Homological Algebra of Racks and Quandles

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For Dad,

who would have been interested.

Wilfred Leslie Jackson (1932–1997)

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Introduction

Overview

In this thesis, I investigate the homological algebra of racks and quandles, and its applications to knot theory.

A **rack** (or **wrack**) is a set equipped with an asymmetric binary operation (often written as exponentiation) satisfying two axioms loosely analogous to the second and third Reidemeister moves. A **quandle** is a rack which satisfies a further axiom, itself similarly analogous to the first Reidemeister move. It is well-known [15] [23] [11] that such objects provide interesting collections of invariants of both classical links and higher-dimensional codimension-2 embedded manifolds.

There is a well-defined notion of homology and cohomology groups of a rack or quandle, defined in terms of an appropriate classifying space (the 'rack space'); these have already been studied [20] [18] [17], as has a related theory, that of 'quandle (co)homology' [11].

My original objective was to generalise these existing theories and show how they could be regarded as derived functors constructed on resolutions in some suitable category of modules, and also in terms of simplicial objects constructed from the free cotriple in the relevant categories.

As will become apparent, it transpired that the derived functor approach yields another set of homology and cohomology theories which in general do not coincide with the existing theories. Furthermore, it seems that the cotriple theory may itself be different to both the existing theory and the derived-functor theory.

In chapter 1, 'extensions' of racks are defined in greater generality than before, and the non-abelian case very briefly considered before we concentrate on the neater and more elegant abelian theory. It is shown that such extensions are classified by Ext groups which are in some sense a more general form of $H^2(BX)$, the second cohomology of the rack space of X. This framework is then specialised to the subcategory Quandle and it is found that the corresponding Ext_Q groups are a more general form of the 'abelian quandle extensions' and 'twisted quandle extensions' of Carter, Saito, and their colleagues. This is followed by further specialisation into the categories InvRack and InvQuandle of, respectively, involutory racks and involutory quandles. It is demonstrated that such extensions are classified by groups Ext_I and Ext_{IQ}.

Chapter 2 is concerned with the development of the theory of modules over a rack X, the definition of which is motivated by the preceding discussion of extensions. It is shown that these objects (essentially the 'extending' objects of the previous chapter) form an abelian category RMod_X and are exactly the 'Beck modules' [5] [2] over X in the category Rack. They are thus suitable coefficient modules for (co)homology theories. The theory is further specialised to the subcategories Quandle, InvRack, and InvQuandle to obtain notions of 'quandle modules' and 'involutory modules'. An analogue of the group ring (the 'rack algebra' or 'wring') is developed, together with the corresponding augmentation module.

The third chapter begins with a discussion and generalisation of the existing (co)homology theories for racks, resulting in the construction of a standard complex which encompasses the definitions of all rack (co)homology theories known so far. In the remainder of this chapter, concepts of projective, injective and flat modules are developed and studied, and new definitions of rack homology and cohomology are devised in terms of derived functors on suitable resolutions of modules. A standard normalised resolution (the 'bar resolution') is constructed, which appears to bear little resemblance to the standard complex constructed earlier. Finally, a brief overview is given of the corresponding approaches to quandle and involutory (co)homology.

Chapter 4 investigates the cotriple approach of Barr and Beck [3] [2]. We begin by proving the tripleability of Rack over Set and then examine the abelianisation functor Diff. A result of Barr [2] is used to investigate the relationship between the (co)homology theories arising from the free cotriple on Rack, and the 'rack space' and 'derived functor' theories investigated earlier. These discussions are then repeated for the quandle and involutory theories. The chapter ends with a brief discussion of a new (co)homology theory for groups, constructed from the cotriple corresponding to the As / Conj adjunction.

The majority of chapter 5 is concerned with proving the existence of well-known exact sequences for rack and quandle (co)homology theories.

In the penultimate chapter, homology and cohomology is computed for trivial and free racks. In particular, it is found that the 'rack space' and 'derived functor' definitions of rack (co)homology do not, in general, coincide.

Chapter 7 is concerned with the application of rack and quandle cohomology theories to the theory of classical links, extending the work of Carter, Saito, and their colleagues. The generalised 'rack space' version of quandle cohomology yields new ambient isotopy invariants of classical links which in particular are shown to detect reversibility. Rack cohomology is similarly applied to produce isotopy invariants of framed classical links. Finally, state-sum invariants are constructed from involutory cohomology and are shown to be isotopy invariants of unoriented classical links.

A short afterword contains brief discussions of some of the questions raised and/or left unanswered by this thesis.

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Declaration

I hereby declare that, to the best of my knowledge, the material contained in this thesis is original and my own work, except where otherwise indicated, cited, or commonly-known. In general, the zeroth section of each chapter should be regarded as expository rather than original. Some aspects of the work in the first three chapters have been duplicated independently by Andruskiewitsch and Graña [1], although with many differences in notation, motivation, and emphasis. Part of section 7.1 has been duplicated independently by Carter, Elhamdadi, Graña and Saito [9] since the submission of the original version of this thesis.

I have not submitted any of this material in partial or complete fulfilment of the requirements for another degree at this or any other university.

Chapter 1

Extensions

A **rack** (or **wrack**) is a set X equipped with an asymmetric binary operation, here written as exponentiation, such that:

(i) For every $a, b \in X$, there is a unique $c \in X$ such that $c^b = a$.

(ii) For every $a, b, c \in X$, the **rack identity**

$$a^{bc} = a^{cb^c}.$$

holds.

In the first of these axioms, the unique element c is often denoted $a^{\bar{b}}$, although $\bar{b}$ should not itself be regarded as an *element* of the rack. Association of exponents should be understood to follow the usual conventions for exponential notation. In particular, in the second axiom, the expressions a^{bc} and a^{cb^c} should be interpreted as $(a^b)^c$ and $(a^c)^{(b^c)}$ respectively.

These two axioms imply that the rack operation is self-right-distributive, and that the function $\pi_b\colon X \longrightarrow X$ defined by $\pi_b(x) = x^b$ is a bijection, and hence a permutation of X.

In addition, if the rack also satisfies

(iii) For every $a \in X, a^a = a$

then it is referred to as a **quandle**.

If X satisfies the condition

(iv) For every $a, b \in X, a^{bb} = a$

then it is said to be an **involutory rack** or **involutory quandle**.

There is an obvious notion of a **homomorphism** of racks – a map $f\colon X \longrightarrow Y$ such that $f(x^y) = f(x)^{f(y)}$ for all $x, y \in X$. There is thus a category Rack whose objects are racks, and whose morphisms are rack homomorphisms.

The subgroup of $\operatorname{Sym} X$ generated by the set $\{\pi_x : x \in X\}$ of permutations of X is called the **operator group** of X and denoted $\operatorname{Op} X$ (although Joyce calls it $\operatorname{Inn} X$). The map $\operatorname{Op}\colon \mathsf{Rack} \longrightarrow \mathsf{Group}$ provides a group $\operatorname{Op} X$ for every rack X, but there is, in general, no corresponding group homomorphism $f_* = \operatorname{Op} f\colon \operatorname{Op} X \longrightarrow \operatorname{Op} Y$ for a given rack homomorphism $f\colon X \longrightarrow Y$. Hence, the operator group is not functorial.

Example 1.1 (Conjugation racks)

Given a group G we may define a rack structure on it by setting

$$g^h = h^{-1}gh$$

for all $g, h \in G$. This rack is called the ***conjugation rack*** *of G and denoted* $\operatorname{Conj} G$*. This construction provides a functor* $\operatorname{Conj}\colon \mathsf{Group} \longrightarrow \mathsf{Rack}$.

This functor Conj has a left adjoint: For any rack X, the **associated group** $\operatorname{As} X$ is defined to be the free group generated by the elements of the rack, modulo relations of the form $a^b = b^{-1}ab$ for all $a, b \in X$. Thus there is a natural isomorphism

$$\operatorname{Hom}_{\mathsf{Group}}(\operatorname{As} X, G) \cong \operatorname{Hom}_{\mathsf{Rack}}(X, \operatorname{Conj} G)$$

for any rack X and any group G.

Example 1.2 (Core racks)

We may define a different rack structure (the ***core rack****) on G by setting*

$$g^h = hg^{-1}h$$

for all $g, h \in G$. This construction is not functorial.

Example 1.3 (Trivial racks)

*The **trivial rack** T_n of order n is the set $\{0, \ldots, n-1\}$ with rack structure:*

$$p^q = p$$

for all $p, q \in T_n$. The infinite trivial rack T_∞ is the set $\mathbb{Z}$ equipped with the same structure.

Example 1.4 (Dihedral racks)

*The **dihedral rack** D_n is the set $\{0, \ldots, n-1\}$ with rack structure*

$$p^q = 2q - p \mod n$$

for all $p, q \in D_n$ and the infinite dihedral rack D_∞ is the set $\mathbb{Z}$ equipped with rack operation

$$p^q = 2q - p$$

for all $p, q \in \mathbb{Z}$. These may also be regarded as the core racks on the cyclic groups $\mathbb{Z}_n$ and $\mathbb{Z}$, respectively.

Example 1.5 (Alexander quandles)

Denote by Λ the ring $\mathbb{Z}[t, t^{-1}]$ of Laurent polynomials in t. Then any Λ–module may be equipped with rack structure

$$a^b = ta + (1-t)b.$$

Example 1.6 (Cyclic racks)

*The **cyclic rack** C_n of order n is the set $\{0, \ldots, n-1\}$ with rack structure*

$$p^q = p + 1 \mod n$$

for $p, q \in C_n$, while the infinite cyclic rack C_∞ is the set $\mathbb{Z}$ with rack structure

$$p^q = p + 1$$

for $p, q \in \mathbb{Z}$. These racks, unlike the examples above, are not quandles.

Example 1.7 (Inverted racks)

Given any rack X, we may form a related rack X^* which has an element x^* for each $x \in X$, and rack operation given by $x^{*y^*} = (x^{\bar{y}})^*$. This rack is called the ***inverted*** *or* ***opposite rack*** *of X. In particular, If X is an involutory rack, then $X^* = X$.*

Racks (or 'wracks') were first investigated by Conway and Wraith [14], and quandles by Joyce [23][1]. A detailed survey may be found in [15].

A **trunk** T is a construct loosely analogous to a category, and consists of a class of **objects** and, for each pair of objects A, B, a set of **morphisms** $\mathrm{Hom}_{\mathsf{T}}(A, B)$. Furthermore T has a number of **preferred squares**

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ C & \xrightarrow[k]{} & D \end{array}$$

of morphisms, a concept analogous to composition in a category.

There is an analogue of a functor between categories: given two trunks S and T, a (**covariant**) **trunk map** $F\colon \mathsf{S} \longrightarrow \mathsf{T}$ determines an object $F(A)$ of T for each object of S; and, for any morphism $f\colon A \longrightarrow B$ of S, a morphism $f_* = F(f)\colon F(A) \longrightarrow F(B)$ in T such that preferred squares are preserved:

$$\begin{array}{ccc} F(A) & \xrightarrow{f_*} & F(B) \\ {\scriptstyle g_*}\downarrow & & \downarrow{\scriptstyle h_*} \\ F(C) & \xrightarrow[k_*]{} & F(D) \end{array}$$

A **contravariant** trunk map behaves analogously to a contravariant functor: for any given morphism $f\colon A \longrightarrow B$ in S, there is a corresponding morphism

[1] It seems that the term 'wrack' originated because Conway and Wraith were studying the structure that remained when the multiplication operation in a group was discarded, leaving only the conjugation structure – thus, the 'wrack and ruin' of the group. Conway recently commented that the spelling was also inspired by the name of his collaborator. The term 'quandle', however, appears to have been coined by Joyce.

$f^* = F(f)\colon\ F(B) \longrightarrow F(A)$ in T, such that squares

$$\begin{array}{ccc} F(A) & \xleftarrow{f^*} & F(B) \\ {\scriptstyle g^*}\uparrow & & \uparrow{\scriptstyle h^*} \\ F(C) & \xleftarrow[k^*]{} & F(D) \end{array}$$

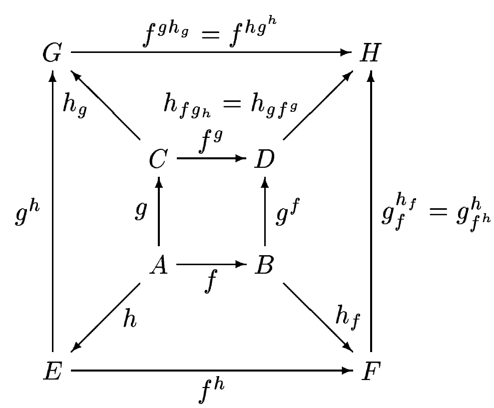
are preferred.

A **corner trunk** is a trunk which satisfies the following two axioms:

(CT1) Given an ordered pair of morphisms $A \xrightarrow{f} B$ and $A \xrightarrow{g} C$, there is a unique object D, and unique morphisms $B \xrightarrow{g^f} D$ and $C \xrightarrow{f_g} D$ such that the following square is preferred:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g^f} \\ C & \xrightarrow[f_g]{} & D \end{array}$$

(CT2) Given three morphisms $A \xrightarrow{f} B$, $A \xrightarrow{g} C$, and $A \xrightarrow{h} E$, there is a unique completion to form a commutative cube:



The corner trunk axioms are roughly analogous to associativity of composition in a category.

Any category C determines a trunk Trunk(C) with the same objects and morphisms as C, and with preferred squares the commutative squares in C. Where there is no ambiguity, we may refer to Trunk(C) as C; in particular we will adopt this abuse of terminology for Trunk(Group) and Trunk(Ab).

Further information on trunks may be found in the papers [16] and [17] by Fenn, Rourke, and Sanderson.

1.1 Extensions and expansions

We begin by devising a suitable general theory of extensions for racks. Rack extensions have been studied before, in particular by Ryder[30] under the name 'expansions'; the constructs which she dubs 'extensions' are in some sense racks formed by disjoint unions, whereby the original rack becomes a subrack of the 'extended' rack. Ryder's notion of rack expansions is somewhat more general than the extensions studied in this and later chapters, as she investigates arbitrary congruences (equivalently, surjective rack homomorphisms onto a quotient rack) whereas we will only examine certain classes of such objects.

Given a rack X we may define a trunk $\mathsf{S}(X)$ which has as objects the elements of X, and one morphism $\alpha_{x,y}\colon x \longrightarrow x^y$ for each ordered pair (x, y) of elements of X. The preferred squares of this trunk are the diagrams

$$\begin{array}{ccc} x & \xrightarrow{\alpha_{x,y}} & x^y \\ {\scriptstyle \alpha_{x,z}}\big\downarrow & & \big\downarrow{\scriptstyle \alpha_{x^y,z}} \\ x^z & \xrightarrow[\alpha_{x^z,y^z}]{} & x^{yz} = x^{zy^z} \end{array}$$

for $x, y, z \in X$. Note that this is the **extended rack trunk** of X, referred to in [16].

A trunk map $\mathcal{G}\colon \mathsf{S}(X) \longrightarrow \mathsf{Group}$, then, determines a group G_x for each element $x \in X$, and a group homomorphism $\phi_{x,y}\colon G_x \longrightarrow G_{x^y}$, for each ordered pair $(x, y) \in X \times X$, such that $\phi_{x^y,z}\phi_{x,y} = \phi_{x^z,y^z}\phi_{x,z}$ for all $x, y, z \in X$; that is, the diagrams

$$\begin{CD} G_x @>\phi_{x,y}>> G_{x^y} \\ @V{\phi_{x,z}}VV @VV{\phi_{x^y,z}}V \\ G_{x^z} @>>\phi_{x^z,y^z}> G_{x^{yz}} = G_{x^z y^z} \end{CD}$$

commute. We may, where convenient, denote such a trunk map by a pair (G, ϕ).

A trunk map $\mathcal{G} = (G, \phi)$ is said to be **thin** if $\phi_{x,u} = \phi_{x,v}$ whenever $x^u = x^v$ for any $x, u, v \in X$.

An **extension** of a rack X by a trunk map $\mathcal{G} = (G, \phi)\colon \mathsf{S}(X) \longrightarrow \mathsf{Group}$ consists of a rack E together with a surjective rack homomorphism $f\colon E \twoheadrightarrow X$ inducing a partition $E = \bigcup_{x \in X} E_x$ (where E_x is the preimage $f^{-1}(x)$), and for each $x \in X$ a right group action of G_x on E_x satisfying the following two conditions:

(i) the G_x action on E_x is simply transitive, which is to say that for any $a, b \in E_x$ there is a unique $g \in G_x$ such that $a \cdot g = b$.

(ii) for any $a \in E_x, g \in G_x$, and $b \in E_y$, $(a \cdot g)^b = (a^b) \cdot \phi_{x,y}(g)$.

Two extensions $f_1\colon E_1 \twoheadrightarrow X$ and $f_2\colon E_2 \twoheadrightarrow X$ of X by the same trunk map $\mathcal{G}$ are **equivalent** if there exists an isomorphism (an **equivalence**) $\theta\colon E_1 \longrightarrow E_2$ which respects the projection map and the group actions:

(i) $f_2\theta(a) = f_1(a)$ for all $a \in E_1$

(ii) $\theta(a \cdot g) = \theta(a) \cdot g$ for all $a \in E_x$ and $g \in G_x$

Example 1.8

The identity homomorphism $\mathrm{Id}\colon X \longrightarrow X$ *may be considered as an extension of* X *by the trunk map* $\mathcal{G} = (1, \mathrm{Id})$, *where 1 denotes the trivial group.*

1.2 Factor systems

We now investigate the classification of rack extensions by factor systems, following broadly the same approach as Grillet [21, III.4]. Let $f\colon E \twoheadrightarrow X$ be an extension of a rack X by a trunk map $\mathcal{G} = (G, \phi)\colon \mathsf{S}(X) \longrightarrow \mathsf{Group}$. A **section** of E is a function (not necessarily a rack homomorphism) $s\colon X \longrightarrow E$ such that $fs = \mathrm{Id}_X$.

Since the G_x act simply and transitively on the E_x, there is a unique $x \in X$ and a unique $g \in G_x$ such that a given element $a \in E_x$ can be written as

$$a = s(x) \cdot g.$$

Since f is a homomorphism, it follows that $s(x)^{s(y)} \in E_{x^y}$ and further that there is a unique $\sigma_{x,y} \in G_{x^y}$ such that

$$s(x)^{s(y)} = s(x^y) \cdot \sigma_{x,y}.$$

The set $\sigma = \{\sigma_{x,y} : x, y \in X\}$ is the **factor set** of the extension E **relative to** the section s.

Similarly, for every $h \in G_y$, $s(x)^{s(y)\cdot h} \in E_{x^y}$, and so there is a unique $\omega(h) \in G_{x^y}$ such that

$$s(x)^{s(y)\cdot h} = (s(x)^{s(y)}) \cdot \omega(h).$$

This determines a map $\omega_{y,x}\colon G_y \longrightarrow G_{x^y}$ which is not, in general a homomorphism, but does satisfy the property $\omega_{y,x}(1) = 1$.

It follows that, for all $x, y \in X, g \in G_x, h \in G_y$,

$$(s(x) \cdot g)^{(s(y)\cdot h)} = s(x)^{s(y)\cdot h} \cdot \phi_{x,y}(g) = \\ s(x)^{s(y)} \cdot \omega_{y,x}(h)\phi_{x,y}(g) = s(x^y) \cdot \sigma_{x,y}\omega_{y,x}(h)\phi_{x,y}(g) \quad (1.1)$$

Hence the rack structure on E is determined completely by the factor set σ and the maps $\omega_{y,x}$.

