Cohomology of racks and quandles

or

You say you want a resolution

1. Racks and quandles

   What are they?

   Why might we care?

2. Extensions and H^2

   What is an extension?

   Classification and second cohomology.

3. Modules over racks

   What is a 'rack module'?

   Coefficients for (co)homology.

4. Work in progress

   What next?

1. Racks and quandles

A **rack** is a set X, equipped with a binary operation (written as exponentiation), such that:

R1 For any $a, b \in X$ there is a unique $c \in X$ such that $a = c^b$.

R2 $a^{bc} = a^{cb^c}$ for any $a, b, c \in X$.

A **quandle** is a rack which satisfies:

Q $a^a = a$ for all $a \in X$.

Originally studied by Conway and Wraith, later by Joyce, and more recently by Fenn, Rourke, and Sanderson, and by Carter and Saito.

In particular, this means that the map $f_b : X \longrightarrow X; x \mapsto x^b$ is a bijection, and right-distributive.

Can think of X as a set with an action by a quotient of $F(X)$: This is the **operator group** $\operatorname{Op} X$.

Denote by $[x]$ the **orbit** of x under this action.

Examples

(i) The **trivial rack** $T_n = \{0, \dots, n-1\}$ with structure $a^b := a$.

(ii) The **cyclic rack** $C_n = \{0, \dots, n-1\}$ with $a^b := a + 1 (\mod n)$.

(iii) The **core rack** Core G of a group G: Define $g^h := hg^{-1}h$.

(iv) The **conjugation rack** Conj G of a group G:
Define $g^h := hgh^{-1}$.

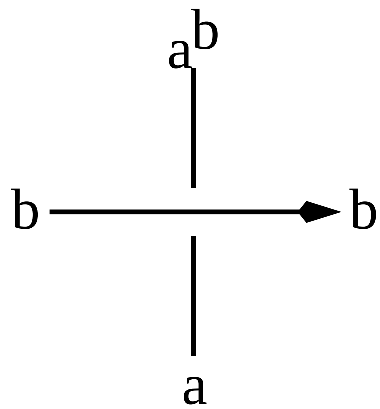
(v) The **dihedral rack** $D_n = \{0, \dots, n-1\}$, with $p^q := 2q - p$.

(vi) **Alexander quandles** are modules over $\Lambda = \mathbb{Z}[t, t^{-1}]$ with rack structure given by $a^b := ta + (1-t)b$.

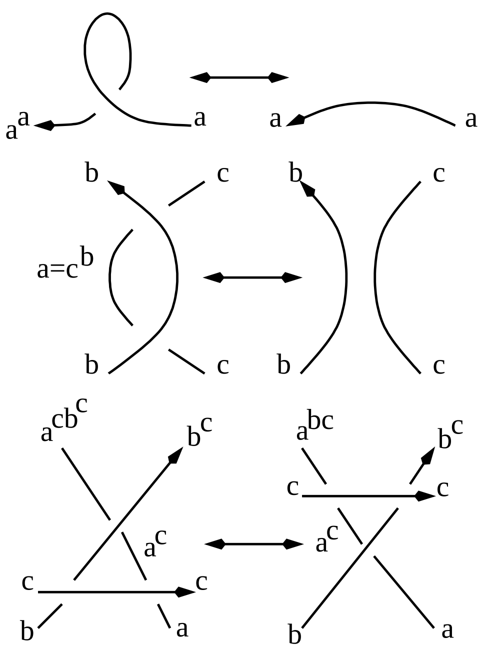
All of these (except for the cyclic racks) are quandles.

Why do we care?

Let $\Gamma_D(L)$ consist of the arcs of a link diagram $D(L)$. Then define a rack structure on $\Gamma_D(L)$ thus:



Then the rack (and quandle) axioms correspond to the Reidemeister moves:



Q: $a^a = a$

R1: $a = c^b$

R2: $a^{bc} = a^{cb^c}$

Cohomology of racks

For any rack X there is a well-defined topological space BBX, the **rack space**, analogous to the classifying space for a group.

