+1} \xleftarrow{d^n} C_n \xleftarrow{d^{n-1}} C_{n-1} \xleftarrow{d^{n-2}} \ldots$$

Cohomology groups:

$$H^n(\mathbf{C}) := \ker d^n / \operatorname{im} d^{n-1}$$

Elements of $\ker d^n$ are n–**cocycles**, and elements of $\operatorname{im} d^n$ are n–**coboundaries**.

CW homology (J H C Whitehead)

Topological space X. Let $C_n(X)$ be the free Abelian group generated by the n–cells of (a CW complex homeomorphic to) X.

Define **boundary map** $\partial_n \colon C_n \to C_{n-1}$ which maps an n–cell in C_n to the oriented formal sum of $(n-1)$–cells which bound it.

Elements of $\ker \partial_n$ are boundaryless 'sums' of n–cells, and elements of $\operatorname{im} \partial_{n+1}$ are ('sums' of) n–cells which bound ('sums' of) $(n+1)$–cells. We find that $\partial_n \partial_{n+1} = 0$.

So $H_n(\mathbf{C})$ 'counts' the n–dimensional 'holes' in X. Denote it $H_n(X)$.

Examples

$$H_i(S^n) = \begin{cases} \mathbb{Z} & \text{if } i = 0, n \\ 0 & \text{otherwise} \end{cases}$$

$$H_i(T^2) = \begin{cases} \mathbb{Z} & \text{if } i = 0, 2 \\ \mathbb{Z} \oplus \mathbb{Z} & \text{if } i = 1 \\ 0 & \text{otherwise} \end{cases}$$

Homology with coefficients

Don't have to use integers to count the holes. Pick any Abelian group A and use that instead.

More precisely, form the chain complex $\mathbf{C}\otimes A$:

$$\ldots \xrightarrow{d_{n+2}\otimes \mathrm{Id}} C_{n+1}\otimes A \xrightarrow{d_{n+1}\otimes \mathrm{Id}} C_n \otimes A \xrightarrow{d_n\otimes \mathrm{Id}} C_{n-1}\otimes A \xrightarrow{d_{n-1}\otimes \mathrm{Id}} \ldots$$

Denote $H_n(\mathbf{C}\otimes A)$ by $H_n(\mathbf{C}; A)$.

Cohomology

Form the *cochain* complex $\mathrm{Hom}(\mathbf{C}, A)$:

$$\ldots \xleftarrow{d^{n+2}} \mathrm{Hom}(C_{n+1}, A) \xleftarrow{d^{n+1}} \mathrm{Hom}(C_n, A) \xleftarrow{d^{n}} \mathrm{Hom}(C_{n-1}, A) \xleftarrow{d^{n-1}} \ldots$$

Elements of $\mathrm{Hom}(C_n, A)$ are homomorphisms $C_n \to A$.

The nth cohomology of this complex is denoted $H^n(X; A)$.

If $A = \mathbb{Z}$ we tend to omit the coefficients and write $H_n(X)$ or $H^n(X)$.

Theorem (Poincaré)
If X is an n–manifold, then

$$H^i(X) \cong H_{n-i}(X).$$

Categories

A **category** C consists of a class Obj C of **objects**, and for each pair X, Y of objects, a set $\mathrm{Hom}_{\mathsf{C}}(X, Y)$ of **morphisms** (satisfying some sort of associative composition law).

Examples

Set: objects are sets, morphisms are functions.
Top: objects are topological spaces, morphisms are continuous maps.
Ab: objects are Abelian groups, morphisms are Abelian group homomorphisms.

Functors

A **functor** is a map $F\colon \mathsf{C} \to \mathsf{D}$, assigning objects to objects and morphisms to morphisms in a way which respects the composition law.

F is **covariant** if $F(f \circ g) = F(f) \circ F(g)$ and **contravariant** if $F(f \circ g) = F(g) \circ F(f)$

Examples

$H_n\colon$ Top $\to$ Ab is a covariant functor.
$H^n\colon$ Top $\to$ Ab is a contravariant functor.

Homology of groups

Given a group G we can form a topological space BG – the **classifying space** of G. This is homotopy-equivalent to the **Eilenberg–MacLane space** $K(G,1)$, which has $\pi_1 \cong G$ and π_n trivial for $n > 1$.

Example

$B\mathbb{Z} \simeq K(\mathbb{Z},1) \simeq S^1$.