We now define another map $\psi_{y,x}\colon G_y \longrightarrow G_{x^y}$ by conjugation with the factor set:

$$\psi_{y,x} = \sigma_{x,y}\omega_{y,x}\sigma_{x,y}^{-1}$$

These new maps $\psi_{y,x}$ share the same drawback as the $\omega_{y,x}$: They satisfy the same condition $\psi_{y,x}(1) = 1$ but are not, in general, group homomorphisms. We may now write the above condition (1.1) in a slightly more elegant form, moving the factor set to the middle of the expression:

$$(s(x) \cdot g)^{(s(y) \cdot h)} = \psi_{y,x}(h)\sigma_{x,y}\phi_{x,y}(g) \tag{1.2}$$

As before, the rack structure on E is now determined by the pair (σ, ψ), which we refer to as the **factor system** of the extension E **relative to** the section s.

Example 1.9

Consider the dihedral rack D_6 (the set $\{0,1,2,3,4,5\}$ with rack operation given by $a^b = 2b - a$) as an extension of the trivial rack $T_1 = \{0\}$ by the dihedral group D_3 (of order 6) with action given by the table:

	e	ρ_1	ρ_2	μ_1	μ_2	μ_3
0	0	1	2	3	4	5
1	1	2	0	5	3	4
2	2	0	1	4	5	3
3	3	4	5	0	1	2
4	4	5	3	2	0	1
5	5	3	4	1	2	0

Let the section $s\colon T_1 \longrightarrow D_6$ send $0 \in T_1$ to $0 \in D_6$. Then the homomorphism $\phi\colon D_3 \longrightarrow D_3$ is given by:

g	e	ρ_1	ρ_2	μ_1	μ_2	μ_3
$\phi(g)$	e	μ_3	μ_2	μ_1	ρ_2	ρ_1

and the map $\omega\colon D_3 \longrightarrow D_3$ by:

h	e	ρ_1	ρ_2	μ_1	μ_2	μ_3
$\omega(h)$	e	ρ_2	μ_2	e	ρ_2	μ_2

The latter is not a homomorphism, since for example

$$\omega(\rho_1^2) = \omega(\rho_2) = \mu_2 \neq \rho_1 = \rho_2^2 = \omega(\rho_1)^2$$

Finally, the factor set σ is trivial.

The following theorem gives necessary and sufficient conditions for factor systems:

Theorem 1.1

Let X be a rack, and let $\mathcal{G} = (G, \phi)\colon \mathsf{S}(X) \longrightarrow \mathsf{Group}$ be a trunk map such that $\phi_{x,y}$ is an isomorphism for all $x, y \in X$.

Let $\sigma = \{\sigma_{x,y} \in G_{x^y} \colon x, y \in X\}$ be a collection of group elements, and let $\psi = \{\psi_{y,x} \colon G_y \longrightarrow G_{x^y} : x, y \in X\}$ be a collection of functions (not necessarily group homomorphisms) such that $\psi_{y,x}(1) = 1$ for all $x, y \in X$.

Furthermore, let $E[\mathcal{G}, \sigma, \psi]$ be the set of ordered pairs (x, g) with $x \in X$, and $g \in G_x$, with rack operation defined for all $x, y \in X, g \in G_x$, and $h \in G_y$ by

$$(x, g)^{(y,h)} = (x^y, \psi_{y,x}(h)\sigma_{x,y}\phi_{x,y}(g)) \tag{1.3}$$

Then $E[\mathcal{G}, \sigma, \psi]$ is an extension of X by $\mathcal{G}$ with factor system (σ, ψ) if

$$\psi_{z,x^y}(k)\sigma_{x^y,z}\phi_{x^y,z}\psi_{y,x}(h)\phi_{x^y,z}(\sigma_{x,y}) = \\ \psi_{y^z,x^z}\left(\psi_{z,y}(k)\sigma_{y,z}\phi_{y,z}(h)\right)\sigma_{x^z,y^z}\phi_{x^z,y^z}\psi_{z,x}(k)\phi_{x^z,y^z}(\sigma_{x,z}) \tag{1.4}$$

for all $x, y, z \in X, h \in G_y$, and $k \in G_z$.

Conversely, if E is an extension of X by $\mathcal{G}$ with factor system (σ, ψ) then (1.4) holds, and E is equivalent to $E[\mathcal{G}, \sigma, \psi]$.

Proof

To prove the first part, we require that $E[\mathcal{G}, \sigma, \psi]$ satisfy the two rack axioms. Firstly, for all $(x, g), (y, h) \in E[\mathcal{G}, \sigma, \psi]$, there must exist a unique element $(z, k) \in E[\mathcal{G}, \sigma, \psi]$ such that $(z, k)^{(y,h)} = (x, g)$. That is,

$$(z^y, \psi_{y,z}(h)\sigma_{z,y}\phi_{z,y}(h)) = (x, g)$$

The required unique element of $E[\mathcal{G}, \sigma, \psi]$ is given by:

$$(z, k) = (x^{\bar{y}}, \phi_{x,y}^{-1}(\sigma_{x,y}^{-1}\psi_{y,x}(h)^{-1}g))$$

which is well-defined by virtue of the hypotheses that X is a rack, and the ϕ

maps are isomorphisms.

Secondly, the rack operation on $E[\mathcal{G},\sigma,\psi]$ must satisfy the rack identity, so that for each $(x,g),(y,h),(z,k) \in E[\mathcal{G},\sigma,\psi]$ the expressions

$$
\begin{aligned}
(x,g)^{(y,h)(z,k)} &= (x^y, \psi_{y,x}(h)\sigma_{x,y}\phi_{x,y}(g))^{(z,k)} \\
&= (x^{yz}, \psi_{z,x^y}(k)\sigma_{x^y,z}\phi_{x^y,z}(\psi_{y,x}(h)\sigma_{x,y}\phi_{x,y}(g)))
\end{aligned}
$$

and

$$
\begin{aligned}
(x,g)^{(z,k)(y,h)^{(z,k)}} &= (x^z, \psi_{z,x}(k)\sigma_{x,z}\phi_{x,z}(g))^{(y^z,\psi_{z,y}(k)\sigma_{y,z}\phi_{y,z}(h))} \\
&= (x^{zy^z}, \psi_{y^z,x^y}(\psi_{z,y}(k)\sigma_{y,z}\phi_{y,z}(h))\sigma_{x^z,y^z}\phi_{x^z,y^z}(\psi_{z,x}(h)\sigma_{x,z}\phi_{x,z}(g)))
\end{aligned}
$$

must be equivalent, which requirement is satisfied if (1.4) holds. So $E[\mathcal{G},\sigma,\psi]$ is a rack.

Define $f\colon E[\mathcal{G},\sigma,\psi] \twoheadrightarrow X$ to be projection onto the first coordinate, and the action of G_x on $E[\mathcal{G},\sigma,\psi]_x$ by $(x,g_1)\cdot g_2 = (x,g_1g_2)$. This action is simply transitive, and satisfies the requirement

$$
\begin{aligned}
((x,g_1)\cdot g_2)^{(y,h)} &= (x,g_1g_2)^{(y,h)} \\
&= (x^y, \psi_{y,x}(h)\sigma_{x,y}\phi_{x,y}(g_1g_2)) \\
&= (x^y, \psi_{y,x}(h)\sigma_{x,y}\phi_{x,y}(g_1)\phi_{x,y}(g_2)) \\
&= (x^y, \psi_{y,x}(h)\sigma_{x,y}\phi_{x,y}(g_1))\cdot\phi_{x,y}(g_2) \\
&= (x,g_1)^{(y,h)}\cdot\phi_{x,y}(g_1)
\end{aligned}
$$

and so $E[\mathcal{G},\sigma,\psi]$ is an extension of X by $\mathcal{G}$.

Now define $s\colon X \twoheadrightarrow E[\mathcal{G},\sigma,\psi]$ by $s(x)=(x,1)$. Then

$$
s(x)^{s(y)} = (x,1)^{(y,1)} = (x^y,\sigma_{x,y}) = s(x^y)\cdot\sigma_{x,y}
$$

and

$$
s(x)^{s(y)\cdot h} = (x,1)^{(y,h)} = (x,\psi_{y,x}(h)\sigma_{x,y}) = s(x)^{s(y)}\cdot\sigma_{x,y}^{-1}\psi_{y,x}(h)\sigma_{x,y},
$$

so (σ, ψ) is the factor system of this extension relative to the section s.

Conversely, let $f\colon E \twoheadrightarrow X$ be an extension of the rack X by a given trunk map $\mathcal{G} = (G, \phi)\colon \mathsf{S}(X) \longrightarrow \mathsf{Group}$, with factor system (σ, ψ) relative to some section $s\colon X \longrightarrow E$.

By the simple transitivity of the G_x-action on E_x, and by condition (1.2), the map $\theta\colon (x, g) \mapsto s(x) \cdot g$ is an isomorphism $E[\mathcal{G}, \sigma, \psi] \cong E$. Since E satisfies the rack axioms, the earlier part of the proof shows that (1.4) holds, and so $E[\mathcal{G}, \sigma, \psi]$ is another extension of X by $\mathcal{G}$. Furthermore, θ preserves projection onto X, and

$$\theta((x, g_1) \cdot g_2) = \theta(x, g_1 g_2) = s(x) \cdot (g_1 g_2) = (s(x) \cdot g_1) \cdot g_2 = \theta(x, g_1) \cdot g_2,$$

so θ is an equivalence of extensions. □

The following proposition gives conditions for two extensions of X by the same trunk map $\mathcal{G}\colon \mathsf{S}(X) \longrightarrow \mathsf{Group}$ to be equivalent:

Proposition 1.2

Let $\mathcal{G} = (G, \phi)\colon \mathsf{S}(X) \longrightarrow \mathsf{Group}$ be a trunk map, and $(\sigma, \psi), (\tau, \omega)$ be two factor systems for G. Then the following are equivalent:

(i) *$E[\mathcal{G}, \sigma, \psi]$ and $E[\mathcal{G}, \tau, \omega]$ are equivalent as extensions of X,*

(ii) *there exists a $u = \{u_x \in G_x : x \in X\}$ such that for all $x, y \in X, h \in G_y$:*

$$\tau_{x,y} = u_{x^y}^{-1} \psi_{y,x}(u_y) \sigma_{x,y} \phi_{x,y}(u_x) \tag{1.5}$$

$$\omega_{y,x}(h) = u_{x^y}^{-1} \psi_{y,x}(u_y h) \psi_{y,x}(u_y)^{-1} u_{x^y} \tag{1.6}$$

(iii) *(σ, ψ) and (τ, ω) are factor systems of the same extension of X by $\mathcal{G}$ relative to different cross-sections.*

Proof

Let $\theta\colon E[\mathcal{G}, \sigma, \psi] \longrightarrow E[\mathcal{G}, \tau, \omega]$ be the hypothesised equivalence. Then it follows that $\theta(x, 1) = (x, u_x)$ for some $u_x \in G_x$, since θ preserves projection onto X.

Furthermore,

$$\theta(x, g) = \theta((x, 1) \cdot g) = \theta(x, 1) \cdot g = (x, u_x) \cdot g = (x, u_x g)$$

for all $g \in G_x$, since θ preserves the right group actions. Then

$$\begin{aligned} \theta((x, g)^{(y,h)}) &= (x^y, \psi_{y,x}(h)\sigma_{x,y}\phi_{x,y}(g)) \\ \theta(x, g)^{\theta(y,h)} &= (x, u_x g)^{(y,u_y h)} \\ &= (x^y, \psi_{y,x}(u_y h)\sigma_{x,y}\phi_{x,y}(u_x)\phi_{x,y}(g)). \end{aligned}$$

Since θ is an isomorphism, it follows that

$$u_{x^y}\omega_{y,x}(h)\tau_{x,y} = \psi_{y,x}(u_y h)\sigma_{x,y}\phi_{x,y}(u_x) \tag{1.7}$$

for all $x, y \in X, h \in G_y$. Setting h to 1 yields:

$$\tau_{x,y} = u_{x^y}^{-1}\psi_{y,x}(u_y)\sigma_{x,y}\phi_{x,y}(u_x)$$

which, when substituted in (1.7) gives

$$\omega_{y,x}(h) = u_{x^y}^{-1}\psi_{y,x}(u_y h)\psi_{y,x}(u_y)^{-1}u_{x^y}$$

This calculation is reversible, showing the equivalence of the first two conditions. Now, given such an equivalence θ, define a new section $t\colon X \longrightarrow E[\mathcal{G}, \sigma, \psi]$ as $t(x) = (x, u_x)$. Then the above argument also shows that

$$t(x)^{t(y)\cdot h} = (x, u_x)^{(y,u_y h)} = t(x^y) \cdot \omega_{y,x}(h)\tau_{x,y}$$

which implies that (τ, ω) is the factor system of $E[\mathcal{G}, \sigma, \psi]$ relative to the section t. Hence (σ, ψ) and (τ, ω) are factor systems of $E[\mathcal{G}, \sigma, \psi]$ relative to different sections. This property holds for any extension equivalent to $E[\mathcal{G}, \sigma, \psi]$ (and hence $E[\mathcal{G}, \tau, \omega]$). Conversely, if (σ, ψ) and (τ, ω) are factor systems of some extension E of X relative to different sections $s, t\colon X \longrightarrow E$, then $t(x) = s(x) \cdot u_x$ for some $u_x \in G_x$. Hence the first and third conditions are equivalent. □

A factor system (σ, ψ) relative to a section $s\colon X \longrightarrow E$ gives an obstruction to s being a rack homomorphism. We now consider the special case where s is a rack homomorphism in order to justify this viewpoint.

Proposition 1.3

For an extension $E \xrightarrow{f} X$ by a trunk map $\mathcal{G} = (G, \phi)\colon \mathsf{S}(X) \longrightarrow \mathsf{Group}$, the following are equivalent:

(i) There exists a rack homomorphism $s\colon X \longrightarrow E$ such that $fs = \mathrm{Id}_X$,

(ii) relative to some section the factor set of E is trivial (that is, $\sigma_{x,y} = 1 \in G_{x^y}$ and $\psi_{y,x}(h) = 1 \in G_{x^y}$, for all $x, y \in X$ and $h \in G_y$),

(iii) relative to any section there exists for the factor system (σ, ψ) of E a family $u = \{u_x : x \in X, u_x \in G_x\}$ such that for all $x, y \in X$,

$$\sigma_{x,y} = \psi_{y,x}(u_y)^{-1} u_{x^y} \phi_{x,y}(u_x)^{-1} \tag{1.8}$$

Proof

Assume that there exists a rack homomorphism $s\colon X \longrightarrow E$ such that $fs = \mathrm{Id}_X$, then by the definition of the factor set σ of E relative to s

$$s(x)^{s(y)} = s(x^y) \cdot \sigma_{x,y}$$

for some $\sigma_{x,y} \in G_{x^y}$, for all $x, y \in X$. But since s is a rack homomorphism, $\sigma_{x,y} = 1 \in G_{x^y}$ for all $x, y \in X$. The converse holds, showing the equivalence of the first two conditions.

Now suppose that there is a section $s\colon X \longrightarrow E$ with trivial factor set. Then any other section of this extension, with factor system (σ, ψ), satisfies the following condition (which is the special case of proposition 1.2 where σ is trivial):

$$1 = u_{x^y}^{-1} \psi_{y,x}(u_y) \sigma_{x,y} \phi_{x,y}(u_x)$$

for some suitable $u = \{u_x : x \in X, u_x \in G_x\}$ Hence

$$\sigma_{x,y} = \psi_{y,x}(u_y)^{-1} u_{x^y} \phi_{x,y}(u_x)^{-1}$$

The converse also holds: if a factor system (σ, ψ) exists satisfying the above condition then it is equivalent to the trivial factor system. Thus the second and third conditions are equivalent. □

Such extensions are termed **split extensions**; a factor system **splits** if it satisfies the third of the above hypotheses.

1.3 Abelian extensions

We now consider the case of a trunk map $\mathcal{A}\colon \mathsf{S}(X) \longrightarrow \mathsf{Ab}$, and restrict the extensions under consideration to the case where the maps $\psi_{y,x}\colon A_y \longrightarrow A_{x^y}$ are homomorphisms. The conditions described in theorem 1.1 then simplify considerably:

Theorem 1.4

Let X be a rack, and let $\mathcal{A} = (A, \phi)\colon \mathsf{S}(X) \longrightarrow \mathsf{Ab}$ such that $\phi_{x,y}$ is an isomorphism for all $x, y \in X$. Furthermore, let $\sigma = \{\sigma_{x,y} \in A_{x^y}\colon x, y \in X\}$ be a collection of group elements, and let $\psi = \{\psi_{y,x}\colon A_y \longrightarrow A_{x^y} : x, y \in X\}$ be a collection of abelian group homomorphisms. As in theorem 1.1 denote by $E[\mathcal{A}, \sigma, \psi]$ the set of ordered pairs (x, a) with $x \in X$ and $a \in A_x$, with rack operation

$$(x, a)^{(y,b)} = (x^y, \psi_{y,x}(b) + \sigma_{x,y} + \phi_{x,y}(a)) \tag{1.9}$$

for all $x, y \in X$, $a \in A_x$, and $b \in A_y$.

Then (σ, ψ) is a factor system for $\mathcal{A}$ iff the following conditions hold:

$$\psi_{z,x^y}(k) = \psi_{y^z,x^z}\psi_{z,y}(k) + \phi_{x^z,y^z}\psi_{z,x}(k) \tag{1.10}$$

$$\phi_{x^y,z}\psi_{y,x}(h) = \psi_{y^z,x^z}\phi_{y,z}(h) \tag{1.11}$$

$$\sigma_{x^y,z} + \phi_{x^y,z}(\sigma_{x,y}) = \psi_{y^z,x^z}(\sigma_{y,z}) + \sigma_{x^z,y^z} + \phi_{x^z,y^z}(\sigma_{x,z}) \tag{1.12}$$

for all $x, y, z \in X, h \in A_y$, and $k \in A_z$.

Furthermore, another factor system (τ, ω) for $\mathcal{A}$ is equivalent to (σ, ψ) iff there exists a family $u = \{u_x : x \in X, u_x \in A_x\}$ such that:

$$\tau_{x,y} = \psi_{y,x}(u_y) + \phi_{x,y}(u_x) - u_{x^y} + \sigma_{x,y} \tag{1.13}$$

and if

$$\omega_{y,x} = \psi_{y,x}$$

for all $x, y \in X$.

A factor system (σ, ψ) *splits iff there exists a family* $u = \{u_x : x \in X, u_x \in A_x\}$ *such that:*

$$\sigma_{x,y} \quad = \quad \psi_{y,x}(u_y) + \phi_{x,y}(u_x) - u_{x^y} \tag{1.14}$$

for all $x, y \in X$.

Proof

In the case where the groups A_x are abelian, and the maps $\psi_{x,y}$ are homomorphisms, condition (1.4) becomes:

$$\begin{aligned}\psi_{z,x^y}(k) + \sigma_{x^y,z} + \phi_{x^y,z}\psi_{y,x}(h) + \phi_{x^y,z}(\sigma_{x,y}) = \\ \psi_{y^z,x^z}\psi_{z,y}(k) + \psi_{y^z,x^z}(\sigma_{y,z}) + \psi_{y^z,x^z}\phi_{y,z}(h) \\ + \sigma_{x^z,y^z} + \phi_{x^z,y^z}\psi_{z,x}(k) + \phi_{x^z,y^z}(\sigma_{x,z})\end{aligned}$$

which splits into the required conditions. Similarly, condition (1.5) reduces to:

$$\tau_{x,y} = \psi_{y,x}(u_y) + \phi_{x,y}(u_x) + \sigma_{x,y} - u_{x^y}$$

and condition (1.8) to:

$$\sigma_{x,y} = \psi_{y,x}(u_y) + \phi_{x,y}(u_x) - u_{x^y}$$

as required. □

We wish to define (abelian) groups $\mathrm{Ext}_\psi(X, \mathcal{A})$ which classify, up to equivalence, extensions of a given rack X by a fixed trunk map $\mathcal{A} = (A, \phi)\colon \mathsf{S}(X) \longrightarrow \mathsf{Ab}$ and a fixed collection of (abelian group) homomorphisms ψ. The preceding result indicates that this is equivalent to classifying factor sets modulo those which split. The maps ψ are very closely connected with the trunk map $\mathcal{A}$ and it seems both sensible and elegant to incorporate them, together with conditions

(1.10) and (1.11), into the definition of our chosen class of trunk maps. This requires the replacement of the trunk $\mathsf{S}(X)$ with a modified version which we will denote $\mathsf{T}(X)$.