Define the cohomology of X to be the cohomology of the rack space of X.
Concretely:

$$\cdots \xrightarrow{\delta^{n-1}} C^n(X;A) \xrightarrow{\delta^n} C^{n+1}(X;A) \xrightarrow{\delta^{n+1}} \cdots$$

where

$$C^n(X;A) = \mathrm{Hom}(FA(X^n), A)$$

and

$$\begin{aligned}(\delta^n f)(x_0,\ldots,x_n) = &\\ &\sum_{i=0}^{n}(-1)^i f(x_0^{x_i},\ldots,x_{i-1}^{x_i},x_{i+1},\ldots,x_n)\\ &\quad - \sum_{i=0}^{n}(-1)^i f(x_0,\ldots,x_{i-1},x_{i+1},\ldots,x_n)\end{aligned}$$

So a 2-cocycle satisfies:

$$f(x^y,z) + f(x,y) = f(x^z,y^z) + f(x,z)$$

and a 2-coboundary satisfies:

$$g(x,y) = h(x) - h(x^y)$$

for some 1-cochain $h \in C^1(X;A)$.

Quandle cohomology and state-sum invariants

Carter, Jelsofsky, Kamada, Langford, and Saito (1999):

Let $C^n(X; A)$ consist of all homomorphisms $FA(X^n) \longrightarrow A$ which are zero on all expressions of the form $(\ldots, x, x, \ldots)$.

This gives a variant cohomology theory H_Q^*.

The CJKLS state-sum invariants are defined as follows:

(i) Take a diagram D of a link L and **colour** it with a given quandle X – this is the same as choosing a homomorphism $C : \Gamma(L) \longrightarrow X$.

(ii) Pick a cocycle $\phi \in H_Q^2(X; A)$, and write the coefficient group A multiplicatively.

(iii) Apply ϕ to each crossing τ of D by taking the labels of the incoming arcs as the arguments of ϕ.

(iv) The **weight** of the crossing τ is then $\phi(x, y)^{\varepsilon(\tau)}$ where $\varepsilon(\tau)$ is the sign of τ.

(v) Take the product (in A) of the weight of every crossing.

(vi) Sum over all colourings of D.

The **state sum** of L corresponding to the cocycle ϕ is

$$\Phi_\phi(L) = \sum_C \prod_\tau \phi(x, y)^{\varepsilon(\tau)}$$

Trunks and trunk maps

A **trunk** T is a construct analogous to a category, having a class of **objects** and, for any two objects A, B, a set $\mathrm{Hom}_{\mathsf{T}}(A, B)$ of **morphisms**.

We don't necessarily require the existence of identity morphisms.

There may be **preferred squares** of objects and morphisms:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ C & \xrightarrow[k]{} & D \end{array}$$

This is the analogue of composition in a category.

We may consider any category C as a trunk with the same objects and morphisms.

A **trunk map** from one trunk to another is the obvious analogue of a functor between categories.

2. Extensions of racks and quandles

Given a rack X, define the **extended rack trunk** $\mathsf{S}(X)$ with one object for each element $x \in X$, one morphism $\alpha_{x,y} : x \longrightarrow x^y$ for each ordered pair (x, y) of elements of X, and preferred squares

$$\begin{array}{ccc} x & \xrightarrow{\alpha_{x,y}} & x^y \\ {\scriptstyle \alpha_{x,z}}\downarrow & & \downarrow{\scriptstyle \alpha_{x^y,z}} \\ x^z & \xrightarrow[\alpha_{x^z,y^z}]{} & x^{yz} = x^{zy^z} \end{array}$$