Extensions of groups

An **extension** of G by a normal subgroup $A \lhd G$ is an (exact) sequence

$$A \overset{i}{\rightarrowtail} E \overset{p}{\twoheadrightarrow} G$$

of groups, so that $A = \ker p$.

$A \rightarrowtail E_1 \twoheadrightarrow G$ and $A \rightarrowtail E_2 \twoheadrightarrow G$ are **equivalent** if there is an isomorphism $E_1 \cong E_2$ which commutes with the projection maps.

It turns out that $H^2(G;A)$ classifies extensions of G by A, modulo equivalence.

As it happens, A needn't be just an ordinary Abelian group – more generally it's a G–module, an Abelian group equipped with a G–action.

There is an equivalent, purely algebraic, way of defining (co)homology of groups.

Resolutions

Given a group G, a G–**free resolution** (over $\mathbb{Z}$) is an exact sequence

$$\ldots \to F_3 \to F_2 \to F_1 \to \mathbb{Z} \to 0$$

of G–modules, where each F_i is free, and $\mathbb{Z}$ is considered to have a trivial G–module structure.

Choose such a resolution (it turns out not to matter which one), and apply $- \otimes_G A$ or $\mathrm{Hom}_G(-, A)$ to it to get a complex of Abelian groups. Now take its (co)homology.

This gives the same results as the topological method.

Racks and quandles

A **quandle** is a set X equipped with a binary operation (written as exponentiation) such that:

R1 $a^a = a$ for all $a \in X$

R2 For any $a, b \in X$ there is a unique $c \in X$ with $c^b = a$

R3 For all $a, b, c \in X$,

$$(a^b)^c = (a^c)^{(b^c)}$$

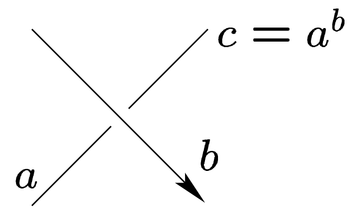
A **rack** is a quandle which only satisfies **R2** and **R3**.

Examples

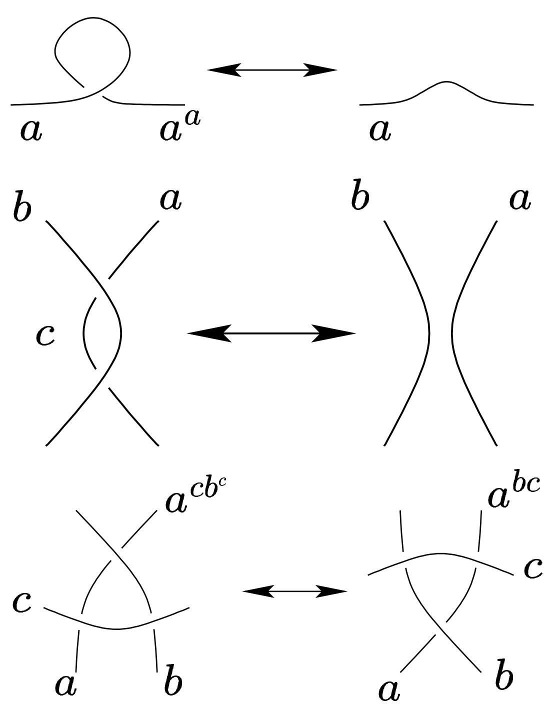
The **trivial rack** T_n is the set $\{0, \ldots, n\}$ with rack operation $a^b = a$.
For a group G, the **conjugation rack** $\mathrm{Conj}\, G$ is (the underlying set of) G with rack operation $g^h = h^{-1}gh$.

How did I get interested in this?

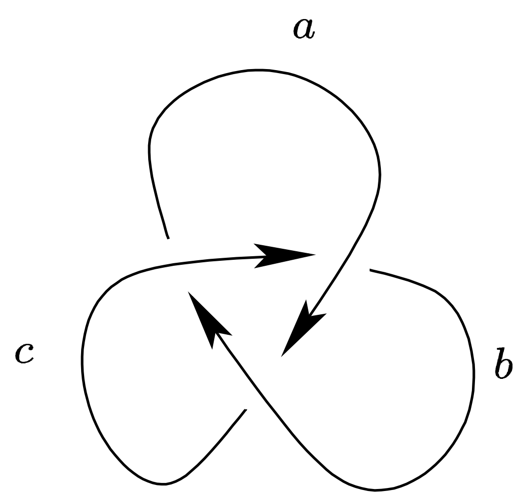
Take a knot diagram. Label each arc with an element of some set, so that at any crossing:



Then the quandle axioms correspond to the Reidemeister moves:



Given any link L we can construct the **fundamental quandle** $\Gamma_Q(L)$ of L.



$$\langle a, b, c : a^a = b^c = a; b^b = c^a = b; c^c = a^b = c\rangle$$

Analogous to the Wirtinger presentation of the knot group $\pi_1(S^3 \setminus 3_1)$

Related concept of the **fundamental rack** $\Gamma(L)$, which preserves framing.