Let $\mathsf{T}(X)$ have, as before, objects $x \in X$, and morphisms $\alpha_{x,y} \colon x \longrightarrow x^y$ for each ordered pair $(x, y) \in X \times X$. Let the squares

$$\begin{array}{ccc} x & \xrightarrow{\alpha_{x,y}} & x^y \\ {\scriptstyle \alpha_{x,z}}\downarrow & & \downarrow{\scriptstyle \alpha_{x^y,z}} \\ x^z & \xrightarrow[\alpha_{x^z,y^z}]{} & x^{yz} = x^{zy^z} \end{array}$$

be preferred for all $x, y, z \in X$, as before.

Furthermore, let $\mathsf{T}(X)$ have additional morphisms $\beta_{y,x} \colon y \longrightarrow x^y$ for all $x, y \in X$, and let all squares

$$\begin{array}{ccc} y & \xrightarrow{\beta_{y,x}} & x^y \\ {\scriptstyle \alpha_{y,z}}\downarrow & & \downarrow{\scriptstyle \alpha_{x^y,z}} \\ y^z & \xrightarrow[\beta_{y^z,x^z}]{} & x^{yz} = x^{zy^z} \end{array}$$

be preferred, for $x, y, z \in X$.

A trunk map $\mathcal{A} \colon \mathsf{T}(X) \longrightarrow \mathsf{Ab}$, then, determines an abelian group A_x for each element x of the rack X, and (abelian group) homomorphisms $\phi_{x,y} \colon A_x \longrightarrow A_{x^y}$ and $\psi_{y,x} \colon A_y \longrightarrow A_{x^y}$ for $x, y \in X$, such that the diagrams

$$\begin{array}{ccc} A_x & \xrightarrow{\phi_{x,y}} & A_{x^y} \\ {\scriptstyle \phi_{x,z}}\downarrow & & \downarrow{\scriptstyle \phi_{x^y,z}} \\ A_{x^z} & \xrightarrow[\phi_{x^z,y^z}]{} & A_{x^{yz}} = A_{x^{zy^z}} \end{array} \qquad \begin{array}{ccc} A_y & \xrightarrow{\psi_{y,x}} & A_{x^y} \\ {\scriptstyle \phi_{y,z}}\downarrow & & \downarrow{\scriptstyle \phi_{x^y,z}} \\ A_{y^z} & \xrightarrow[\psi_{y^z,x^z}]{} & A_{x^{yz}} = A_{x^{zy^z}} \end{array}$$

commute. We may, where convenient and unambiguous, denote such a trunk map $\mathcal{A}$ by a triple (A, ϕ, ψ).

The following result is a modified version of theorem 1.4:

Theorem 1.5

Let X be a rack, and let $\mathcal{A} = (A, \phi, \psi)$ be a trunk map $\mathsf{T}(X) \longrightarrow \mathsf{Ab}$ such that

$\phi_{x,y}$ is an isomorphism $A_x \to A_{x^y}$, and

$$\psi_{z,x^y}(k) = \psi_{y^z,x^z}\psi_{x,y}(k) + \phi_{x^y,z^y}\psi_{z,x}(k) \qquad (1.15)$$

for all $x, y, z \in X$ and $k \in A_z$. Then extensions of X by $\mathcal{A}$ are in bijective correspondence with factor sets σ satisfying the condition

$$\sigma_{x^y,z} + \phi_{x^y,z}(\sigma_{x,y}) = \psi_{y^z,x^z}(\sigma_{y,z}) + \sigma_{x^z,y^z} + \phi_{x^z,y^z}(\sigma_{x,z}) \qquad (1.16)$$

Let τ be another factor set corresponding to an extension of X by $\mathcal{A}$. Then the extension corresponding to τ is equivalent to the extension corresponding to σ iff there exists a family $u = \{u_x : x \in X, u_x \in A_x\}$ such that

$$\tau_{x,y} = \psi_{y,x}(u_y) + \phi_{x,y}(u_x) + \sigma_{x,y}$$

for all $x, y \in X$. Furthermore, a factor set σ splits iff it is equivalent to the trivial factor set. That is, if there exists a family $u = \{u_x : x \in X, u_x \in A_x\}$ such that

$$\sigma_{x,y} = \psi_{y,x}(u_y) + \phi_{x,y}(u_x) - u_{x^y}$$

As hoped, given a trunk map $\mathcal{A}\colon \mathsf{T}(X) \to \mathsf{Ab}$ satisfying the hypotheses of theorem 1.5, we may define an abelian group $\operatorname{Ext}(X, \mathcal{A})$ consisting of equivalence classes of factor sets σ of extensions of X by $\mathcal{A}$, together with a suitable addition operation.

Theorem 1.6

Let X be a rack, and $\mathcal{A} = (A, \phi, \psi)$ be a trunk map $\mathsf{T}(X) \to \mathsf{Ab}$ such that $\phi_{x,y}\colon A_x \cong A_{x^y}$ and

$$\psi_{z,x^y}(k) = \psi_{y^z,x^z}\psi_{z,y}(k) + \phi_{x^z,y^z}\psi_{z,x}(k)$$

for all $x, y, z \in X$, and $k \in A_z$.

Then there is an abelian group $\operatorname{Ext}(X, \mathcal{A})$ classifying extensions $f\colon E \twoheadrightarrow X$ of X by $\mathcal{A}$ up to equivalence.

Proof

Define $Z(X, \mathcal{A})$ to consist of extensions of X by $\mathcal{A}$. As just shown, these are determined by factor sets σ satisfying condition (1.16).

Defining addition pointwise, so that $(\sigma+\tau)_{x,y} = \sigma_{x,y} + \tau_{x,y}$, we see that this set forms a group, with the trivial factor set $\sigma = 0$ as identity (which may be seen to satisfy (1.16), and corresponds to the appropriate split extension).

Furthermore, if a factor set σ satisfies (1.16) then so does the corresponding inverse factor set $-\sigma = \{-\sigma_{x,y} : \sigma_{x,y} \in \sigma\}$. The set $Z(X, \mathcal{A})$ is thus closed under addition, and is thus an abelian group (since all the A_x are).

We may also define groups $B(X, \mathcal{A})$ consisting of equivalence classes of split extensions. These are determined by factor sets σ satisfying condition (1.14), which form a subgroup of $Z(X, \mathcal{A})$, since substituting (1.14) into the left- and right-hand sides of (1.16) gives equality:

$$\begin{aligned}
&\sigma_{x^y,z} + \phi_{x^y,z}(\sigma_{x,y}) = \\
&\qquad \psi_{z,x^y}(u_z) + \phi_{x^y,z}(u_{x^y}) - u_{x^{yz}} \\
&\qquad + \phi_{x^y,z}\psi_{y,x}(u_y) + \phi_{x^y,z}\phi_{x,y}(u_x) - \phi_{x^y,z}(u_{x^y}) \\
&\qquad = \psi_{z,x^y}(u_z) - u_{x^{yz}} + \phi_{x^y,z}\psi_{y,x}(u_y) + \phi_{x^y,z}\phi_{x,y}(u_x)
\end{aligned}$$

and

$$\begin{aligned}
&\psi_{y^z,x^z}(\sigma_{y,z}) + \sigma_{x^z,y^z} + \phi_{x^z,y^z}(\sigma_{x,z}) = \\
&\qquad \psi_{y^z,x^z}\psi_{z,y}(u_z) + \psi_{y^z,x^z}\phi_{y,z}(u_y) - \psi_{y^z,x^z}(u_{y^z}) \\
&\qquad + \psi_{y^z,x^z}(u_{y^z}) + \phi_{x^z,y^z}(u_{x^z}) - u_{x^{zy^z}} \\
&\qquad + \phi_{x^z,y^z}\psi_{z,x}(u_z) + \phi_{x^z,y^z}\phi_{x,z}(u_x) - \phi_{x^z,y^z}(u_{x^z}) \\
&= \psi_{y^z,x^z}\psi_{z,y}(u_z) + \psi_{y^z,x^z}\phi_{y,z}(u_y) - u_{x^{zy^z}} + \phi_{x^z,y^z}\psi_{z,x}(u_z) + \phi_{x^z,y^z}\phi_{x,z}(u_x)
\end{aligned}$$

Finally, we define $\mathrm{Ext}(X, \mathcal{A}) = Z(X, \mathcal{A})/B(X, \mathcal{A})$. □

We will see later that the trunk map $\mathcal{A} = (A, \phi, \psi)$ plays the rôle of a coefficient module for homology and cohomology.

In the case where the groups A_x are all the same group A, the ϕ-isomorphisms the identity map Id_A, and the ψ-homomorphisms are all zero, then this definition reduces to that of the second cohomology group $H^2(X;A)$ with coefficients in the abelian group A, with the factor sets σ fulfilling the rôle of 2-cocycles.

Example 1.10

Let $\mathbb{Z}_n$ denote the trunk map $(\mathbb{Z}_n, \mathrm{Id}, 0)\colon \mathsf{T}(T_2) \longrightarrow \mathsf{Ab}$. We now compute $\mathrm{Ext}(T_2, \mathbb{Z}_n)$. The group $Z(T_2, \mathbb{Z}_n)$ consists of factor sets σ satisfying the condition (1.11), which we evaluate as x, y, z range over $T_2 = \{0, 1\}$ to obtain conditions on the $\sigma_{x,y}$. Doing this, we find that

$$\begin{aligned} Z(T_2, \mathbb{Z}_n) &= \{\sigma : \sigma_{0,0}, \sigma_{0,1}, \sigma_{1,0}, \sigma_{1,1} \in \mathbb{Z}_n\} \\ &\cong \mathbb{Z}_n \oplus \mathbb{Z}_n \oplus \mathbb{Z}_n \oplus \mathbb{Z}_n \\ Z(T_2, \mathbb{Z}) &\cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \end{aligned}$$

The subgroups $B(T_2, \mathbb{Z}_n)$ and $B(T_2, \mathbb{Z})$ consist of factor sets τ satisfying condition (1.14). In these cases, the only split factor sets are trivial: $\tau = 0$. Hence:

$$\begin{aligned} \mathrm{Ext}(T_2, \mathbb{Z}_n) &= \mathbb{Z}_n \oplus \mathbb{Z}_n \oplus \mathbb{Z}_n \oplus \mathbb{Z}_n \\ \mathrm{Ext}(T_2, \mathbb{Z}) &= \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \end{aligned}$$

This may be seen to agree, as remarked above, with the second cohomology of the rack T_2 with coefficients in cyclic groups.

Example 1.11

Let D_n denote the trunk map $(\mathbb{Z}_n, \phi, \psi)$ where $\phi_{x,y}$ is multiplication by -1, and $\psi_{y,x}$ is multiplication by 2. Denote by D_∞ the trunk map $(\mathbb{Z}, \phi, \psi)$. A routine check confirms that these are trunk maps $\mathsf{T}(T_2) \longrightarrow \mathsf{Ab}$, and satisfy the condition (1.15). Then as before we consider (1.11) as x, y, z range over T_2, and find that

$$Z(T_2, D_n) = \{\sigma : 2\sigma_{0,0} = 2\sigma_{1,1} = 0,\ 2\sigma_{0,1} = -2\sigma_{1,0}\}$$

Hence

$$\begin{aligned}
Z(T_2, D_{2k}) &= \{\sigma : \sigma_{0,0}, \sigma_{1,1} \in \{0, k\}, \sigma_{0,1} + \sigma_{1,0} \in \{0, k\}\} \\
&\cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{2k} \\
Z(T_2, D_{2k+1}) &= \{\sigma : \sigma_{0,0} = \sigma_{1,1} = 0, \sigma_{0,1} = -\sigma_{1,0}\} \\
&\cong \mathbb{Z}_{2k+1} \\
Z(T_2, D_\infty) &\cong \mathbb{Z}
\end{aligned}$$

Considering the form of the split factor sets, we find that

$$\begin{aligned}
B(T_2, D_n) &= \{\tau : \tau_{0,0} = \tau_{1,1} = 0, \tau_{0,1} = -\tau_{1,0}\} \\
&= \mathbb{Z}_n
\end{aligned}$$

and thus

$$\begin{aligned}
\mathrm{Ext}(T_2, D_{2k}) &\cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \\
\mathrm{Ext}(T_2, D_{2k+1}) &= 0 \\
\mathrm{Ext}(T_2, D_\infty) &= 0
\end{aligned}$$

This example demonstrates extensions by a nontrivial trunk map, and is analogous to the case of group extensions by a nontrivial G-module.

Example 1.12

By a similar calculation, we find that

$$\begin{aligned}
Z(C_2, \mathbb{Z}_n) &= \{\sigma : \sigma_{0,0} = \sigma_{0,1}, \sigma_{1,0} = \sigma_{1,1}\} \\
&\cong \mathbb{Z}_n \oplus \mathbb{Z}_n \\
B(C_2, \mathbb{Z}_n) &= \{\tau : \tau_{0,0} = \tau_{0,1} = -\tau_{1,0} = -\tau_{1,1}\} \\
&\cong \mathbb{Z}_n \\
Z(C_2, \mathbb{Z}) &\cong \mathbb{Z} \oplus \mathbb{Z} \\
B(C_2, \mathbb{Z}) &\cong \mathbb{Z}
\end{aligned}$$

and hence

$$\begin{aligned}\mathrm{Ext}(C_2, \mathbb{Z}_n) &\cong \mathbb{Z}_n \\ \mathrm{Ext}(C_2, \mathbb{Z}) &\cong \mathbb{Z}\end{aligned}$$

This, again, agrees with the second cohomology with coefficients in a cyclic group.

Example 1.13

Further computation yields:

$$\begin{aligned}\mathrm{Ext}(C_2, D_{2k}) &\cong \mathbb{Z}_2 \\ \mathrm{Ext}(C_2, D_{2k+1}) &= 0 \\ \mathrm{Ext}(C_2, D_\infty) &= 0\end{aligned}$$

which again shows the effect of extending by a nontrivial trunk map.

All of the above examples consider the case where the groups A_x are the same. The next calculation considers a case where they are not.

Example 1.14

Let $D_{2,4}$ *denote the trunk map* $D\colon \mathsf{T}(T_2) \longrightarrow \mathsf{Ab}$ *where* $D_0 = \mathbb{Z}_2$ *and* $D_1 = \mathbb{Z}_4$, *with the usual dihedral structure maps* $\phi_{x,y} = -\mathrm{Id}\colon D_x \longrightarrow D_{x^y} = D_x$ *and* $\psi_{y,x} = 2\mathrm{Id}\colon D_y \longrightarrow D_{x^y} = D_x$ *for all* $x, y \in T_2$. *This satisfies criterion (1.15), and thus we can calculate* $\mathrm{Ext}(T_2, D_{2,4})$.

It transpires that

$$\begin{aligned}Z(T_2, D_{2,4}) &= \{\sigma : \sigma_{0,0} = 0; \sigma_{1,1} = 0, 2; \sigma_{0,1} \in \mathbb{Z}_2, \sigma_{1,0} \in \mathbb{Z}_4; 2\sigma_{0,1} + 2\sigma_{1,0} = 0\} \\ &\cong \mathbb{Z}_2 \oplus \mathbb{Z}_4 \\ B(T_2, D_{2,4}) &= \{\sigma : \sigma_{0,0} = \sigma_{0,1} = \sigma_{1,1} = 0; \sigma_{1,0} = 0, 2\} \\ &\cong \mathbb{Z}_2\end{aligned}$$

and so $\mathrm{Ext}(T_2, D_{2,4}) \cong \mathbb{Z}_4$.

1.4 Quandle extensions

The preceding sections have been concerned with the theory of extensions in the category Rack. We now investigate extensions in the category Quandle.

Theorem 1.7

Let X be a quandle, and $\mathcal{A} = (A, \phi, \psi)$ be a trunk map $\mathsf{T}(X) \longrightarrow \mathsf{Ab}$ satisfying the hypotheses of theorem 1.5, and the additional requirement that

$$\psi_{x,x}(g) + \phi_{x,x}(g) \quad = \quad g \tag{1.17}$$

for all $x \in X$ and $g \in A_x$.

Then extensions $f\colon E \twoheadrightarrow X$ of X by $\mathcal{A}$, such that E is also a quandle, are in bijective correspondence with factor sets σ satisfying hypothesis (1.11) of theorem 1.5 and the requirement

$$\sigma_{x,x} \quad = \quad 0 \tag{1.18}$$

for all $x \in X$.

Proof

Following the reasoning of theorem 1.1, the quandle condition on E is equivalent to:

$$(x, g)^{(x,g)} = (x, g)$$

for every $x \in X$ and $g \in G_x$, or equivalently:

$$(x^x, \psi_{x,x}(g) + \sigma_{x,x} + \phi_{x,x}(g)) = (x, g).$$

So E is a quandle iff $\psi_{x,x}(g) + \sigma_{x,x} + \phi_{x,x}(g) = g$; that is, if $\sigma_{x,x} = 0$ and $\psi_{x,x}(g) + \phi_{x,x}(g) = g$ for all $g \in A_x$, and $x \in X$. □

In [13], Carter, Saito and Kamada define the notion of an **abelian extension** of a quandle. Let X be a quandle, A an abelian group, and $f \in H^2_Q(X; A)$ a 2-cocycle on X. Then we may define a quandle operation on the set $A \times X$ by

$$(a, x)^{(b,y)} = (a + f(x, y), x^y)$$

for all $x, y \in X$.

This may readily be seen to be the special case of the above theory where the groups $A_x = A$ are all the same, $\phi_{x,y} = \mathrm{Id}_A$, and $\psi_{y,x} = 0_A$ for all $x, y \in X$. In the terminology of the next chapter, they are quandle extensions by a 'trivial homogeneous quandle module'.

In [10], Carter, Saito and Elhamdadi define a **twisted extension** of a quandle as follows: Let X be a quandle, let A be an Alexander quandle as defined in example 1.5, and let $f \in H^2_{TQ}(X; A)$. Then we may define a quandle operation on $X \times A$ by:

$$(a, x)^{(b,y)} = (a^b + f(x, y), x^y) = (ta + f(x, y) + (1 - t)b, x^y)$$

which may also be seen to be a special case of the theory described earlier. For, let $\mathcal{A}$ be the trunk map $\mathsf{T}(X) \longrightarrow \mathsf{Ab}$ such that

$$\begin{aligned} A_x &= \mathbb{Z}[t, t^{-1}]/h(t) \\ \phi_{x,y}(a) &= ta \\ \psi_{y,x}(b) &= (1 - t)b \end{aligned}$$

for all $x, y \in X, a \in A_x$, and $b \in A_y$, and where $h(t)$ is a Laurent polynomial in one variable t. In terminology to be introduced in the next chapter, they are quandle extensions by a 'homogeneous Alexander module'.

We can define groups $\mathrm{Ext}_Q(X, \mathcal{A})$ consisting of equivalence classes of such quandle extensions. As with the more general rack extension groups $\mathrm{Ext}(X, \mathcal{A})$, these are isomorphic to the appropriate second cohomology groups $H^2_Q(X, A)$ in the case where $\mathcal{A} = (A, \mathrm{Id}, 0)$.