A trunk map $G : \mathsf{S}(X) \longrightarrow \mathsf{Group}$ determines a group G_x for each element of X, and homomorphisms $\phi_{x,y} : G_x \longrightarrow G_{x^y}$ for each ordered pair (x, y) of elements of X, such that:

$$\phi_{x^y,z}\phi_{x,y} = \phi_{x^z,y^z}\phi_{x,z}$$

That is, the diagram

$$\begin{array}{ccc} G_x & \xrightarrow{\phi_{x,y}} & G_{x^y} \\ {\scriptstyle \phi_{x,z}}\downarrow & & \downarrow{\scriptstyle \phi_{x^y,z}} \\ G_{x^z} & \xrightarrow[\phi_{x^z,y^z}]{} & G_{x^{yz}} = G_{x^{zy^z}} \end{array}$$

commutes.

An **extension** of a rack X by a trunk map $G = (G, \phi) : \mathsf{S}(X) \longrightarrow \mathsf{Group}$ consists of a rack E together with a surjective homomorphism $f : E \longrightarrow X$ inducing a partition $E = \bigcup_{x \in X} E_x$ (where E_x is the preimage $f^{-1}(x)$) and, for each $x \in X$, a right group action of G_x on E_x such that:

(i) G_x acts simply transitively on E_x – that is, for any $a, b \in E_x$, there is a unique $g \in G_x$ such that $a \cdot g = b$.

(ii) $(a \cdot g)^b = (a^b) \cdot \phi_{x,y}(g)$ for any $a \in G_x$, $g \in G_x$, and $b \in E_y$.

Two projections $E_1 \xrightarrow{f_1} X$ and $E_2 \xrightarrow{f_2} X$ are **equivalent** if there exists an isomorphism (**equivalence**) $\theta : E_1 \longrightarrow E_2$ which commutes with the projection maps and the group actions:

(i) $f_2\theta(a) = f_1(a)$ for all $a \in E_1$.

(ii) $\theta(a \cdot g) = \theta(a) \cdot g$ for all $a \in E_x$ and $g \in G_x$.

Factor systems

Let $E \xrightarrow{f} X$ be an extension of X by $G = (G, \phi)$ and let $s : X \longrightarrow E$ be a function (not necessarily a rack homomorphism) such that $fs = \mathrm{Id}_X$.

Since the G_x act simply transitively on the E_x, there is a unique $x \in X$ and a unique $g \in G_x$ such that a given element a can be written as

$$a = s(x) \cdot g.$$

Since f *is* a rack homomorphism, it follows that $s(x)^{s(y)} \in E_{x^y}$, and that there is a unique $\sigma_{x,y} \in G_{x^y}$ such that

$$s(x)^{s(y)} = s(x^y) \cdot \sigma_{x,y}.$$

The family $\sigma = \{\sigma_{x,y} : x, y \in X\}$ is the **factor set** of E **relative to** s. It is the obstruction to s being a rack homomorphism.

Similarly, there is a unique $\omega(h) \in G_{x^y}$ such that

$$s(x)^{s(y)\cdot h} = s(x)^{s(y)} \cdot \omega(h)$$

for any $h \in G_y$.

This determines a map $\omega_{y,x} : G_y \longrightarrow G_{x^y}$ such that $\omega_{y,x}(1) = 1$, but which is not in general a group homomorphism. From the second condition on the G_x-actions, it follows that:

$$(s(x) \cdot g)^{(s(y)\cdot h)} = s(x^y) \cdot \sigma_{x,y}\omega_{y,x}(h)\phi_{x,y}(g)$$

Hence, the rack structure on E is determined completely by the factor set σ and the maps $\omega_{y,x}$.

We define a new collection of maps

$$\psi_{y,x} = \sigma_{x,y}\omega_{y,x}\sigma_{x,y}^{-1}$$

and rewrite this last condition in a slightly more elegant form:

$$s(x)^{s(y)\cdot h} = s(x^y) \cdot \psi_{y,x}(h)\sigma_{x,y}\phi_{x,y}(g)$$

The ψ maps share the same drawback as the ω maps, in that while $\psi_{y,x}(1) = 1$, they are not in general group homomorphisms.