<u>The rack space</u> (Fenn–Rourke–Sanderson)

For any rack X we can construct a topological space (the **rack space**) BX. This is analogous to the classifying space for a group.

We thus have a homology theory for racks and quandles – set $H_n(X;A) = H_n(BX;A)$ and $H^n(X;A) = H^n(BX;A)$.

<u>State sum invariants</u> (Carter–Saito–. . .)

It turns out that (a variation on) the second cohomology of BX gives rise to an interesting and powerful class of knot and link invariants. More generally, we can define invariants of arbitrary codimension–2 embeddings.

<u>Question</u>

Is there a purely algebraic definition of rack and quandle homology?

Extensions of racks and quandles

Start with defining and classifying extensions of racks by some suitable object. First of all, figure out what these extending objects are.

Abelian groups will do, but they're for wimps – we want the most general possible such object. This is tricky, but fortunately the *really* difficult bit has already been done.

In the 1960s, Jon Beck devised a general definition of a 'module' over a fixed object X in an arbitrary category C.

They are the 'Abelian group objects' in the 'slice category' C/X.

Do this for Group and you end up with G–modules. Similarly for LieAlg, $\mathsf{AssocAlg}$ and $\mathsf{CommRing}$.

In particular, these objects form an **Abelian category**, which is essentially a category in which you can form chain complexes and exact sequences – and hence define homology.

A **rack module** $\mathcal{A}$ turns out to be a slightly more complex object than just an Abelian group with some extra structure. It consists of one Abelian group A_x for each element $x \in X$ and **structure maps**

$$\phi_{x,y} \colon A_x \to A_{x^y}$$
$$\psi_{y,x} \colon A_y \to A_{x^y}$$

for all $x, y \in X$, such that each ϕ–map is an isomorphism, and

$$\begin{aligned} \phi_{x^y,z}\phi_{x,y} &= \phi_{x^z,y^z}\phi_{x,z} \\ \phi_{x^y,z}\psi_{y,x} &= \psi_{y^z,x^z}\phi_{y,z} \\ \psi_{z,x^y} &= \phi_{x^z,y^z}\psi_{z,x} + \psi_{y^z,x^z}\psi_{z,y} \end{aligned}$$

for all $x, y, z \in X$.

The ϕ–maps are the analogue of the group action for a G–module.

Not all of the A–groups need be isomorphic, but in general if x and y are in the same **orbit** of X, then $A_x \cong A_y$. If all the A–groups are isomorphic, the module is **homogeneous**, otherwise it's **heterogeneous**.

A **homomorphism** $f\colon \mathcal{A} \to \mathcal{B}$ of X–modules is a collection of Abelian group homomorphisms $f_x\colon A_x \to B_x$ which are natural with respect to the structure maps.

There is an elegant description of rack modules in terms of ‘functors’ defined on ‘trunks’ (constructs analogous to categories).

So we now have a category RMod_X of objects suitable for extending X with.

If X is a quandle, then there is a related concept of a **quandle module**, which has the additional requirement

$$\phi_{x,x} + \psi_{x,x} = \mathrm{Id}_{A_x}$$

for all $x \in X$. These form a category QMod_X.

Examples

Any Abelian group A can be regarded as a rack module $\mathcal{A}$ – set $A_x = A$, $\phi_{x,y} = \mathrm{Id}_A$ and $\psi_{y,x} = 0_A$.

Dihedral modules:

Set $A_x = \mathbb{Z}_n$, $\phi_{x,y}\colon\ i \mapsto -i$, and $\psi_{y,x}\colon\ i \mapsto 2i$.

Alexander modules:

Pick a Laurent polynomial $h(t)$.

Set $A_x = \mathbb{Z}_n[t, t^{-1}]/h(t)$, $\phi_{x,y}\colon\ f(t) \mapsto tf(t)$, and $\psi_{y,x}\colon\ f(t) \mapsto (1-t)f(t)$.

There are heterogeneous generalisations of these examples.