Example 1.15

Calculation of the groups Ext_Q *follows much the same procedure as the groups* Ext *in the previous section, except with the extra condition (1.18). We may obtain quandle extensions by the (trivial) trunk maps* $\mathbb{Z}$ *and* $\mathbb{Z}_n$ *since they may* *be seen to satisfy (1.17). Hence:*

$$\begin{aligned}\mathrm{Ext}_Q(T_2,\mathbb{Z}_n) &\cong \mathbb{Z}_n\oplus\mathbb{Z}_n\\ \mathrm{Ext}_Q(T_2,\mathbb{Z}) &\cong \mathbb{Z}\oplus\mathbb{Z}\end{aligned}$$

Example 1.16

The trunk maps D_n and D_∞ defined in the previous section also satisfy condition (1.17) and so we may define and calculate the corresponding extension groups:

$$\begin{aligned}\mathrm{Ext}_Q(T_2,D_{2k}) &\cong \mathbb{Z}_2\\ \mathrm{Ext}_Q(T_2,D_{2k+1}) &= 0\\ \mathrm{Ext}_Q(T_2,D_\infty) &= 0\end{aligned}$$

Note that, as with the previous example, these are different to the groups of extensions by the corresponding trivial trunk maps.

1.5 Involutory extensions

We may also consider the case of extensions in the category $\mathsf{InvRack}$ of involutory racks, and the category $\mathsf{InvQuandle}$ of involutory quandles.

Theorem 1.8

Let X be an involutory rack, and let $\mathcal{A}=(A,\phi,\psi)$ be a trunk map $\mathsf{T}(X)\longrightarrow\mathsf{Ab}$ satisfying the hypotheses of theorem 1.5, as well as the requirements

$$\phi_{x^y,y}\phi_{x,y} = \mathrm{Id}_{A_x} \tag{1.19}$$

$$\psi_{y,x}(h)+\phi_{x^y,y}\psi_{y,x}(h) = 0 \tag{1.20}$$

for all $x,y\in X$ and $h\in A_y$.

Then extensions $f\colon E\twoheadrightarrow X$ of X by $\mathcal{A}$ such that E is also an involutory rack, are in bijective correspondence with factor sets σ satisfying hypothesis (1.11) of theorem 1.5, together with the requirement

$$\sigma_{x^y,y}+\phi_{x^y,y}(\sigma_{x,y}) = 0 \tag{1.21}$$

for all $x, y \in X$.

Proof

Following the reasoning of theorem 1.1, the involutory condition on E is equivalent to:

$$(x,g)^{(y,h)(y,h)} = (x,g)$$

for every $x \in X$ and $g \in G_x$, or equivalently:

$$(x^{yy}, \psi_{y,x^y}(h) + \sigma_{x^y,y} + \phi_{x^y,y}\psi_{y,x}(h) + \ \phi_{x^y,y}(\sigma_{x,y}) + \phi_{x^y,y}\phi_{x,y}(g)) = (x,g)$$

The first coordinate of this splits neatly into the required three conditions. □

This result, as with theorem 1.7 in the previous section, gives additional conditions on the trunk maps $\mathcal{A} = (A, \phi, \psi)$, as well as on the factor sets σ.

As before, we may define groups $\mathrm{Ext}_I(X, \mathcal{A})$ consisting of equivalence classes of such involutory extensions. Defining $H^2_I(X; \mathcal{A}) = \mathrm{Ext}_I(X, \mathcal{A})$, it seems natural to suspect that state sums derived from such 2-cocycles will provide unoriented regular isotopy invariants of knots and links. We will investigate such applications in chapter 7.

Example 1.17

As with quandle extensions, the Ext_I *groups are calculated in the same way as the ordinary (rack)* Ext *groups, but with condition (1.21) taken into consideration. So*

$$\begin{aligned} Z_I(T_2, \mathbb{Z}_n) &= \{\sigma : \sigma_{0,0}, \sigma_{0,1}, \sigma_{1,0}, \sigma_{1,1} \in \mathbb{Z}_n, 2\sigma_{0,0} = 2\sigma_{0,1} = 2\sigma_{1,0} = 2\sigma_{1,1}\} \\ B_I(T_2, \mathbb{Z}_n) &= 0 \end{aligned}$$

hence

$$\begin{aligned} \mathrm{Ext}_I(T_2, \mathbb{Z}_{2k}) &\cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \\ \mathrm{Ext}_I(T_2, \mathbb{Z}_{2k+1}) &= 0 \\ \mathrm{Ext}_I(T_2, \mathbb{Z}) &= 0 \end{aligned}$$

Example 1.18

Involutory extensions of the trivial rack T_2 by 'dihedral' trunk maps are classified as follows:

$$\begin{aligned}\mathrm{Ext}_I(T_2, D_{2k}) &\cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \\ \mathrm{Ext}_I(T_2, D_{2k+1}) &= 0 \\ \mathrm{Ext}_I(T_2, D_\infty) &= 0\end{aligned}$$

In particular, all extensions of T_2 by odd dihedral trunk maps split.

The following two examples show that involutory extensions of the cyclic rack C_2 by the integral and dihedral trunk maps split:

Example 1.19

$$\begin{aligned}\mathrm{Ext}_I(C_2, \mathbb{Z}_n) &= 0 \\ \mathrm{Ext}_I(C_2, \mathbb{Z}) &= 0\end{aligned}$$

Example 1.20

$$\begin{aligned}\mathrm{Ext}_I(C_2, D_n) &= 0 \\ \mathrm{Ext}_I(C_2, D_\infty) &= 0\end{aligned}$$

Furthermore, we may combine the Ext_I and Ext_Q groups by requiring such extensions (of an involutory quandle) to satisfy both the conditions for involutory extensions and quandle extensions. Such Ext_{IQ} groups lead to state-sums which are unoriented ambient isotopy invariants of knots and links.

Chapter 2

Modules

In his doctoral thesis [5], Jonathan Beck devised a general definition for coefficient modules of algebraic (co)homology theories. This chapter is concerned with the definition of the appropriate objects for rack (co)homology theories (and the related quandle and involutory theories).

Let a category C have finite products and a terminal object T. Then an **abelian group object** in C is an object A equipped with

(i) A multiplication (or addition) morphism $m\colon\ A \times A \longrightarrow A$

(ii) An inverting morphism $r\colon\ A \longrightarrow A$

(iii) A unit morphism $s\colon\ T \longrightarrow A$

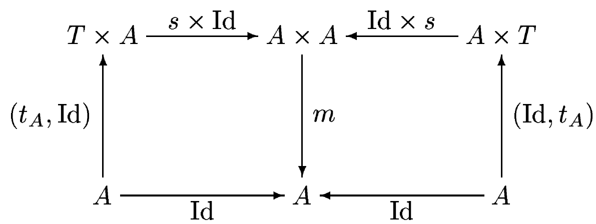
such that the following diagrams commute:

(i)

$$\begin{array}{ccc} A \times A \times A & \xrightarrow{m \times \mathrm{Id}} & A \times A \\ \Big\downarrow{\scriptstyle \mathrm{Id} \times m} & & \Big\downarrow{\scriptstyle m} \\ A \times A & \xrightarrow[m]{} & A \end{array}$$

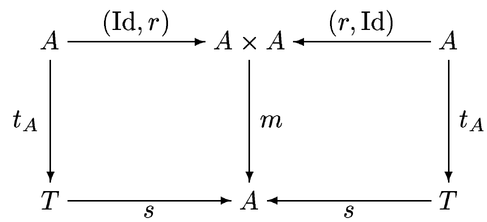
This is a categorical encoding of the associativity of the multiplication operation.

(ii)



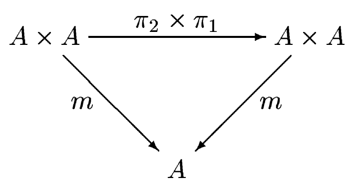
where t_A is the unique morphism $A \longrightarrow T$. This is the categorical analogy of multiplication by the identity element in a group.

(iii)



This is the categorical analogy of multiplication of an element by its inverse.

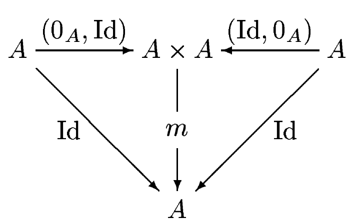
(iv)



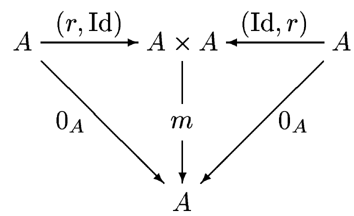
where $\pi_i \colon A \times A \longrightarrow A$ is the canonical projection morphism onto the ith coordinate. This is the categorification of the commutativity of the multiplication operation.

In the case where C has a zero object, the composition $s \circ t_A$ is the zero map $0_A \colon A \longrightarrow A$, and so the second and third diagrams above become

(ii)

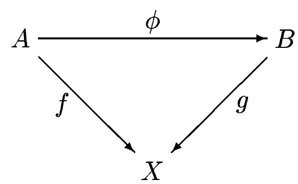


(iii)



We denote by $\mathsf{Ab}(\mathsf{C})$ the (sub)category of abelian group objects in C.

For any category C and object X, the **slice category** C/X is the category whose objects are morphisms $f\colon A \longrightarrow X$ in C — often referred to as '(objects of C) **over** X' — and whose morphisms are commutative triangles



We may refer to ϕ as a **morphism over** X.

Given a fixed object X in C, a **Beck module over** X (**in the category** C) is an abelian group object in the slice category C/X.

The multiplication morphism is then defined on the fibred product (or pullback) $f\colon A \times_X A \longrightarrow X$ in C/X.

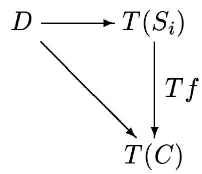
The category $\mathsf{Ab}(\mathsf{Group}/G)$, of Beck modules over a fixed group G, is equivalent to the category ${}_G\mathsf{Mod}$ of (left) G–modules. Analogous results hold for the categories LieAlg, $\mathsf{AssocAlg}$ and $\mathsf{CommRing}$; for details refer to [2] and [28].

It thus seems natural to suspect that the abelian group objects in the category Rack/X are exactly the split extensions examined in the previous chapter, and that the rôle of coefficient modules for rack (co)homology theories may be played by the appropriate trunk maps described earlier. And if this is the case, it seems reasonable to suspect that the corresponding objects in the various subcategories of Rack are the related refined forms of such trunk maps.

This chapter, then, is concerned with justifying this suspicion and developing the theory of such objects so that we may then investigate (co)homology theories from the usual Cartan/Eilenberg derived functor approach [8], and also from

Barr and Beck's cotriple construction [3].

Now let $T\colon \mathsf{C} \longrightarrow \mathsf{D}$ be a covariant functor. A **solution set** (**for an object** D **of** D, **with respect to** T) is a set of objects $\{S_i\}_{i\in I}$, indexed by some set I, such that for any object C of C and any morphism $D \longrightarrow T(C)$ in D there are morphisms $D \longrightarrow T(S_i)$ and $f\colon S_i \longrightarrow C$ for some $i \in I$ such that the diagram



commutes.

The following result, on the existence of adjoint functors, is due to Freyd [19]:

Theorem 2.1 (Adjoint Functor Theorem)

Let $T\colon \mathsf{C} \longrightarrow \mathsf{D}$ be a covariant functor, where C is complete and nonempty. Then T has a left adjoint iff

(i) T preserves products and equalisers, and

(ii) T has solution sets.

2.1 Rack modules

Let X be a rack, and let $\mathsf{T}(X)$ be the trunk defined in section 1.3. Now let $\mathcal{A} = (A, \phi, \psi)$ be a trunk map $\mathsf{T}(X) \longrightarrow \mathsf{Ab}$ such that

(i) The homomorphisms $\phi_{x,y}\colon A_x \longrightarrow A_{x^y}$ are all (abelian group) isomorphisms.

(ii) The identity

$$\psi_{z,x^y}(c) = \psi_{y^z,x^z}\psi_{z,y}(c) + \phi_{x^z,y^z}\psi_{z,x}(c)$$

holds for all $x, y, z \in X$ and $c \in A_z$.

Then we refer to $\mathcal{A}$ as a **rack module** (**over** X), or (where there is no ambiguity) an X–**module**. Note that all the maps $\psi_{y,x}\colon A_y \longrightarrow A_{x^y}$ are (abelian group) homomorphisms, since $\mathcal{A}$ is a trunk map $\mathsf{T}(X) \longrightarrow \mathsf{Ab}$.

If x and y are in the same orbit of X (that is, if there exists a word $w \in \mathrm{Op}(X)$ such that $x^w = y$) then $A_x \cong A_y$. This need not be the case, however, if x and y are not in the same orbit. Rack modules where all the groups A_x are isomorphic are said to be **homogeneous**; otherwise they are **heterogeneous**. If X is a transitive rack, for example, all X–modules must be homogeneous. Dihedral racks of even order, however, have two orbits and hence may give rise to heterogeneous modules.

A rack module $\mathcal{A}$ of the form $(A, \mathrm{Id}, 0)$ (so that $\phi_{x,y}$ is the identity on $A_x = A_{x^y}$, and $\psi_{y,x}$ the zero map $A_y \longrightarrow A_{x^y} = A_x$) is said to be **trivial**.

The **zero** X–module 0 is the trivial rack module where A_x is the trivial group for all $x \in X$.

Example 2.1 (Abelian groups)

Any abelian group A may be considered as a homogeneous trivial rack module $\mathcal{M} = (M, \phi, \psi)$ over an arbitrary rack X by defining $M_x = A$, $\phi_{x,y} = Id_A$, and $\psi_{y,x} = 0_A$.

Example 2.2 (Alexander modules)

Let $h(t)$ be a Laurent polynomial in one variable t, and let X be an arbitrary rack. Then there exists a class of X–modules defined by setting A_x to the quotient group $\mathbb{Z}_n[t, t^{-1}]/h(t)$, and letting $\phi_{x,y}(a) = ta$ and $\psi_{y,x}(b) = (1-t)b$ for all $x, y \in X, a \in A_x$, and $b \in A_y$. The group $\mathbb{Z}_n[t, t^{-1}]/h(t)$ is finite if the coefficients of the highest- and lowest-degree terms of the polynomial h are units of $\mathbb{Z}_n$. The case where $A_x = \mathbb{Z}[t, t^{-1}]/h(t)$ for all $x \in X$ is also a rack module over X.

Example 2.3 (Dihedral modules)

*Given an arbitrary rack X, we may define a family of rack modules as follows: Let D_n denote the module where $A_x = \mathbb{Z}_n, \phi_{x,y}(a) = -a$, and $\psi_{y,x}(b) = 2b$ for all $x, y \in X, a \in A_x$, and $b \in A_y$. This is the n^{th} **dihedral module** over X. The **infinite dihedral module** D_∞ is defined similarly, with $A_x = \mathbb{Z}$.*

This formulation agrees (up to differences in notation) with the independent work of Andruskiewitsch and Graña [1], although they only consider homogeneous ('indecomposable') cases.

Given a rack X and an X–module $\mathcal{A} = (A, \phi, \psi)$, we may construct a new rack as follows: Let $\mathcal{A} \rtimes X$ be the set $\{(a, x) : a \in A_x, x \in X\}$ equipped with rack operation $(a, x)^{(b,y)} = (\phi_{x,y}(a) + \psi_{y,x}(b), x^y)$. This is the **semi-direct product** of $\mathcal{A}$ with X.

Given two rack modules $\mathcal{A} = (A, \phi, \psi)$ and $\mathcal{B} = (B, \chi, \omega)$ over X, a **homomorphism** (or X–**map**) is a natural transformation $f\colon \mathcal{A} \longrightarrow \mathcal{B}$. That is, a collection of abelian group homomorphisms $f_x\colon A_x \longrightarrow B_x$ such that

(i) $f_{x^y}\phi_{x,y}(a) = \chi_{x,y} f_x(a)$

(ii) $f_{x^y}\psi_{y,x}(b) = \omega_{y,x} f_y(b)$

for all $x, y \in X$, and all $a \in A_x$ and $b \in A_y$.

An X–module $\mathcal{A}$ is said to be a **submodule** of an X–module $\mathcal{B}$ if there is an inclusion monomorphism $i\colon \mathcal{A} \hookrightarrow \mathcal{B}$. This requires that each A_x is a subgroup of B_x in a natural way. That is, $\phi_{x,y}$ and $\psi_{y,x}$ are the restrictions, respectively, of $\chi_{x,y}$ to A_x and $\omega_{y,x}$ to A_y.

We may then form the quotient $\mathcal{C} = (C, \eta, \nu) = \mathcal{B}/\mathcal{A}$ by setting $C_x = B_x/A_x$, and letting $\eta_{x,y}$ and $\nu_{y,x}$ be the obvious quotient maps:

$$\begin{aligned}\eta_{x,y}(a + A_x) &= \chi_{x,y}(a) + A_{x^y} \in C_{x^y}\\ \nu_{y,x}(b + A_y) &= \omega_{y,x}(b) + A_{x^y} \in C_{x^y}\end{aligned}$$

As hoped, the rack modules over a given rack X, together with the X–maps between them, form a category, which we shall denote by RMod_X.

Proposition 2.2

Given a (fixed) rack X, there is a category RMod_X with objects the rack modules over X, and morphisms the X–module homomorphisms.

Proof

Let composition be defined as usual: Given $f\colon \mathcal{A} \longrightarrow \mathcal{B}$ and $g\colon \mathcal{B} \longrightarrow \mathcal{C}$, define $g \circ f\colon \mathcal{A} \longrightarrow \mathcal{C}$ by $(g \circ f)_x = g_x \circ f_x\colon A_x \longrightarrow C_x$.

This is itself an X–module homomorphism, since

$$
\begin{aligned}
&(g \circ f)_{x^y} \circ \phi_{x,y}(a) = \\
&\qquad g_{x^y} \circ f_{x^y} \circ \phi_{x,y}(a) = g_{x^y} \circ \chi_{x,y} \circ f_x(a) = \eta_{x,y} \circ g_x \circ f_x(a) \\
&\qquad\qquad = \eta_{x,y} \circ (g \circ f)_x(a)
\end{aligned}
$$

and

$$
\begin{aligned}
&(g \circ f)_{x^y} \circ \psi_{y,x}(b) = \\
&\qquad g_{x^y} \circ f_{x^y} \circ \psi_{y,x}(b) = g_{x^y} \circ \omega_{y,x} \circ f_y(b) = \nu_{y,x} \circ g_y \circ f_y(b) \\
&\qquad\qquad = \nu_{y,x} \circ (g \circ f)_y(b)
\end{aligned}
$$

Associativity of this composition follows from the fact that composition of abelian group homomorphisms is associative. Furthermore, for each X–module $\mathcal{A}$ there is an identity homomorphism $\mathrm{Id}_{\mathcal{A}}\colon \mathcal{A} \longrightarrow \mathcal{A}$ defined by $(\mathrm{Id}_{\mathcal{A}})_x = \mathrm{Id}_{A_x}$.
□

Given two X–modules $\mathcal{A} = (A, \phi, \psi)$ and $\mathcal{B} = (B, \chi, \omega)$, we may form their Cartesian product $\mathcal{A} \times \mathcal{B}$ by setting $(\mathcal{A} \times \mathcal{B})_x = A_x \times B_x$ for each $x \in X$, and defining new maps $\rho_{x,y}\colon (\mathcal{A}\times\mathcal{B})_x \longrightarrow (\mathcal{A}\times\mathcal{B})_{x^y}$ and $\lambda_{y,x}\colon (\mathcal{A}\times\mathcal{B})_y \longrightarrow (\mathcal{A}\times\mathcal{B})_{x^y}$ for all $x, y \in X$ by $\rho_{x,y}(a, b) = (\phi_{x,y}(a), \chi_{x,y}(b))$ for all $a \in A_x, b \in B_x$, and $\lambda_{y,x}(c, d) = (\psi_{y,x}(c), \omega_{y,x}(d))$ for all $c \in A_y, d \in B_y$.