As before, the rack structure on E is determined completely by the factor set σ and the ψ maps. We call the pair (σ, ψ) the **factor system** of E **relative to** s.

An example: Consider the dihedral rack D_6 as an extension of the trivial rack $T_1 = \{0\}$ by the dihedral group D_3 (of order 6) with action given by

	e	ρ_1	ρ_2	μ_1	μ_2	μ_3
0	0	1	2	3	4	5
1	1	2	0	5	3	4
2	2	0	1	4	5	3
3	3	4	5	0	1	2
4	4	5	3	2	0	1
5	5	3	4	1	2	0

and section $s : 0 \mapsto 0$.

Then $\phi_{0,0} : D_3 \longrightarrow D_3$ is given by:

g	e	ρ_1	ρ_2	μ_1	μ_2	μ_3
$\phi(g)$	e	μ_3	μ_2	μ_1	ρ_2	ρ_1

and $\omega_{0,0} : D_3 \longrightarrow D_3$ by:

h	e	ρ_1	ρ_2	μ_1	μ_2	μ_3
$\omega(h)$	e	ρ_2	μ_2	e	ρ_2	μ_2

The latter is not a homomorphism, since for example

$$\omega(\rho_1^2) = \omega(\rho_2) = \mu_2 \neq \rho_1 = \rho_2^2 = \omega(\rho_1)^2$$

Finally, the factor set σ is trivial.

<u>Theorem 1</u> (the horrid condition)

Let $G = (G, \phi) : \mathsf{S}(X) \longrightarrow \mathsf{Group}$ with $\phi_{x,y}$ an isomorphism for all $x, y \in X$.

Let $\sigma = \{\sigma_{x,y} \in G_{x^y} : x, y \in X\}$ be a family of group elements.

Let $\psi = \{\psi_{y,x} : G_y \longrightarrow G_{x^y} : x, y \in X\}$ be a collection of functions such that $\psi_{y,x}(1) = 1$.

Furthermore, let $E[G, \sigma, \psi]$ be the set of ordered pairs (x, g) with $x \in X$ and $g \in G_x$, with rack operation

$$(x, g)^{(y,h)} = (x^y, \psi_{y,x}(h)\sigma_{x,y}\phi_{x,y}(g))$$

Then $E[G, \sigma, \psi]$ is an extension of X by G with factor system (σ, ψ) if

$$\psi_{z,x^y}(k)\sigma_{x^y,z}\phi_{x^y,z}\psi_{y,x}(h)\phi_{x^y,z}(\sigma_{x,y}) = \psi_{y^z,x^z}\left(\psi_{z,y}(k)\sigma_{y,z}\phi_{y,z}(h)\right)\sigma_{x^z,y^z}\phi_{x^z,y^z}\psi_{z,x}(k)\phi_{x^z,y^z}(\sigma_{x,z})$$

for all $x, y, z \in X, h \in G_y$, and $k \in G_z$.

Conversely, if E is an extension of X by G with factor system (σ, ψ) then this condition holds, and E is equivalent to $E[G, \sigma, \psi]$.

Theorem 2 (equivalence of extensions)

Let $G = (G, \phi) : \mathsf{S}(X) \longrightarrow$ Group with $\phi_{x,y}$ an isomorphism for all $x, y \in X$.

Let (σ, ψ) and (τ, ω) be two factor systems for G.

Then the following are equivalent:

(i) $E[G, \sigma, \psi]$ and $E[G, \tau, \omega]$ are equivalent as extensions of X.

(ii) There exists a $u = \{u_x \in G_x : x \in X\}$ such that:

$$\begin{aligned} \tau_{x,y} &= u_{x^y}^{-1}\psi_{y,x}(u_y)\sigma_{x,y}\phi_{x,y}(u_x) \\ \omega_{y,x}(h) &= u_{x^y}^{-1}\psi_{y,x}(u_y h)\psi_{y,x}(u_y)^{-1}u_{x^y} \end{aligned}$$

(iii) (σ, ψ) and (τ, ω) are factor systems of the same extension of X by G relative to different cross-sections.