Equivalence classes of extensions of a group G by a G–module A correspond to equivalence classes of **factor sets** $\sigma\colon G \times G \to A$.

These form a group $\mathrm{Ext}(G, A)$, which is isomorphic to $H^2(G; A)$.

There's a nice (vaguely) sensible notion of an **extension** of a rack X by an X–module $\mathcal{A}$, and it turns out that these are also classified by factor sets, which are (a bit like) maps $\sigma\colon X \times X \to \mathcal{A}$.

These also form a group $\mathrm{Ext}(X, \mathcal{A})$, which is isomorphic to $H^2(X; A)$ in the case where $\mathcal{A}$ is just an Abelian group A.

So define $H^2(X; \mathcal{A}) = \mathrm{Ext}(X, \mathcal{A})$. By analogy, we can define $H^n(X; \mathcal{A})$ and $H_n(X; \mathcal{A})$ – generalising the rack space homology and cohomology groups for coefficients in an X–module rather than just an Abelian group.

Even more generally, we can classify non-Abelian extensions of racks and quandles, but they're a bit too heroic.

Alternatively, construct an X–free resolution of $\mathbb{Z}$:

$$\ldots \to \mathcal{F}_3 \to \mathcal{F}_2 \to \mathcal{F}_1 \to \mathbb{Z} \to 0$$

and apply $\mathrm{Hom}_X(-, \mathcal{A})$ or $- \otimes_X \mathcal{A}$ to it.

Weirdly, it turns out that this construction gives a different (but equally valid) homology theory to the rack space method.

BT_m has the homotopy type of (and hence the same homology groups as)

$$\Omega(\vee_m S^2),$$

the loop space on a wedge of m copies of S^2. In particular, $H_n(BT_m)$ is nontrivial for $n \geqslant 2$.

On the other hand,

$$0 \to \mathcal{I}T_m \to \mathbb{Z}T_m \to \mathbb{Z} \to 0$$

is a T_m–free resolution of $\mathbb{Z}$ with trivial higher homology.

It gets even better...

Cotriple homology

There's another construction of homology theories due to Barr and Beck.

A **cotriple** or **standard construction** in a category C is a functor $\perp\colon \mathsf{C} \to \mathsf{C}$ equipped with some extra structure. Pick an object X of C and iteratively apply $\perp$ to it. This gives a sequence

$$\ldots \to \perp^3 X \to \perp^2 X \to \perp X \to X$$

of objects in C. Now take a functor $T\colon \mathsf{C} \to \mathsf{A}$ where A is an Abelian category. Apply T to this sequence to get a chain complex

$$\ldots \to T(\perp^3 X) \to T(\perp^2 X) \to T(\perp X) \to T(X)$$

in A, and take its homology (or cohomology if T is contravariant).

Doing this gives $H_*(X;T)_\perp$ (or $H^*(X;T)_\perp$), the **cotriple (co)homology** of X with coefficient functor T.

Example

Let $U\colon \mathsf{Group} \to \mathsf{Set}$ be the 'forgetful' functor, and let $F\colon \mathsf{Set} \to \mathsf{Group}$ be the 'free group' functor.

The composition $FU\colon \mathsf{Group} \to \mathsf{Group}$ turns out to be a cotriple.

Then $H_*(G; \mathsf{Diff}_G(-) \otimes_G A)_{FU}$ coincides with ordinary group homology $H_{*+1}(G; A)$.
And $H^*(G; \mathsf{Der}_G(-, A))_{FU}$ coincides with ordinary group cohomology $H^{*+1}(G; A)$.

There are similar results for Lie algebras, associative algebras and commutative rings.

But the analogous cotriple construction for racks and quandles doesn't seem to give the same results as either of the other two theories.

Applications

As mentioned earlier, $H^n(X; A)$ yields an interesting and powerful family of invariants of embeddings $S^{n-1} \hookrightarrow S^{n+1}$.

The basic idea is that an element of $H^2(X; A)$ is a function $f\colon X \times X \to A$. Take a diagram of the link L, colour each arc consistently with an element of the rack X – this is the same as choosing a homomorphism $c\colon \Gamma(L) \to X$.

Now apply f to each crossing. This gives a collection of elements of A. Multiply them together. Do this for all X–colourings c and sum the results. This gives a polynomial invariant $\Phi_f(L)$ of L.

Can generalise this to embedded n–manifolds in $\mathbb{R}^{n+2}$.

With a bit of extra work, we can extend this construction to use the more generalised rack space cohomology theory.