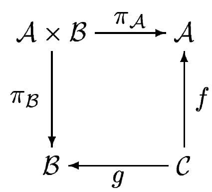
A routine, but tedious, calculation confirms that $\mathcal{A}\times\mathcal{B}$ is indeed an X–module. This product operation extends, in the obvious way, to arbitrarily large collections of rack modules.

Proposition 2.3

The Cartesian product is the categorical product in RMod_X.

Proof

Let $\pi_{\mathcal{A}}\colon \mathcal{A} \times \mathcal{B} \longrightarrow \mathcal{A}$ and $\pi_{\mathcal{B}}\colon \mathcal{A} \times \mathcal{B} \longrightarrow \mathcal{B}$ be the canonical projection maps. Then for any other X–module $\mathcal{C}$ with X–maps $f\colon \mathcal{C} \longrightarrow \mathcal{A}$ and $g\colon \mathcal{C} \longrightarrow \mathcal{B}$, there is a unique X–map $h = (f, g)\colon \mathcal{C} \longrightarrow \mathcal{A} \times \mathcal{B}$ making the diagram



commute. □

Theorem 2.4

The category RMod_X *is an abelian category.*

Proof

The category of X–modules may be seen to be an additive category, since any morphism set $\mathrm{Hom}_{\mathsf{RMod}_X}(\mathcal{A}, \mathcal{B})$ may be given an additive abelian group structure by defining, for any two X–maps $f, f'\colon \mathcal{A} \longrightarrow \mathcal{B}$, the sum $(f + f')\colon \mathcal{A} \longrightarrow \mathcal{B}$ by $(f + f')_x(a) = f_x(a) + f'_x(a)$ for any $a \in A_x$ and all $x \in X$. This addition distributes over composition.

Furthermore, as noted above, we have a well-defined (up to isomorphism) product operation, and a zero object (the zero X–module 0 defined earlier), and so RMod_X is an additive category.

To confirm that it is also an abelian category, we must show further that:

(i) every X–map has a kernel and a cokernel,

(ii) every monic X–map is the kernel of its cokernel, and

(iii) every epic X–map is the cokernel of its kernel.

Given an X–map $f\colon \mathcal{B} = (B, \beta, \zeta) \longrightarrow \mathcal{C} = (C, \gamma, \eta)$ we may define an X–module $\mathcal{A} = (A, \alpha, \varepsilon)$ by $A_x = \{a \in B_x : f_x(a) = 0\}$, with $\alpha_{x,y} = \beta_{x,y}|_{A_x}$ and $\varepsilon_{y,x} = \zeta_{y,x}|_{A_y}$.

Then $\mathcal{A}$ is a submodule of $\mathcal{B}$, and the inclusion $\iota\colon \mathcal{A} \hookrightarrow \mathcal{B}$ is the (categorical) kernel of f.

We may define another X–module $\mathcal{D} = (D, \delta, \xi)$ where $D_x = C_x/\operatorname{im} f_x$, and with homomorphisms $\delta_{x,y}(a) = \gamma_{x,y}(a) + \operatorname{im} f_x$ and $\xi_{y,x}(a) = \eta_{y,x}(a) + \operatorname{im} f_y$.

Then $\mathcal{D}$ is a quotient of $\mathcal{C}$, and the canonical projection $\pi\colon \mathcal{C} \longrightarrow \mathcal{D}$ is the categorical cokernel of f.

To confirm that our category fulfils the second criterion, take a monic X–map $\mu\colon \mathcal{A} \longrightarrow \mathcal{B}$ for two arbitrary X–modules $\mathcal{A} = (A, \phi, \psi)$ and $\mathcal{B} = (B, \chi, \omega)$. Observe (by the first part of the proof) that the embedding $\iota\colon \operatorname{im}\mu \longrightarrow \mathcal{B}$ is a kernel of the quotient map $\pi\colon \mathcal{A} \longrightarrow \mathcal{B}/\operatorname{im}\mu$. Since μ is monic, it is injective and hence the X–map $\mu'\colon \mathcal{A} \longrightarrow \operatorname{im}\mu$ given by $\mu'_x(a) = \mu_x(a)$ (for all $x \in X$) is an isomorphism. But since kernels are unique up to composition with an isomorphism, and since $\mu = \iota\mu'$, it follows that μ is a kernel of its cokernel, the quotient map π.

Finally, given an epic X–map $\varepsilon\colon \mathcal{A} \longrightarrow \mathcal{B}$, we must show that it is the cokernel of its kernel. A kernel of ε is the inclusion $\iota\colon \ker\varepsilon \longrightarrow \mathcal{A}$. If we have another X–map $\eta\colon \mathcal{A} \longrightarrow \mathcal{C}$ such that $\eta\iota = 0$ then $\ker\varepsilon \subseteq \ker\eta$, so that $\varepsilon(a) = \varepsilon(b)$ implies that $\eta(a) = \eta(b)$. But since ε is epic (and hence surjective) we can define an X–map $\theta\colon \mathcal{B} \longrightarrow \mathcal{C}$ by $\theta_x(\varepsilon_x(a)) = \eta_x(a)$ for all $a \in A_x$ and $x \in X$. Then $\theta\varepsilon = \eta$ and so ε is a cokernel of the inclusion map ι.

Hence RMod_X is an abelian category. □

We have just shown that our category RMod_X is a suitable environment for the practice of homological algebra, as the axioms for an abelian category essentially guarantee the existence of the necessary constructs to define chain complexes and exact sequences. In the next section we show that, in addition, rack modules are suitable coefficient objects for homology and cohomology theories of racks.

The following facts about abelian categories are stated without proof:

Theorem 2.5

(i) *An abelian category has pullbacks (and hence also finite products, finite intersections, and equalisers) and pushouts (and hence also finite sums, finite unions, and coequalisers).*

(ii) *An abelian category is uniquely factorisable, which is to say that any morphism f may be expressed uniquely as a composition $\mu \circ \varepsilon$ where μ is monic and ε epic.*

(iii) *An abelian category is balanced; that is, every bimorphism (morphism* *which is both epic and monic) is an isomorphism.*

Curious readers are directed to any suitable book on category theory or homological algebra (such as [25], [6], or [31]).

2.2 Beck modules

The following result justifies our use of the term 'rack module' to denote the objects of RMod_X:

Theorem 2.6

The category RMod_X *of rack modules over a (fixed) rack* X *is equivalent to the category* $\mathsf{Ab}(\mathsf{Rack}/X)$ *of abelian group objects in the slice category* Rack/X*. That is, the rack modules just described are exactly the Beck modules in* Rack*.*

Proof

We must construct a functor $\mathsf{RMod}_X \longrightarrow \mathsf{Ab}(\mathsf{Rack}/X)$ which is bijective on objects and morphisms.

Let T, then, be the map which assigns the split extension $p\colon \mathcal{A} \rtimes X \twoheadrightarrow X$ to the X–module $\mathcal{A}$, where p is projection onto the second coordinate.

This is functorial, since, given an X–map $f\colon \mathcal{A} \longrightarrow \mathcal{B}$, we obtain a morphism $T(f)\colon \mathcal{A} \rtimes X \longrightarrow \mathcal{B} \rtimes X$ defined by $T(f)(a,x) = (f_x(a), x)$ for $a \in A_x, x \in X$. Furthermore,

$$T(fg)(a,x) = ((fg)_x(a), x) = (f_x g_x(a), x) = T(f)(g_x(a), x) = T(f)T(g)(a,x)$$

for all $a \in A_x, x \in X$.

To show that $p\colon \mathcal{A} \rtimes X \twoheadrightarrow X$ is an abelian group object in the slice category Rack/X we must construct well-defined multiplication and inverse morphisms, and a section $s\colon X \longrightarrow \mathcal{A} \rtimes X$.

So let $r(a,x) = (-a,x)$, $m((a_1,x),(a_2,x)) = (a_1 + a_2, x)$, and $s(x) = (0,x)$.

Then

$$\begin{aligned} m(m((a_1,x),(a_2,x)),(a_3,x)) &= m((a_1+a_2,x),(a_3,x)) \\ &= (a_1+a_2+a_3,x) \\ &= m((a_1,x),(a_2+a_3,x)) = m((a_1,x),m((a_2,x),(a_3,x))), \end{aligned}$$

$$\begin{aligned} m(s(x),(a,x)) = m((0,x),(a,x)) &= (a,x) \\ &= m((a,x),(0,x)) = m((a,x),s(x)), \end{aligned}$$

$$\begin{aligned} m(r(a,x),(a,x)) = m((-a,x),(a,x)) &= (0,x) \\ &= m((a,x),(-a,x)) = m((a,x),r(a,x)) \end{aligned}$$

and

$$m((a_1,x),(a_2,x)) = (a_1+a_2,x) = (a_2+a_1,x) = m((a_2,x),(a_1,x))$$

Now, we define the inverse functor $T^{-1}\colon \mathsf{Ab}(\mathsf{Rack}/X) \longrightarrow \mathsf{RMod}_X$. Given an abelian group object $p\colon R \longrightarrow X$ in Rack/X, let $R_x = p^{-1}(x)$ for each $x \in X$. Then each of the R_x has an abelian group structure defined in terms of m and r: for $u, v \in R_x$, let $u+v = m(u,v) \in R_x$, and $u^{-1} = r(u) \in R_x$. The preimage R_x is closed under this addition operation because m and r are morphisms in Rack/X and hence commute with the projection map p. Now define a collection of maps $\rho_{x,y}\colon R_x \longrightarrow R_{x^y}$ for all $x, y \in X$ by $\rho_{x,y}(u) = u^{s(y)} \in R_{x^y}$ for each $u \in R_x$.

This is a homomorphism of abelian groups, since $\rho_{x,y}(s(x)) = s(x)^{s(y)} = s(x^y)$ (which is the identity in R_{x^y}) and, for any $u_1, u_2 \in R_x$,

$$\begin{aligned} \rho_{x,y}(u_1+u_2) = m(u_1,u_2)^{s(y)} &= m(u_1,u_2)^{m(s(y),s(y))} \\ &= m(u_1^{s(y)}, u_2^{s(y)}) = \rho_{x,y}(u_1) + \rho_{x,y}(u_2) \end{aligned}$$

It is also an isomorphism, since exponentiation by a fixed element of a rack is a bijection.

Furthermore, for any $x, y, z \in X$, and any $u \in R_x$,

$$\rho_{x^y,z}\rho_{x,y}(u) \quad = \quad u^{s(y)s(z)} \quad = \quad u^{s(z)s(y)^{s(z)}} \quad = \quad u^{s(z)s(x^z)} \quad = \quad \rho_{x^z,y^z}\rho_{x,z}(u)$$

Now define another collection of maps $\lambda_{y,x} \colon R_y \longrightarrow R_{x^y}$ for all $x, y \in X$ by $\lambda_{y,x}(v) = s(x)^v \in R_{x^y}$ for each $v \in R_y$. Then these, too, are abelian group homomorphisms, since $\lambda_{y,x}(s(y)) = s(x)^{s(y)} = s(x^y)$ (which is the identity in R_{x^y}) and, for any $v_1, v_2 \in R_y$,

$$\begin{aligned}\lambda_{y,x}(v_1 + v_2) = s(x)^{m(v_1,v_2)} &= m(s(x), s(x))^{m(v_1,v_2)} \\ &= m(s(x)^{v_1}, s(x)^{v_2}) = \lambda_{y,x}(v_1) + \lambda_{y,x}(v_2)\end{aligned}$$

Furthermore, for any $x, y, z \in X, v \in R_y$, and $w \in R_z$,

$$\rho_{x^y,z}\lambda_{y,x}(v) \quad = \quad s(x)^{vs(z)} \quad = \quad s(x)^{s(z)v^{s(z)}} \quad = \quad s(x^z)^{v^{s(z)}} \quad = \quad \lambda_{y^z,x^z}\rho_{y,z}(v)$$

and

$$\begin{aligned}\lambda_{z,x^y}(w) = s(x^y)^w = s(x)^{s(y)w} &= s(x)^{ws(y)^w} \\ &= m(s(x), s(x))^{m(s(z),w)m(s(y),s(y))^{m(w,s(z))}} \\ &= m\left(s(x)^{s(z)s(y)^w}, s(x)^{ws(y)^{s(z)}}\right) \\ &= s(x)^{s(z)s(y)^w} + s(x)^{ws(y)^{s(z)}} \\ &= s(x^z)^{s(y)^w} + s(x)^{ws(y^z)} \\ &= \lambda_{y^z,x^z}\lambda_{z,y}(w) + \rho_{x^z,y^z}\lambda_{z,x}(w)\end{aligned}$$

Thus, an abelian group object $R \longrightarrow X$ in Rack/X, determines a rack module $\mathcal{R} = (R, \rho, \lambda)$ over X.

Now given two such abelian group objects $p \colon R \longrightarrow X$ and $p' \colon R' \longrightarrow X$, together with a homomorphism $f \colon R \longrightarrow R'$ which commutes with the respective projection maps, we may construct two X–modules $S(R) = (R, \rho, \lambda)$ and $S(R') = (R', \rho', \lambda')$ as detailed above, and an X–map $S(f) \colon S(R) \longrightarrow S(R')$ as follows:

For each $x \in X$, let $S(f)_x(u) = f(u)$ for each $u \in R_x$.

Note that $S(f)_x\colon R_x \longrightarrow R'_x$ since f commutes with the projection maps p and p'. It is also a natural transformation of trunk maps $\mathsf{T}(X) \longrightarrow \mathsf{Ab}$, since

$$S(f)_{x^y}\rho_{x,y}(u) = f(\rho_{x,y}(u)) = f(u^{s(y)}) = f(u)^{f(s(y))} = f(u)^{s'(y)} = \rho'_{x,y}S(f)_x(u)$$

and

$$\begin{aligned} S(f)_{x^y}\lambda_{y,x}(v) = f(\lambda_{y,x}(v)) = f(s(x)^v) \\ = f(s(x))^{f(v)} = s'(x)^{f(v)} = \lambda'_{x,y}S(f)_y(v) \end{aligned}$$

for all $x, y \in X, u \in R_x$, and $v \in R_y$.

Given another abelian group object $p''\colon R'' \longrightarrow X$, and another slice morphism $f'\colon R' \longrightarrow R''$, we obtain a trunk map $S(R'')$ and another corresponding X–map $S(f')\colon S(R') \longrightarrow S(R'')$. Then

$$S(f'f)_x(u) = (f'f)(u) = S(f')_x(f(u)) = S(f')_xS(f)_x(u)$$

for all $x \in X$ and $u \in R_x$.

Hence the map $S\colon \mathsf{Ab}(\mathsf{Rack}/X) \longrightarrow \mathsf{RMod}_X$ is a functor and may be seen to be the inverse of T since, for any X–module $\mathcal{A} = (A, \phi, \psi)$, it follows that $ST(A_x) = S(\{(a,x) : a \in A_x\}) = S(p^{-1}(x)) = A_x$; and for any given X–map $f\colon \mathcal{A} \longrightarrow \mathcal{A}'$, it follows that $ST(f_x)(a) = S(f(a,x)) = f_x(a)$. Finally, TS is also the identity functor on $\mathsf{Ab}(\mathsf{Rack}/X)$ since for any such abelian group object $f\colon R \twoheadrightarrow X$ we may construct the corresponding rack module $\mathcal{A} = S(R)$, where $A_x = R_x = \{u \in R : f(u) = x\}$ with addition given by $m|_{R_x}$. Then $T(\mathcal{A}) = \{(a,x) : a \in R_x, x \in X\}$ which is precisely the original abelian group object $f\colon R \twoheadrightarrow X$. □

In particular, our definition of rack module is suitable for use as a coefficient object for homology and cohomology theories defined in the category Rack.

We may now introduce a notational convenience which may serve to simplify matters in later sections. Let X be a rack, $\mathcal{A} = (A, \phi, \psi)$ a rack module over

X, and $w = y_1 y_2 \dots y_n$ a word in $\mathrm{As}\,X$. Then we may denote the composition $\phi_{x^{y_1 \cdots y_{n-1}}, y_n} \phi_{x^{y_1 \cdots y_{n-2}}, y_{n-1}} \dots \phi_{x, y_1}$ by $\phi_{x,w}$. This is well-defined, as the following lemma shows:

Lemma 2.7

Let X be a rack, $\mathcal{A} = (A, \phi, \psi)$ an X–module, and $w \in \mathrm{As}\,X$. If $y_1 \dots y_n$ and $z_1 \dots z_m$ are two different representative words for w, then the compositions

$$\phi_{x^{y_1 \cdots y_{n-1}}, y_n} \phi_{x^{y_1 \cdots y_{n-2}}, y_{n-1}} \dots \phi_{x, y_1}$$

and

$$\phi_{x^{z_1 \cdots z_{m-1}}, z_m} \phi_{x^{z_1 \cdots z_{m-2}}, z_{m-1}} \dots \phi_{x, z_1}$$

are equal for all $x \in X$.

Furthermore, $\phi_{x,1} = \mathrm{Id}_{A_x}$ (where 1 denotes the identity in $\mathrm{As}\,X$).

Proof

As shown in theorem 2.6, there is a bijective correspondence between X–modules and abelian group objects in the slice category Rack/X, given by the functor $T\colon \mathsf{RMod}_X \longrightarrow \mathsf{Ab}(\mathsf{Rack}/X)$ which takes $\mathcal{A} = (A, \phi, \psi)$ to $\pi_2\colon \mathcal{A} \rtimes X \twoheadrightarrow X$. Recall that $R_x = T(A_x) = \pi_2^{-1}(x)$ has the structure of an abelian group. For any $x, y \in X$, $T(\phi_{x,y})\colon R_x \longrightarrow R_{x^y}$ is given by the map $r \mapsto r^{s(y)}$

It then follows that

$$\begin{aligned} T(\phi_{x^{y_1 \cdots y_{n-1}}, y_n} \phi_{x^{y_1 \cdots y_{n-2}}, y_{n-1}} \dots \phi_{x, y_1})(r) \\ &= r^{s(y_1) \dots s(y_n)} = r^{s(y_1 \dots y_n)} \\ &= r^{s(z_1 \dots z_m)} = r^{s(z_1) \dots s(z_m)} \\ &= T(\phi_{x^{z_1 \cdots z_{m-1}}, z_m} \phi_{x^{z_1 \cdots z_{m-2}}, z_{m-1}} \dots \phi_{x, z_1}(r)) \end{aligned}$$

where the equality in the second and third lines follows from the functoriality of the associated group.

The final statement follows from the observation

$$T(\phi_{x,1})(r) = r^1 = r = T(\mathrm{Id}_{A_x})(r)$$

where 1 is the identity in $\operatorname{As} X$. □

2.3 Free modules

Let $\mathsf{D}(X)$ denote the **discrete trunk** on X, having one object x for each element of X, and no morphisms or preferred squares. Then let $\mathsf{Set}^{\mathsf{D}(X)}$ denote the category whose objects are trunk maps $\mathcal{S}\colon \mathsf{D}(X) \longrightarrow \mathsf{Set}$, and whose morphisms are natural transformations between such trunk maps. An object of $\mathsf{Set}^{\mathsf{D}(X)}$, then, is simply a collection of sets S_x indexed by the elements of the rack X.