Corollary 3 (split extensions)

The following are equivalent:

(i) There exists a rack homomorphism $X \xrightarrow{s} E$ such that $fs = \mathrm{Id}_X$,

(ii) relative to some section the factor set of E is trivial,

(iii) relative to any section there exists a family $u = \{u_x : x \in X, u_x \in G_x\}$ such that

$$\sigma_{x,y} = \psi_{y,x}(u_y)^{-1}u_{x^y}\phi_{x,y}(u_x)^{-1}$$

Abelian extensions

The abelian case is much nicer:

Theorem 1A

Let $A = (A, \phi) : \mathsf{S}(X) \longrightarrow \mathsf{Ab}$ and let (σ, ψ) be a factor set such that the ψ maps are rack homomorphisms.

Then $E[A, \sigma, \psi]$ is an extension of X with factor system (σ, ψ) iff

$$\begin{aligned} \psi_{z,x^y}(k) &= \psi_{y^z,x^z}\psi_{z,y}(k) + \phi_{x^z,y^z}\psi_{z,x}(k) \\ \phi_{x^y,z}\psi_{y,x}(h) &= \psi_{y^z,x^z}\phi_{y,z}(h) \\ \sigma_{x^y,z} + \phi_{x^y,z}(\sigma_{x,y}) &= \psi_{y^z,x^z}(\sigma_{y,z}) + \sigma_{x^z,y^z} + \phi_{x^z,y^z}(\sigma_{x,z}) \end{aligned}$$

Theorem 2A

Two factor systems (τ, ω) and (σ, ψ) for A are equivalent iff there exists a family $u = \{u_x : x \in X, u_x \in A_x\}$ such that:

$$\tau_{x,y} = \psi_{y,x}(u_y) + \phi_{x,y}(u_x) - u_{x^y} + \sigma_{x,y}$$

and if

$$\omega_{y,x} = \psi_{y,x}$$

Corollary 3A

A factor system (σ, ψ) splits iff there exists a family $u = \{u_x : x \in X, u_x \in A_x\}$ such that:

$$\sigma_{x,y} = \psi_{y,x}(u_y) + \phi_{x,y}(u_x) - u_{x^y}$$

Now amalgamate the trunk map $A = (A, \phi)$ with the ψ homomorphisms:

Define $\mathsf{T}(X)$ similarly to $\mathsf{S}(X)$ but with additional morphisms $\beta_{y,x} : y \longrightarrow x^y$ and preferred squares

$$\begin{array}{ccc} y & \xrightarrow{\;\beta_{y,x}\;} & x^y \\ {\scriptstyle \alpha_{y,z}}\big\downarrow & & \big\downarrow{\scriptstyle \alpha_{x^y,z}} \\ y^z & \xrightarrow[\;\beta_{y^z,x^z}\;]{} & x^{yz} = x^{zy^z} \end{array}$$

We can now classify extensions of a rack X by a trunk map

$$A = (A, \phi, \psi) : \mathsf{T}(X) \longrightarrow \mathsf{Ab}.$$

Let $\mathrm{Ext}(X, A)$ consist of the factor sets σ such that

$$\sigma_{x^y,z} + \phi_{x^y,z}(\sigma_{x,y}) = \psi_{y^z,x^z}(\sigma_{y,z}) + \sigma_{x^z,y^z} + \phi_{x^z,y^z}(\sigma_{x,z})$$

modulo the split factor sets:

$$\tau_{x,y} = \psi_{y,x}(u_y) - u_{x^y} + \phi_{x,y}(u_x)$$

This has an obvious abelian group structure.

If A is the trivial trunk map with $A_x = A$, $\psi_{y,x} = 0$, and $\phi_{x,y} = \mathrm{Id}$,
then $\mathrm{Ext}(X, A) = H^2(X; A)$.