Now, given a rack module $\mathcal{A}$, there is a forgetful functor $U\colon \mathsf{RMod}_X \longrightarrow \mathsf{Set}^{\mathsf{D}(X)}$ defined in the obvious way: $(UA)_x$ is the underlying set of the abelian group A_x, and we simply discard the structure maps $\phi_{x,y}$ and $\psi_{y,x}$.

Theorem 2.8

For any rack X, the forgetful functor $U\colon \mathsf{RMod}_X \longrightarrow \mathsf{Set}^{\mathsf{D}(X)}$ has a left adjoint $F\colon \mathsf{Set}^{\mathsf{D}(X)} \longrightarrow \mathsf{RMod}_X$.

Proof

To prove that U has a left adjoint, we must first check that RMod_X and U satisfy the hypotheses of theorem 2.1.

The category RMod_X is complete, since it has products and (by theorem 2.5) equalisers.

The functor U may be seen to preserve products, since for an arbitrary collection $\{\mathcal{A}_i = (A^i, \phi^i, \psi^i) : i \in I\}$ of X–modules,

$$U\left(\prod_{i\in I}\mathcal{A}_i\right)_x = U\left(\prod_{i\in I}A^i_x\right) = \prod_{i\in I}U\left(A^i_x\right) = \prod_{i\in I}U\left(\mathcal{A}_i\right)_x$$

Let $\mathcal{B}$ and $\mathcal{C}$ be X–modules. The equaliser of $f, g\colon \mathcal{B} \longrightarrow \mathcal{C}$ is (isomorphic to) the X–submodule $\mathcal{A}$ of $\mathcal{B}$ where $A_x = \{b \in B_x : f_x(b) = g_x(b)\}$ for each $x \in X$, together with the canonical inclusion map $i\colon \mathcal{A} \hookrightarrow \mathcal{B}$. Then $U\mathcal{A}$ determines a set $U(\mathcal{A})_x = U(A_x)$ for each $x \in X$, and $U(i)_x$ is the obvious inclusion function $U(i_x)\colon U(A_x) \hookrightarrow U(B_x)$ for all $x \in X$. This may be seen to be the equaliser of the pair $Ug, Uh\colon U\mathcal{B} \longrightarrow U\mathcal{C}$, and hence U preserves equalisers.

Showing that U satisfies the solution set condition is rather more involved. Given a trunk map $\mathcal{S}\colon \mathsf{D}(X) \longrightarrow \mathsf{Set}$, let $\mathcal{M} = (M, P, \Lambda)$ be defined as follows:

For all $x \in X$, let M_x be the free abelian group generated by all symbols of the forms

(i) $\rho_{x^{\bar{w}},w}(a)$ where $w \in \operatorname{As} X$ and $a \in S_{x^{\bar{w}}}$.

(ii) $\rho_{x^{\bar{w}},w}\lambda_{y,x^{\bar{w}\bar{y}}}(b)$ where $w \in \operatorname{As} X$, $y \in X$ and $b \in S_y$

modulo the relations

(i) $\rho_{x^u,v}\rho_{x,u} = \rho_{x^v,u^v}\rho_{x,v} = \rho_{x,uv}$

(ii) $\rho_{x^y,v}\lambda_{y,x} = \lambda_{y^v,x^v}\rho_{y,v}$

(iii) $\lambda_{z,x^y} = \lambda_{y^z,x^z}\lambda_{z,y} + \rho_{x^z,y^z}\lambda_{z,x}$

(iv) $\rho_{x,w}(p+q) = \rho_{x,w}(p) + \rho_{x,w}(q)$

(v) $\lambda_{y,x}(s+t) = \lambda_{y,x}(s) + \lambda_{y,x}(t)$

for all $x, y, z \in X$; $u, v, w \in \operatorname{As} X$; $p, q \in M_x$; and $s, t \in M_y$.

The structure maps are defined as follows:

$$\begin{aligned} P_{x,y}\colon\ M_x &\longrightarrow M_{x^y}; \qquad p \mapsto \rho_{x,y}p \\ \Lambda_{y,x}\colon\ M_y &\longrightarrow M_{x^y}; \qquad s \mapsto \lambda_{y,x}s \end{aligned}$$

for all $x, y \in X$; $p \in M_x$; and $s \in M_y$.

Any string of ρ and λ symbols may, by means of relation (ii) be converted to a form where all the ρ symbols are grouped together, followed by all the λ symbols. Relation (i) may then be used to collapse the ρ symbols into one, and relations (ii), (iii), (iv), and (v) may be used iteratively to reduce this compound symbol into a $\mathbb{Z}$–linear combination of symbols consisting of a single ρ, and symbols consisting of a single ρ followed by a single λ.

The map $P_{x,w}$ is an abelian group isomorphism $M_x \cong M_{x^w}$, as the image $P_{x,w}(M_x)$ is the free abelian group generated by the set

$$\{\rho_{x,w}\rho_{x^{\bar{v}},v}(p), \rho_{x,w}\rho_{x^{\bar{v}},v}\lambda_{y,x^{\bar{v}\bar{y}}}(q) : y \in X, v \in \operatorname{As} X, p \in S_{x^{\bar{v}}}, q \in S_y\}$$

modulo the relations described above. But this is the free abelian group generated by

$$\{\rho_{x^{\bar{v}},vw}(p), \rho_{x^{\bar{v}},vw}\lambda_{y,x^{\bar{v}\bar{y}}}(q) : y \in X, v \in \mathrm{As}\, X, p \in S_{x^{\bar{v}}}, q \in S_y\}$$

modulo the above relations, which is exactly M_{x^y}.

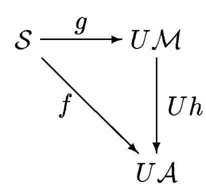
Furthermore, relation (ii) ensures that the maps $\Lambda_{y,x}$ satisfy the condition

$$\Lambda_{z,x^y} = \Lambda_{y^z,x^z}\Lambda_{z,y} + P_{x^z,y^z}\Lambda_{z,x}$$

for all $x, y, z \in X$, and so $\mathcal{M}$ is an X–module.

Now let $\mathcal{A} = (A, \phi, \psi)$ be any X–module, and $f\colon \mathcal{S} \longrightarrow U\mathcal{A}$ an arbitrary natural transformation. Then let $g\colon \mathcal{S} \longrightarrow U\mathcal{M}$ and $h\colon \mathcal{M} \longrightarrow \mathcal{A}$ be defined so that $g_x(p) = (p) = \rho_{x,1}(p)$ and $h_x(p) = f_x(p) \in A_x$ for all $p \in S_x$ and $x \in X$.

Then the diagram



commutes, and thus $\{\mathcal{M}\}$ is a solution set for $\mathcal{S}$ relative to U.

Thus, by theorem 2.1, the functor $U\colon \mathsf{RMod}_X \longrightarrow \mathsf{Set}^{\mathsf{D}(X)}$ has a well-defined left adjoint $F\colon \mathsf{Set}^{\mathsf{D}(X)} \longrightarrow \mathsf{RMod}_X$. □

We may now define the **free** X**–module** functor $F\colon \mathsf{Set}^{\mathsf{D}(X)} \longrightarrow \mathsf{RMod}_X$ to be the left adjoint to U, in the sense that $\mathrm{Hom}_{\mathsf{RMod}_X}(F\mathcal{S}, \mathcal{A}) \cong \mathrm{Hom}_{\mathsf{Set}^{\mathsf{D}(X)}}(\mathcal{S}, U\mathcal{A})$. Let $\mathcal{V}_x$ be the trunk map $\mathsf{D}(X) \longrightarrow \mathsf{Set}$ with

$$(\mathcal{V}_x)_y = \begin{cases} \{*\} & \text{if } y = x \\ \emptyset & \text{otherwise} \end{cases}.$$

That is, $\mathcal{V}_x$ maps the object x in $\mathsf{D}(X)$ to a singleton set $\{*\}$ and everything else to the empty set $\emptyset$. Every trunk map $\mathcal{S}\colon \mathsf{D}(X) \longrightarrow \mathsf{Set}$ may be described as a coproduct of maps of this type:

$$\mathcal{S} = \coprod_{y \in X} \coprod_{s \in S_y} \mathcal{V}_y$$

The dual form of the adjoint functor theorem ensures that the free functor $F\colon \mathsf{Set}^{\mathsf{D}(X)} \longrightarrow \mathsf{RMod}_X$ preserves coproducts (since F has a right adjoint U). Thus

$$F\mathcal{S} = \coprod_{y \in X} \coprod_{s \in S_y} F\mathcal{V}_y.$$

In particular, the structure of the X–module $F\mathcal{S}$ is determined completely by the structure of the individual X–modules $F\mathcal{V}_x$ for all $x \in X$.

The group $(F\mathcal{V}_x)_y$ is the free abelian group generated by the set

$$\{\rho_{x,w}(*), \rho_{y^{\bar{v}},v}\lambda_{x,y^{\bar{v}\bar{x}}}(*) : v, w \in \operatorname{As} X, x^w = y\}$$

modulo the relations listed in theorem 2.8, with structure maps

$$P_{y,z}\colon \begin{cases} \rho_{x,w}(*) & \mapsto \; \rho_{y,z}\rho_{x,w}(*) \\ & = \; \rho_{x,wz}(*) \\ \rho_{y^{\bar{v}},v}\lambda_{x,y^{\bar{v}\bar{x}}}(*) & \mapsto \; \rho_{y,z}\rho_{y^{\bar{v}},v}\lambda_{x,y^{\bar{v}\bar{x}}}(*) \\ & = \; \rho_{y^{\bar{v}},vz}\lambda_{x,y^{\bar{v}\bar{x}}}(*) \end{cases}$$

$$\Lambda_{y,z}\colon \begin{cases} \rho_{x,w}(*) & \mapsto \; \lambda_{y,z}\rho_{x,w}(*) \\ & = \; \rho_{y^{\bar{w}x},w}\lambda_{x,y^{\bar{w}}}(*) \\ \rho_{y^{\bar{v}},v}\lambda_{x,y^{\bar{v}\bar{x}}}(*) & \mapsto \; \lambda_{y,z}\rho_{y^{\bar{v}},v}\lambda_{x,y^{\bar{v}\bar{x}}}(*) \\ & = \; \rho_{z^{y\bar{v}},v}\lambda_{x,z^{y\bar{v}\bar{x}}}(*) - \rho_{z^{\bar{v}},vy}\lambda_{x,z^{\bar{v}\bar{x}}}(*) \end{cases}$$

The map $P_{y,z}\colon (F\mathcal{V}_x)_y \longrightarrow (F\mathcal{V}_x)_{y^z}$ is an isomorphism of abelian groups, since

$$P_{y,z}(F\mathcal{V}_x)_y = \{\rho_{x,wz}(*), \rho_{y^{\bar{v}},vz}\lambda_{x,y^{\bar{v}\bar{x}}}(*) : v, w \in \operatorname{As} X, x^w = y\} = (F\mathcal{V}_x)_{y^z}$$

The free X–module $F\mathcal{S}$ may be constructed as a direct sum of the modules $F\mathcal{V}_x$. Indeed, $F\mathcal{S}$ is exactly the module $\mathcal{M}$ constructed in the proof to theorem 2.8, and is said to be the **free X–module** with **basis $\mathcal{S}$**.

2.4 The rack algebra $\mathbb{Z}X$

In particular, if $\mathcal{S}\colon \mathsf{D}(X) \longrightarrow \mathsf{Set}$ is the trunk map with S_x the singleton set $\{*\}$ for each $x \in X$, then the free module $F\mathcal{S}$ is analogous to the group ring $\mathbb{Z}G$ in group (co)homology, and the enveloping algebra A^e for Lie algebra (co)homology. We shall denote this module (the **rack algebra**) by $\mathbb{Z}X$.

A typical element of $(\mathbb{Z}X)_x$ is of the form

$$\sum_{w\in \mathrm{As}\, X} n_w \rho_{x^{\bar{w}},w}(*) + \sum_{t\in X, v\in \mathrm{As}\, X} m_{t,v}\rho_{x^{\bar{v}},v}\lambda_{t,x^{\bar{v}\bar{t}}}(*)$$

where $n_w, m_{t,v} \in \mathbb{Z}$.

The composition of the structure maps in $\mathbb{Z}X$ yields a multiplicative structure as follows:

$$\begin{array}{lclcl}
\rho_{x,v}(*)\cdot\rho_{x^{\bar{u}},u}(*) & = & \rho_{x,v}\rho_{x^{\bar{u}},u}(*) & = & \rho_{x^{\bar{u}},uv}(*) \\
\rho_{x,v}(*)\cdot\rho_{x^{\bar{u}},u}\lambda_{s,x^{\bar{u}\bar{s}}}(*) & = & \rho_{x,v}\rho_{x^{\bar{u}},u}\lambda_{s,x^{\bar{u}\bar{s}}}(*) & = & \rho_{x^{\bar{u}},uv}\lambda_{s,x^{\bar{u}\bar{s}}}(*) \\
\rho_{x,v}\lambda_{t,x^{\bar{t}}}(*)\cdot\rho_{t^{\bar{u}},u}(*) & = & \rho_{x,v}\lambda_{t,x^{\bar{t}}}\rho_{t^{\bar{u}},u}(*) & = & \rho_{x^{\bar{u}},uv}\lambda_{t^{\bar{u}},x^{\bar{t}\bar{u}}}(*) \\
\rho_{x,v}\lambda_{t,x^{\bar{t}}}(*)\cdot\rho_{t^{\bar{u}},u}\lambda_{s,t^{\bar{u}\bar{s}}}(*) & = & \rho_{x,v}\lambda_{t,x^{\bar{t}}}\rho_{t^{\bar{u}},u}\lambda_{s,t^{\bar{u}\bar{s}}}(*) & = & \rho_{x^{\bar{u}},uv}\lambda_{s,x^{\bar{u}\bar{s}}}(*) \\
 & & & & -\rho_{x^{\bar{t}\bar{u}},utv}\lambda_{s,x^{\bar{t}\bar{u}\bar{s}}}(*)
\end{array}$$

where $x, s, t \in X$ and $u, v \in \mathrm{As}\, X$.

This product operation is associative and distributes over addition, giving $\mathbb{Z}X$ a structure analogous to that of a preadditive category or 'ring with several objects'[26]. A general object of this type (an X–module with an associative and distributive product structure) might be termed a **wring**, in reference to the etymological origin of the word '(w)rack'.

Regarding $\mathbb{Z}$ as a trivial X–module, the **augmentation map** $\varepsilon\colon \mathbb{Z}X \longrightarrow \mathbb{Z}$ is defined by

$$\varepsilon_x\left(\sum_{w\in \mathrm{As}\, X} n_w\rho_{x^{\bar{w}},w}(*) + \sum_{y\in X, v\in \mathrm{As}\, X} m_{y,v}\rho_{x^{\bar{v}},v}\lambda_{y,x^{\bar{v}\bar{y}}}(*)\right) = \sum_{w\in \mathrm{As}\, X} n_w$$

The **augmentation module** of X is the kernel $\mathcal{I}X = \ker\varepsilon$ of this map.

The group $(\mathcal{I}X)_x$ is free abelian, generated by symbols of the forms

(i) $\rho_{x^{\bar{v}},v}(*) - \rho_{x^{\bar{w}},w}(*)$

(ii) $\rho_{x^{\bar{v}},v}\lambda_{y,x^{\bar{v}\bar{y}}}(*)$

where $x, y \in X$ and $v, w \in \operatorname{As} X$, subject to the usual rewriting relations.

The structure maps of $\mathcal{I}X$ are the obvious restrictions of the free structure maps P and Λ of $\mathbb{Z}X$.

2.5 Right X–modules

We wish to define a tensor product operation on the category of X–modules. In the more conventional case of modules over a ring R, the tensor product in its simplest form is a bifunctor $\otimes_R\colon \mathsf{Mod}_R \times {}_R\mathsf{Mod} \longrightarrow \mathsf{Ab}$. More generally, it may be considered as a bifunctor $\otimes_R\colon {}_S\mathsf{Mod}_R \times {}_R\mathsf{Mod}_T \longrightarrow {}_S\mathsf{Mod}_T$. Our first task, then, is to devise the correct notion of a 'left' or 'right' rack module.

The category RMod_X, defined earlier in this chapter, consists of (a certain class of) *covariant* trunk maps $\mathsf{T}(X) \longrightarrow \mathsf{Ab}$. A *contravariant* trunk map $\mathcal{A} = (A, \phi, \psi)\colon \mathsf{T}(X) \longrightarrow \mathsf{Ab}$ determines an abelian group A_x for each element $x \in X$, and homomorphisms $\phi^{x,y}\colon A_{x^y} \longrightarrow A_x$ and $\psi^{y,x}\colon A_{x^y} \longrightarrow A_y$, such that

$$\begin{aligned} \phi^{x,y}\phi^{x^y,z} &= \phi^{x,z}\phi^{x^z,y^z} \\ \psi^{y,x}\phi^{x^y,z} &= \phi^{y,z}\psi^{y^z,x^z} \end{aligned}$$

for all $x, y \in X$.

Let RMod^X be the category whose objects are the contravariant trunk maps $\mathcal{A} = (A, \phi, \psi)\colon \mathsf{T}(X) \longrightarrow \mathsf{Ab}$ satisfying the condition

$$\psi^{z,x^y} = \psi^{z,y}\psi^{y^z,x^z} + \psi^{z,x}\phi^{x^z,y^z} \tag{2.1}$$

for all $x, y, z \in X$, and whose morphisms are natural transformations.

We may refer to objects in RMod_X as **left** X–modules, and objects in RMod^X as **right** X–modules. As with the more familiar setting of modules over a ring R, we may consider a left X–module to be a right X^*–module:

Theorem 2.9

The categories RMod_X *and* RMod^{X^*} *are equivalent, for any rack* X.

Proof

Given an arbitrary left X–module $\mathcal{A} = (A, \phi, \psi)$, we may define a right X–module $\mathcal{B} = (B, \chi, \omega)$ as follows. For each $x \in X$, let $B_{x^*} = A_x$. Define abelian group homomorphisms $\chi^{x^*,y^*}\colon\ B_{x^*y^*} \longrightarrow B_{x^*}$ and $\omega^{y^*,x^*}\colon\ B_{x^*y^*} \longrightarrow B_{y^*}$ by setting $\chi^{x^*,y^*}(a) = \phi_{x^{\bar{y}},y}(a)$, and $\omega^{y^*,x^*}(a) = \psi_{x^{\bar{y}},y^{y\bar{x}\bar{y}}}(a)$ for all elements $x, y \in X$ and $a \in A_{x^{\bar{y}}} = B_{x^*y^*}$.

Then $\mathcal{B} = (B, \chi, \omega)$ is a right X^*–module, since

$$\begin{aligned}\chi^{x^*,y^*}\chi^{x^*y^*,z^*} &= \phi_{x^{\bar{y}},y}\phi_{x^{\bar{y}\bar{z}},z} = \phi_{x^{\bar{y}\bar{z}y},z^y}\phi_{x^{\bar{y}\bar{z}},y} \\ &= \phi_{x^{\bar{z}},z}\phi_{x^{\bar{y}\bar{z}},y^{\bar{z}}} = \phi_{x^{\bar{z}},z}\phi_{x^{\bar{z}\overline{y^{\bar{z}}}},y^{\bar{z}}} = \chi^{x^*,z^*}\chi^{x^*z^*,y^*z^*}\end{aligned}$$

$$\omega^{y^*,x^*}\chi^{x^*y^*,z^*} = \psi_{x^{\bar{y}},y^{y\bar{x}\bar{y}}}\phi_{x^{\bar{y}\bar{z}},z} = \phi_{y^{\bar{z}},z}\psi_{x^{\bar{y}\bar{z}},y^{y\bar{x}\bar{y}\bar{z}}} = \chi^{y^*,z^*}\omega^{y^*z^*,x^*z^*}$$

and

$$\begin{aligned}\omega^{z^*,x^*y^*} &= \psi_{x^{\bar{y}\bar{z}},z^{zy\bar{x}\bar{y}\bar{z}}} = \psi_{y^{\bar{z}},z^{z\bar{y}\bar{z}}}\psi_{x^{\bar{y}\bar{z}},y^{y\bar{x}\bar{y}\bar{z}}} + \phi_{z^{z\bar{y}\bar{z}},y^{\bar{z}}}\psi_{x^{\bar{y}\bar{z}},z^{z\bar{x}\bar{y}\bar{z}}} \\ &= \psi_{y^{\bar{z}},z^{z\bar{y}\bar{z}}}\psi_{x^{\bar{y}\bar{z}},y^{y\bar{x}\bar{y}\bar{z}}} + \psi_{x^{\bar{z}},z^{z\bar{x}\bar{z}}}\phi_{x^{\bar{y}\bar{z}},y^{\bar{z}}} \\ &= \omega^{z^*,y^*}\omega^{y^*z^*,x^*z^*} + \omega^{z^*,x^*}\chi^{x^*z^*,y^*z^*}\end{aligned}$$

Given another left X–module $\mathcal{C} = (C, \gamma, \eta)$, and an X–map $f\colon\ \mathcal{A} \longrightarrow \mathcal{C}$, we may construct another right X^*–module $\mathcal{D} = (D, \delta, \theta)$ as above, together with an X^*–map $g\colon\ \mathcal{B} \longrightarrow \mathcal{D}$ as follows. Let $g_{x^*} = f_x$ for all $x \in X$. Then

$$\delta^{x^*,y^*}g_{x^*y^*} = \gamma_{x^{\bar{y}},y}f_{x^{\bar{y}}} = f_x\phi_{x^{\bar{y}},y} = g_{x^*}\chi^{x^*,y^*}$$

and

$$\theta^{y^*,x^*}g_{x^*y^*} = \eta_{x^{\bar{y}},y^{y\bar{x}\bar{y}}}f_{x^{\bar{y}}} = f_y\psi_{x^{\bar{y}},y^{y\bar{x}\bar{y}}} = g_{y^*}\omega^{y^*,x^*}$$

which confirms that g is a natural transformation $\mathcal{B} \longrightarrow \mathcal{D}$.

This assignment $\mathcal{A} \mapsto \mathcal{B}$ and $f \mapsto g$ just described determines a covariant functor $F\colon\ \mathsf{RMod}_X \longrightarrow \mathsf{RMod}^{X^*}$. Given a third left X–module $\mathcal{E}$ and another X^*–map $h\colon\ \mathcal{C} \longrightarrow \mathcal{E}$, then $F(hf) = F(h)F(f)\colon\ F(\mathcal{A}) \longrightarrow F(\mathcal{E})$.

Now, given an arbitrary right X^*–module $\mathcal{B} = (B, \chi, \omega)$, we may construct a left X–module $\mathcal{A} = (A, \phi, \psi)$ as follows. As before, set $A_x = B_{x^*}$ for all $x \in X$. Let $\phi_{x,y}(a) = \chi^{x^* \overline{y^*}, y^*}(a)$ and $\psi_{y,x}(b) = \omega^{x^* \overline{y^*}, y^* y^* \overline{x^*}\, \overline{y^*}}(b)$, for all $x, y \in X$, $a \in B_{x^*} = A_x$ and $b \in B_{y^*} = A_y$. Three calculations very similar to the ones above confirm that this is indeed a left X–module.

Given another right X^*–module $\mathcal{D} = (D, \delta, \theta)$ and an X^*–map $g\colon \mathcal{B} \longrightarrow \mathcal{D}$ we may construct another left X–module $\mathcal{C} = (C, \gamma, \eta)$ and an X–map f in the obvious manner.

The assignment $\mathcal{B} \mapsto \mathcal{A}$ and $g \mapsto f$ just described determines a covariant functor $G\colon \mathsf{RMod}^{X^*} \longrightarrow \mathsf{RMod}_X$ which may be seen to be the inverse of F. □

Corollary 2.10

For any rack X, the categories RMod^X and RMod_{X^*} are equivalent.

Proof

This follows immediately from the observation that $X^{**} = X$. □

The appropriate definition of an (X, X)–**bimodule** is not entirely obvious, and the more general concept of an (X, Y)–**bimodule** even less so. However, any (left) X–module $\mathcal{A} = (A, \phi, \psi)$ may be regarded as a trivial right X–module $\mathcal{A} = (A, \alpha, \beta)$ by defining right structure maps $\alpha^{x,y} = \mathrm{Id}_{x^y,x}\colon A_{x^y} \longrightarrow A_x$ and $\beta^{y,x} = 0_{x^y,y}\colon A_{x^y} \longrightarrow A_y$. An arbitrary right X–module may similarly be given a trivial left X–module structure. The category TMod_X, of trivial X–modules, may thus be regarded as lying in the intersection of RMod_X and RMod^X.

For any trunk map $\mathcal{S}\colon \mathsf{D}(X) \longrightarrow \mathsf{Set}$ we may define the **free right X–module** on $\mathcal{S}$ in the obvious way. Let F_x be the free abelian group generated by symbols of the forms

(i) $\rho^{x,w}(a)$ where $w \in \mathrm{As}\, X$ and $a \in S_{x^w}$.

(ii) $\lambda^{x,y}\rho^{y^x,w}(b)$ where $y \in X$, $w \in \mathrm{As}\, X$, and $b \in S_{y^{xw}}$.

modulo the relations

(i) $\rho^{x,u}\rho^{x^u,v} = \rho^{x,v}\rho^{x^v,u^v} = \rho^{x,uv}$

(ii) $\lambda^{y,x}\rho^{x^y,u} = \rho^{y,u}\lambda^{y^u,x^u}$

(iii) $\lambda^{z,x^y} = \lambda^{z,y}\lambda^{y^z,x^z} + \lambda^{z,x}\rho^{x^z,y^z}$

(iv) $\rho^{x,w}(p+q) = \rho^{x,w}(p) + \rho^{x,w}(q)$

(v) $\lambda^{y,x}(s+t) = \lambda^{y,x}(s) + \lambda^{y,x}(t)$

where $x, y, z \in X$; $u, v, w \in \operatorname{As} X$; $p, q \in F_{x^w}$; and $s, t \in F_{x^y}$. The structure maps are defined as follows:

$$\begin{aligned} P^{x,y}\colon\ F_{x^y} \longrightarrow F_x \quad &; \quad p \mapsto \rho^{x,y}p \\ \Lambda^{y,x}\colon\ F_{x^y} \longrightarrow F_y \quad &; \quad p \mapsto \lambda^{y,x}p \end{aligned}$$

for all $x, y \in X$ and $p \in F_{x^y}$.

2.6 Tensor products

Let $\mathcal{A} = (A, \phi, \psi)$ be a right X–module, $\mathcal{B} = (B, \chi, \omega)$ a left X–module, and C an abelian group.

Then $\mathcal{H} = (H, \eta, \zeta) = \operatorname{Hom}_{\mathsf{Ab}}(\mathcal{B}, C)$ has a canonical right X–module structure, defined as follows: Let $H_x = \operatorname{Hom}_{\mathsf{Ab}}(B_x, C)$, and define structure maps $\eta^{x,y}\colon\ H_{x^y} \longrightarrow H_x$ and $\zeta^{y,x}\colon\ H_{x^y} \longrightarrow H_y$ to be

$$\begin{aligned} \eta^{x,y}(f) &= f\chi_{x,y} \\ \zeta^{y,x}(g) &= g\omega_{y,x} \end{aligned}$$

for all $x, y \in X$, $f\colon\ B_x \longrightarrow C$, and $g\colon\ B_y \longrightarrow C$.

This construction gives a functor $H_{\mathcal{B}}\colon\ \mathsf{Ab} \longrightarrow \mathsf{RMod}^X$, and we define the **tensor product** to be its left adjoint, so

$$\operatorname{Hom}_{\mathsf{RMod}^X}(\mathcal{A}, H_{\mathcal{B}}(C)) = \operatorname{Hom}_{\mathsf{RMod}^X}(\mathcal{A}, \operatorname{Hom}_{\mathsf{Ab}}(\mathcal{B}, C)) \cong \operatorname{Hom}_{\mathsf{Ab}}(\mathcal{A} \otimes_X \mathcal{B}, C)$$

An X–**biadditive function** is an abelian group homomorphism

$$f\colon\ \bigcup_{x \in X} (A_x \times B_x) \longrightarrow C$$

such that

(i) $f(a_1 + a_2, b) = f(a_1, b) + f(a_2, b)$

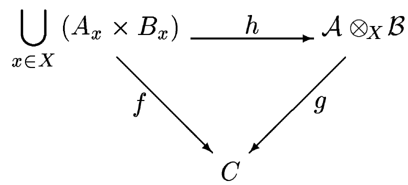
(ii) $f(a, b_1 + b_2) = f(a, b_1) + f(a, b_2)$

(iii) $f(na, b) = f(a, nb) = nf(a, b)$

(iv) $f(\phi^{x,y}c, b) = f(c, \chi_{x,y}b)$

(v) $f(\psi^{y,x}c, d) = f(c, \omega_{y,x}d)$

for all $x, y \in X$; $a, a_1, a_2 \in A_x$; $c \in A_{x^y}$; $b, b_1, b_2 \in B_x$; $d \in B_y$; and $n \in \mathbb{Z}$. Then the tensor product $\mathcal{A} \otimes_X \mathcal{B}$ is the solution of the following universal mapping problem:



For any abelian group C and X–biadditive function f there exists a unique homomorphism g making the diagram commute.

Theorem 2.11

The tensor product $\mathcal{A} \otimes_X \mathcal{B}$ exists, and is unique up to isomorphism.

Proof

Let D be the free abelian group with basis $\bigcup_{x \in X} (A_x \times B_x)$. That is, let D be generated by symbols of the form (a, b) where $a \in A_x$ and $b \in B_x$ for all $x \in X$. Let E be the subgroup of D generated by elements of the forms

(i) $(a_1 + a_2, b) = (a_1, b) + (a_2, b)$

(ii) $(a, b_1 + b_2) = (a, b_1) + (a, b_2)$

(iii) $(na, b) = (a, nb) = n(a, b)$

(iv) $(\phi^{x,y}c, b) = (c, \chi_{x,y}b)$

(v) $(\psi^{y,x}c, d) = (c, \omega_{y,x}d)$

for all $x, y \in X$; $a, a_1, a_2 \in A_x$; $c \in A_{x^y}$; $b, b_1, b_2 \in B_x$; $d \in B_y$; and $n \in \mathbb{Z}$. Now let $\mathcal{A} \otimes_X \mathcal{B}$ be the quotient D/E, and denote the coset $(a, b) + E$ by $a \otimes b$. The homomorphism $h\colon (a, b) \mapsto a \otimes b$ is X–biadditive, and ensures the existence of a unique homomorphism $g\colon \mathcal{A} \otimes_X \mathcal{B} \to C$ making the above diagram commute.

Uniqueness of the tensor product follows by the usual universality argument.

We must now prove that this abelian group $\mathcal{A} \otimes_X \mathcal{B}$ satisfies the required adjointness condition.

An element f of $\mathrm{Hom}_{\mathsf{RMod}^X}(\mathcal{A}, \mathrm{Hom}_{\mathsf{Ab}}(\mathcal{B}, C))$ assigns, to each $x \in X$ and $a \in A_x$, an abelian group homomorphism $f_x(a)\colon B_x \to C$, in a natural way. That is,

$$\begin{array}{lcllcl}
f_x(a_1 + a_2)(b) & = & f_x(a_1)(b) + f_x(a_2)(b) & & \\
f_x(a)(b_1 + b_2) & = & f_x(a)(b_1) + f_x(a)(b_2) & & \\
f_x(na)(b) & = & n f_x(a)(b) & = & f_x(a)(nb) \\
f_x(\phi^{x,y}(c)(b) & = & (\eta^{y,x} f_{x^y}(c))\,(b) & = & f_{x^y}(c)(\chi_{x,y} b) \\
f_y(\psi^{x,y}(c)(d) & = & (\zeta^{y,x} f_{x^y}(c))\,(d) & = & f_{x^y}(c)(\omega_{y,x} d)
\end{array}$$

for all $x, y \in X$; $a, a_1, a_2 \in A_x$; $c \in A_{x^y}$; $b, b_1, b_2 \in B_x$; $d \in B_y$; and $n \in \mathbb{Z}$. The first three identities follow from the fact that each f_x is an abelian group homomorphism, and the remaining two from the fact that f is a homomorphism of right X–modules (and hence a natural transformation of contravariant trunk maps $\mathsf{T}(X) \to \mathsf{Ab}$).

The required natural isomorphism is

$$\tau_{\mathcal{A},C}\colon \mathrm{Hom}_{\mathsf{Ab}}(\mathcal{A} \otimes_X \mathcal{B}, C) \longrightarrow \mathrm{Hom}_{\mathsf{RMod}^X}(\mathcal{A}, \mathrm{Hom}_{\mathsf{Ab}}(\mathcal{B}, C))$$

defined by

$$f(a \otimes b) \mapsto f_x(a)(b)$$

for all $x \in X, a \in A_x$, and $b \in B_x$. □

In certain cases, the tensor product may be regarded as an X–module:

Proposition 2.12

For any left X–module $\mathcal{A} = (A, \phi, \psi)$, the tensor product $\mathbb{Z}X \otimes_X \mathcal{A}$ has a canonical (left) X–module structure such that $\mathbb{Z}X \otimes_X \mathcal{A} \cong \mathcal{A}$.

Proof

The tensor product $\mathbb{Z}X \otimes_X \mathcal{A}$ is the free abelian group generated by symbols of the form $r \otimes a$ where $r \in (\mathbb{Z}X)_x$ (with $\mathbb{Z}X$ considered as a right X–module) and $a \in A_x$, for all $x \in X$, such that

(i) $P^{x,y}(*) \otimes a = (*) \otimes \phi_{x,y}(a)$

(ii) $\Lambda^{y,x}(*) \otimes b = (*) \otimes \psi_{y,x}(b)$

where $a \in A_x$ and $b \in A_y$. This group has a canonical left X–module structure defined as follows: For each $x \in X$, let B_x be the free abelian group generated by symbols of the form $(*) \otimes a$ with $a \in A_x$. We may then define homomorphisms

$$\chi_{x,y} \colon B_x \longrightarrow B_{x^y} \quad ; \quad (*) \otimes a \mapsto \rho^{x,y}(*) \otimes a = (*) \otimes \phi_{x,y}(a)$$
$$\omega_{y,x} \colon B_y \longrightarrow B_{x^y} \quad ; \quad (*) \otimes b \mapsto \lambda^{y,x}(*) \otimes b = (*) \otimes \psi_{y,x}(b)$$

which satisfy the usual criteria for the structure maps of a left X–module. Furthermore, this X–module $\mathcal{B} = (B, \chi, \omega)$ may readily be seen to be isomorphic to $\mathcal{A}$. □

Proposition 2.13

Let $\mathcal{V}_x$ be the trunk map $\mathsf{D}(X) \longrightarrow \mathsf{Set}$ where

$$(\mathcal{V}_x)_y = \begin{cases} \{(*)\} & \text{if } x = y \\ \emptyset & \text{otherwise} \end{cases}$$

for some fixed $x \in X$, and let $\mathcal{A} = (A, \phi, \psi)$ be an arbitrary (left) X–module. Then $F\mathcal{V}_x \otimes_X \mathcal{A} \cong A_x$ where $F\mathcal{V}_x$ denotes the free right X–module on $\mathcal{V}_x$.

Proof

Observe that $F\mathcal{V}_x \otimes_X \mathcal{A}$ is the free abelian group generated by symbols of the forms:

(i) $\rho^{z,w}(*) \otimes a$ where $z \in X$, $a \in A_z$, and $w \in \operatorname{As} X$ such that $z^w = x$

(ii) $\lambda^{z,y}\rho^{y^z,w}(*) \otimes a$ where $y, z \in X$, $a \in A_z$, and $w \in \operatorname{As} X$ such that $y^{zw} = x$.

modulo the usual X–bilinearity relations. That is, $F\mathcal{V}_x \otimes_X \mathcal{A}$ is the free abelian group generated by symbols of the form $(*) \otimes a$ where $a \in A_x$, modulo the relations

(i) $(*) \otimes a_1 + (*) \otimes a_2 = (*) \otimes (a_1 + a_2)$

(ii) $n\,((*) \otimes a) = (n(*)) \otimes a = (*) \otimes na$

for $a, a_1, a_2 \in A_x$ and $n \in \mathbb{Z}$. But this group is isomorphic to A_x by the map $(*) \otimes a \mapsto a$.

□

The tensor product, then, provides a bifunctor

$$\otimes_X \colon \mathsf{RMod}^X \times \mathsf{RMod}_X \longrightarrow \mathsf{Ab}$$

which is covariant in both variables. Composing this with the obvious trivialisation functor $T\colon \mathsf{RMod}_X \longrightarrow \mathsf{TMod}_X$ yields a bifunctor

$$\otimes \colon \mathsf{TMod}_X \times \mathsf{TMod}_X \longrightarrow \mathsf{Ab}$$

Given a trivial right X–module $\mathcal{A}$ and a trivial left X–module $\mathcal{B}$, the tensor product $\mathcal{A} \otimes \mathcal{B}$ is the free abelian group generated by symbols $a \otimes b$ for all $a \in A_x$ and $b \in B_x$, and all $x \in X$, modulo the relations

(i) $(a_1 + a_2) \otimes b = a_1 \otimes b + a_2 \otimes b$

(ii) $a \otimes (b_1 + b_2) = a \otimes b_1 + a \otimes b_2$

for all $a, a_1, a_2 \in A_x$ and $b, b_1, b_2 \in B_x$.

The product $\mathbb{Z}X \otimes \mathcal{B}$ has a canonical (left) X–module structure analogous to that of $\mathbb{Z}X \otimes_X \mathcal{B}$, and determines a functor $\mathbb{Z}X \otimes - \colon \mathsf{TMod}_X \longrightarrow \mathsf{RMod}_X$ which is the left adjoint to the trivialisation functor $T\colon \mathsf{RMod}_X \longrightarrow \mathsf{TMod}_X$.

Let $f\colon X \longrightarrow Y$ be a rack homomorphism. Then there is a well-defined functor $f^*\colon \mathsf{RMod}_Y \longrightarrow \mathsf{RMod}_X$ which assigns to any Y–module $\mathcal{A} = (A, \phi, \psi)$ the X–

module $f^*\mathcal{A}$ where

$$\begin{aligned}(f^*A)_x &= A_{f(x)} \\ (f^*\phi)_{x,y} &= \phi_{f(x),f(y)}\colon A_{f(x)} \longrightarrow A_{f(x^y)} \\ (f^*\psi)_{y,x} &= \psi_{f(y),f(x)}\colon A_{f(y)} \longrightarrow A_{f(x^y)}\end{aligned}$$

If $\mathcal{B} = (B, \chi, \omega)$ is another Y–module and $\nu\colon \mathcal{A} \longrightarrow \mathcal{B}$ a Y–map, there is an X–map $f^*\nu\colon f^*\mathcal{A} \longrightarrow f^*\mathcal{B}$ such that

$$(f^*\nu)_x = \nu_{f(x)}\colon A_{f(x)} \longrightarrow B_{f(x)}$$

This is the **change of racks** functor induced by f, and has as left adjoint the functor $f_*\colon \mathsf{RMod}_X \longrightarrow \mathsf{RMod}_Y$ given by $f_* = \mathbb{Z}Y \otimes_X -$, where $\mathbb{Z}Y$ is the rack algebra of Y considered as a (right) X–module via f, so that

$$\mathrm{Hom}_{\mathsf{RMod}_X}(\mathcal{A}, f^*\mathcal{B}) \cong \mathrm{Hom}_{\mathsf{RMod}_Y}(\mathbb{Z}Y \otimes_X \mathcal{A}, \mathcal{B})$$

2.7 Quandle modules

We now investigate the appropriate specialisation of rack modules to the subcategory $\mathsf{Quandle}$. As before, let X be a fixed quandle, and let $\mathsf{T}(X)$ denote the trunk constructed from X as described in section 1.3.