Quandle extensions

Given a quandle X, require that E is also a quandle.

Then the trunk map $A : \mathsf{T}(X) \longrightarrow \mathsf{Ab}$ must satisfy

$$\phi_{x,x} + \psi_{x,x} = \mathrm{Id}_{A_x}$$

and the factor set σ must satisfy the condition

$$\sigma_{x,x} = 0$$

in addition to the conditions in theorem 2A.

For the trivial trunk map $A = (A, \mathrm{Id}, 0)$, this recovers the second quandle cohomology $H^2_Q(X; A)$ of Carter, Saito, *et alii*.

As before, we may define groups $\mathrm{Ext}_Q(X, A)$ classifying such extensions.

Involutory extensions

An **involutory rack** is a rack X for which $a^{bb} = a$ for all $a, b \in X$. Dihedral racks, in particular, are involutory.

Considering extensions in this subcategory provides the following additional conditions on the trunk maps:

$$\begin{aligned} \phi_{x^y,y}\phi_{x,y} &= \mathrm{Id}_{A_x} \\ \psi_{y,x^y} + \phi_{x^y,y}\psi_{y,x} &= 0 \end{aligned}$$

and the condition

$$\sigma_{x^y,y} + \phi_{x^y,y}(\sigma_{x,y}) = 0$$

on the factor sets.

Again, there are groups $\mathrm{Ext}_I(X, A)$ classifying such extensions, and groups $\mathrm{Ext}_{IQ}(X, A)$ classifying the hybrid 'involutory quandle extensions'.

It doesn't appear unreasonable to expect that state-sums derived from these groups will be unoriented invariants under regular (or ambient) isotopy.

3. Modules

We must characterise the suitable coefficient objects for rack (co)homology and the related theories.

Jon Beck devised a general approach for doing this for a given object X in a category C:

Let A $=$ Ab(C$/X$) denote the category of abelian group objects in the slice category C$/X$. Then this category is equivalent to the desired category.

For example: If C $=$ Group, then the objects of Ab(Group$/G$) are of the form

$$A \rtimes G \longrightarrow G$$

where A is a G-module.

The idea is that the **Beck modules** are the kernels of split extensions of the chosen object.

Rack modules

It transpires that the Beck modules over X in Rack are exactly the previously-discussed trunk maps $\mathsf{T}(X) \longrightarrow \mathsf{Ab}$, and that they form an abelian category RMod_X.

An interesting quirk is that in general, the group A_x need only be isomorphic to A_y if x and y are in the same orbit of X.

So an X-module consists not just of one abelian group (equipped with some sort of rack action structure) but one for each orbit of X.

Modules where all of the groups are isomorphic are **homogeneous**; otherwise they are **heterogeneous**.

This also follows for the other specialisations mentioned.

Carter, Saito and Kamada's notion of abelian quandle extensions corresponds in this framework to an extension of a quandle by a trivial homogeneous quandle module.

Carter, Saito and Elhamdadi's 'twisted quandle extensions' correspond to extensions by homogeneous **Alexander modules**:

$$\begin{aligned} A_x &= \mathbb{Z}[t, t^{-1}]/h(t) \\ \phi_{x,y}(a) &= ta \\ \psi_{y,x}(b) &= (1-t)b \end{aligned}$$

where $h(t)$ is a Laurent polynomial in t.

4. Further work

(i) Derived functors.

Homology and higher cohomology groups.

Ext^n and Tor_n groups.

Projective and injective resolutions.

(ii) Cotriple (co)homology.

The free cotriple.

The conjugation cotriple.

(iii) Spectral sequences.

Is there a Lyndon/Hochschild/Serre spectral sequence?

(iv) Applications.

State-sum invariants in knot theory and the topology of 3-manifolds.

Computations, via algebraic and geometric methods.

(v) Augmented racks.

Is there a corresponding theory for augmented racks and quandles?