Then a **quandle module** is a trunk map $\mathcal{A} = (A, \phi, \psi)\colon \mathsf{T}(X) \longrightarrow \mathsf{Ab}$ as before, but with the additional requirement that

$$\psi_{x,x}(a) + \phi_{x,x}(a) = a \tag{2.2}$$

for all $x \in X$ and $a \in A_x$.

Note that the earlier examples of trivial, dihedral, and Alexander rack modules also fulfil this criterion, and are hence quandle modules.

There is a corresponding notion of a **homomorphism** of quandle modules, which (as before) may be regarded as natural transformations between quandle module trunk maps.

Theorem 2.14

The category QMod_X *of quandle modules is abelian.*

Proof

The proof is identical to that of theorem 2.4. □

Furthermore, the objects just described are exactly the Beck modules in the category $\mathsf{Quandle}$:

Theorem 2.15

There is an equivalence of categories $\mathsf{QMod}_X \cong \mathsf{Ab}(\mathsf{Quandle}/X)$.

Proof

As in the proof of theorem 2.6, we identify the quandle module $\mathcal{A} = (A, \phi, \psi)$ with the split extension $\mathcal{A} \rtimes X \longrightarrow X$. The only additional modification is to take into account criterion (2.2).

So, if X is a quandle, then this additional requirement serves to ensure that $\mathcal{A} \rtimes X$ is itself a quandle, as shown in theorem 1.7.

Conversely, if $R \longrightarrow X$ is an abelian group object in $\mathsf{Ab}(\mathsf{Quandle}/X)$, then we may construct a rack module $\mathcal{R} = (R, \rho, \lambda)$ over X. It remains only to show that $\mathcal{R}$ satisfies (2.2):

$$\begin{aligned}\lambda_{x,x}(a) + \rho_{x,x}(a) = m(s(x)^a, a^{s(x)}) &= m(s(x), a)^{m(a,s(x))} \\ &= m(s(x), a)^{m(s(x),a)} = m(s(x)^{s(x)}, a^a) = m(s(x), a) = a\end{aligned}$$

□

For any given trunk map $\mathcal{S}\colon \mathsf{D}(X) \longrightarrow \mathsf{Set}$, there is a **free quandle module** $F\mathcal{S} = \mathcal{F} = (F, \rho, \lambda)$, which is defined similarly to that for rack modules, except with the additional relation $\rho_{x,x} + \lambda_{x,x} = \mathrm{Id}_{F_x}$ for all $x \in X$.

For any quandle X there is an analogous **quandle algebra** $\mathbb{Z}X$ which is defined to be the free quandle module on the singleton trunk map $\mathcal{S}\colon \mathsf{D}(X) \longrightarrow \mathsf{Set}$, equipped with the same multiplicative structure as the rack algebra.

The **augmentation map** $\varepsilon\colon \mathbb{Z}X \longrightarrow \mathbb{Z}$ and the **augmentation module** $\mathcal{I}X = \ker\varepsilon$ are also defined analogously.

There is a tensor product bifunctor $\otimes_X\colon \mathsf{QMod}^X \times \mathsf{QMod}_X \longrightarrow \mathsf{Ab}$ which has analogous properties to that defined for rack modules. Furthermore, there is an obvious trivialisation functor $T\colon \mathsf{QMod}_X \longrightarrow \mathsf{TMod}_X$ which has left adjoint $\mathbb{Z}X \otimes -\colon \mathsf{TMod}_X \longrightarrow \mathsf{QMod}_X$. Also, given an arbitrary quandle homomorphism $f\colon X \longrightarrow Y$ there is a **change of quandles** functor $f^*\colon \mathsf{QMod}_Y \longrightarrow \mathsf{QMod}_X$ with left adjoint $\mathbb{Z}Y \otimes_X -\colon \mathsf{QMod}_X \longrightarrow \mathsf{QMod}_Y$.

2.8 Involutory rack and quandle modules

Finally, we turn our attention to rack modules in the categories InvRack and InvQuandle.

An **involutory rack module** over an involutory rack X is an ordinary rack module $\mathcal{A} = (A, \phi, \psi)$, as defined earlier in this chapter which satisfies the additional requirements:

$$\phi_{x^y,y}\phi_{x,y} = \mathrm{Id}_{A_x} \tag{2.3}$$

$$\psi_{y,x^y}(b) + \phi_{x^y,y}\psi_{y,x}(b) = 0 \tag{2.4}$$

for all $x, y \in X$, and $b \in A_y$.

An **involutory quandle module** over an involutory quandle X is (as might be expected) a quandle module over X which satisfies the additional criteria for involutory rack modules.

Theorem 2.16

The category IRMod_X, of involutory rack modules, is abelian.

Proof

The proof is identical to that of theorem 2.4. □

Corollary 2.17

The category IQMod_X, of involutory quandle modules, is abelian.

Furthermore, the objects just described are exactly the Beck modules in the category $\mathsf{InvRack}$ of involutory racks:

Theorem 2.18

There is an equivalence of categories $\mathsf{IRMod}_X \cong \mathsf{Ab}(\mathsf{InvRack}/X)$.

Proof

As in the proofs of theorems 2.6 and 2.15 we assign to an involutory rack module $\mathcal{A} = (A, \phi, \psi)$ the split extension $\mathcal{A} \rtimes X \longrightarrow X$. The only additional modification is to take into account criteria (2.3) and (2.4).

So, if X is an involutory rack, then these additional requirements guarantee that $\mathcal{A} \rtimes X$ is itself an involutory rack.

Conversely, if $R \longrightarrow X$ is an abelian group object in $\mathsf{Ab}(\mathsf{InvRack}/X)$, then we may construct a rack module $\mathcal{R} = (R, \rho, \lambda)$ over X as described in theorem 2.6. The only additional step is to check that $\mathcal{R}$ satisfies the additional requirements listed above:

$$\rho_{x^y,y}\rho_{x,y}(a) = a^{s(y)s(y)} = a$$

and

$$\begin{aligned}\lambda_{y,x^y}(b) + \rho_{x^y,y}\lambda_{y,x}(b) &= m(s(x^y)^b, s(x)^{bs(y)}) \\ &= m(s(x), s(x))^{m(s(y)b,bs(y))} = s(x)^{m(s(y)b,\bar{b}\overline{s(y)})} \\ &= s(x)^{m(s(y)b,\overline{s(y)b})} = s(x) = 0 \in R_x\end{aligned}$$

□

Corollary 2.19

There is an equivalence between the category IQMod_X *of involutory quandle modules, and the category* $\mathsf{Ab}(\mathsf{InvQuandle}/X)$ *of abelian group objects in the slice category* $\mathsf{InvQuandle}/X$.

Given a trunk map $\mathcal{S}\colon\ \mathsf{D}(X) \longrightarrow \mathsf{Set}$, the **free involutory rack module** $F\mathcal{S} = \mathcal{F} = (F, \rho, \lambda)$ on $\mathcal{S}$ is defined similarly to that for rack modules, except with the additional relations (2.3) and (2.4) on the formal structure maps.

For any involutory rack X there is an analogous **involutory rack algebra** $\mathbb{Z}X$ which is defined to be the free involutory rack module on the singleton trunk map $\mathcal{S}\colon \mathsf{D}(X) \longrightarrow \mathsf{Set}$, equipped with the same multiplicative structure as the rack algebra. The **augmentation map** $\varepsilon\colon \mathbb{Z}X \to \mathbb{Z}$ and the **augmentation module** $\mathcal{I}X = \ker\varepsilon$ are also defined in the obvious manner.

There is a tensor product bifunctor $\otimes_X\colon \mathsf{IRMod}^X \times \mathsf{IRMod}_X \longrightarrow \mathsf{Ab}$ which has analogous properties to those defined for rack and quandle modules. The obvious trivialisation functor $T\colon \mathsf{IRMod}_X \longrightarrow \mathsf{TMod}_X$ has a left adjoint given by $\mathbb{Z}X \otimes -\colon \mathsf{TMod}_X \longrightarrow \mathsf{IRMod}_X$. Also, any homomorphism $f\colon X \longrightarrow Y$ of involutory racks gives rise to a change of racks functor $f^*\colon \mathsf{IRMod}_Y \longrightarrow \mathsf{IRMod}_X$ with left adjoint $\mathbb{Z}X \otimes_X -\colon \mathsf{IRMod}_X \longrightarrow \mathsf{IRMod}_Y$.

Given a trunk map $\mathcal{S}\colon \mathsf{D}(X) \longrightarrow \mathsf{Set}$, the **free involutory quandle module** $F\mathcal{S}$ has the obvious definition, and the **involutory quandle algebra** $\mathbb{Z}X$, augmentation map $\varepsilon\colon \mathbb{Z}X \to \mathbb{Z}$ and **augmentation module** $\mathcal{I}X = \ker\varepsilon$ are defined accordingly.

The tensor product $\otimes_X\colon \mathsf{IQMod}^X \times \mathsf{IQMod}_X \longrightarrow \mathsf{Ab}$ has an analogous definition to the other variants discussed in this chapter, and the trivialisation functor $T\colon \mathsf{IQMod}_X \longrightarrow \mathsf{TMod}_X$ has $\mathbb{Z}X \otimes -\colon \mathsf{TMod}_X \longrightarrow \mathsf{IQMod}_X$ as left adjoint. Given an arbitrary homomorphism $f\colon X \longrightarrow Y$ of involutory quandles, there is a change of quandles functor $f^*\colon \mathsf{IQMod}_Y \longrightarrow \mathsf{IQMod}_X$ with the left adjoint $\mathbb{Z}X \otimes_X -\colon \mathsf{IQMod}_X \longrightarrow \mathsf{IQMod}_Y$.

Chapter 3

Resolutions

This chapter is concerned with the definition of higher homology and cohomology groups of racks, quandles, and their involutory variants. Some of the results in this chapter are well-known results in homological algebra, but for completeness we check that they still hold in the category of rack modules.

A **chain complex C** is a sequence

$$\dots \xrightarrow{d_{n+1}} C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \dots$$

of modules such that $d_n d_{n+1} = 0$ (equivalently, $\operatorname{im} d_{n+1} \subseteq \ker d_n$) for all n.

A complex is said to be **acyclic** if it is exact. That is, if $\operatorname{im} d_{n+1} = \ker d_n$ for all n.

Given a complex of the form

$$C = \dots \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0 \xrightarrow{\varepsilon} M \longrightarrow 0$$

we may delete $M = \operatorname{coker}(C_1 \longrightarrow C_0)$ without losing any information, and denote this **deleted complex** by $\mathbf{C}_M$.

Given two complexes **C** and **D**, a **chain map** is a family of maps $f_n \colon C_n \longrightarrow D_n$

such that the diagram

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{d_{n+2}} & C_{n+1} & \xrightarrow{d_{n+1}} & C_n & \xrightarrow{d_n} & C_{n-1} & \xrightarrow{d_{n-1}} & \cdots \\
 & & \downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & \\
\cdots & \xrightarrow[e_{n+2}]{} & D_{n+1} & \xrightarrow[e_{n+1}]{} & D_n & \xrightarrow[e_n]{} & D_{n-1} & \xrightarrow[e_{n-1}]{} & \cdots
\end{array}$$

commutes.

A chain map $f\colon \mathbf{C} \longrightarrow \mathbf{D}$ is **nullhomotopic** if there are maps $s_n\colon C_n \longrightarrow D_{n+1}$ such that

$$f_n = e_{n+1}s_n + s_{n-1}d_n$$

for all n:

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{d_{n+2}} & C_{n+1} & \xrightarrow{d_{n+1}} & C_n & \xrightarrow{d_n} & C_{n-1} & \xrightarrow{d_{n-1}} & \cdots \\
 & & & {\scriptstyle s_n}\swarrow & \downarrow{\scriptstyle f_n} & \swarrow{\scriptstyle s_{n-1}} & & & \\
\cdots & \xrightarrow[e_{n+2}]{} & D_{n+1} & \xrightarrow[e_{n+1}]{} & D_n & \xrightarrow[e_n]{} & D_{n-1} & \xrightarrow[e_{n-1}]{} & \cdots
\end{array}$$

If $f, g\colon \mathbf{C} \longrightarrow \mathbf{D}$ are two chain maps, then f is **homotopic** to g (written $f \simeq g$) if $(f - g)$ is nullhomotopic.

Theorem 3.1 (Comparison Theorem)

Given a diagram

$$\begin{array}{ccccccccccc}
\cdots & \xrightarrow{d_3} & P_2 & \xrightarrow{d_2} & P_1 & \xrightarrow{d_1} & P_0 & \xrightarrow{\varepsilon} & A & \longrightarrow & 0 \\
 & & & & & & & & \downarrow{\scriptstyle f} & & \\
\cdots & \xrightarrow[e_3]{} & E_2 & \xrightarrow[e_2]{} & E_1 & \xrightarrow[e_1]{} & E_0 & \xrightarrow[\eta]{} & B & \longrightarrow & 0
\end{array}$$

where the rows are complexes, each P_n in the top row is projective, and the bottom row is acyclic, then there is a chain map $g\colon \mathbf{P}_A \longrightarrow \mathbf{E}_B$ making the diagram

$$\begin{array}{ccccccccccc}
\cdots & \xrightarrow{d_3} & P_2 & \xrightarrow{d_2} & P_1 & \xrightarrow{d_1} & P_0 & \xrightarrow{\varepsilon} & A & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle g_2} & & \downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle g_0} & & \downarrow{\scriptstyle f} & & \\
\cdots & \xrightarrow[e_3]{} & E_2 & \xrightarrow[e_2]{} & E_1 & \xrightarrow[e_1]{} & E_0 & \xrightarrow[\eta]{} & B & \longrightarrow & 0
\end{array}$$

commute. Furthermore, any two such chain maps are homotopic.

Such a chain map g is called a **chain map over** f.

Given a covariant functor T, we may define its **left derived functors** by

$$(L_nT)(A) = H_n(T\mathbf{P}_A) = \ker Td_n / \operatorname{im} Td_{n+1}$$

That is, choose a projective resolution $\mathbf{P}$ for a given module A, and let $\mathbf{P}_A$ denote the corresponding deleted complex. Apply T to this complex and take homology. Next, given a map $f\colon A \longrightarrow B$, there is a chain map $g\colon \mathbf{P}_A \longrightarrow \mathbf{P}_B$ over f, by the comparison theorem. Define

$$(L_nT)f = H_n(Tg) = (Tg)_*\colon (L_nT)A \longrightarrow (L_nT)B$$

Similarly, given a contravariant functor T, we may define its **right derived functors** by

$$(R^nT)(A) = H^n(T\mathbf{P}_A) = \ker Td_n / \operatorname{im} Td_{n-1}$$

In other words, choose a projective resolution $\mathbf{P}$ for the chosen module A, form its deleted complex, apply T and take cohomology.

As before, a map $f\colon A \longrightarrow B$ determines a chain map $\mathbf{P}_A \longrightarrow \mathbf{P}_B$ by the comparison theorem, so we may define

$$(R^nT)f = H^n(Tg) = (Tg)^*\colon (R^nT)B \longrightarrow (R^nT)A.$$

Theorem 3.2

Left and right derived functors are independent of the choice of projective resolution.

We begin by developing the work in the preceding chapters in order to devise a generalised chain complex of rack modules which reduces to the existing definitions [17] [11] [10] [1] in the appropriate special cases.

The remainder of the chapter is concerned with the development of a different approach to rack and quandle (co)homology using the methods of Cartan and Eilenberg [8] to define derived functors from (projective) resolutions.

There is a generalised method for constructing projective resolutions described by MacLane [24, §IX.5-7] in the context of relative homological algebra.

A **relative abelian category** is a pair of abelian categories A and B together with an additive, exact, and faithful covariant functor $U\colon \mathsf{A} \longrightarrow \mathsf{B}$. A **resolvent pair** of categories is a relative abelian category $U\colon \mathsf{A} \longrightarrow \mathsf{B}$ together with a left adjoint $F\colon \mathsf{B} \longrightarrow \mathsf{A}$. Let $\varepsilon\colon \mathrm{Id}_{\mathsf{B}} \longrightarrow UF$ denote the counit of this adjunction.

Now, given such a resolvent pair, there is an exact sequence

$$B \xrightarrow{\varepsilon_B} UF(B) \xrightarrow{\pi_B} \operatorname{coker} \varepsilon_B$$

where π_B is the natural quotient map $UF(M) \longrightarrow UF(M)/\varepsilon_B(B)$. Denote this cokernel by $R(B)$; then $R\colon \mathsf{B} \longrightarrow \mathsf{B}$ is a covariant functor.

Apply UF to $R(B)$ to get the diagram

$$\begin{array}{ccccccc} B & \overset{\varepsilon_B}{\rightarrowtail} & UF(B) & \overset{\pi_B}{\twoheadrightarrow} & R(B) & \longrightarrow & 0 \\ & & & \overset{s_B}{\searrow} & \downarrow \varepsilon_{RB} & & \\ & & & & UFR(B) & & \end{array}$$

in which the map $s_B = \varepsilon_{RB}\pi_B$ is a natural transformation $UF \longrightarrow UFR$ such that

Lemma 3.3 ([24] Lemma IX.7.1)

The morphisms ε_B and s_B induce, for every object A in B a left exact sequence of abelian groups:

$$0 \longrightarrow \mathrm{Hom}_{\mathsf{A}}(FR(M), A) \xrightarrow{s_B^*} \mathrm{Hom}_{\mathsf{B}}(UF(B), U(A)) \xrightarrow{\varepsilon_B^*} \mathrm{Hom}_{\mathsf{B}}(B, U(A))$$

Now, each object C of A yields a sequence of objects $M_n = R^n U(C)$ of B. The standard resolution consists of the (relatively) free objects

$$B_n(C) = FR^n U(C)$$

of A, for $n = 0, 1, 2, \ldots$

We may now define morphisms between the corresponding 'forgotten' or 'neglected' objects of B:

$$U(C) \xrightarrow{s_{-1}} UB_0(C) \xrightarrow{s_0} UB_1(C) \xrightarrow{s_1} UB_2(C) \xrightarrow{s_2} \cdots$$

where $s_{-1} = \varepsilon_{UC}$ and

$$s_n = s_{M_n} \colon UF(M_n) \longrightarrow UFR(M_n) = UB_{n+1}(C).$$

These morphisms may be used to recursively define unique boundary maps $\partial_n \colon B_n(C) \longrightarrow B_{n-1}(C)$ using the condition

$$\partial_{n+1} s_n + s_{n-1} \partial_n = \mathrm{Id}_{B_n(C)}$$

such that $\mathbf{B} = \{B_n(C) : n = 0, 1, 2, \ldots\}$ becomes a chain complex and a (relatively) free resolution (the **(normalised) bar resolution**) of C with s as contracting homotopy.

To confirm that this description agrees with the usual formulation of group homology, consider the relative abelian category $U \colon {}_G\mathsf{Mod} \longrightarrow \mathsf{Ab}$, where U is the obvious 'underlying abelian group' functor which discards the G–module structure. This has left adjoint $\mathbb{Z}G \otimes_{\mathbb{Z}} -$, and the corresponding adjunction has counit ε where $\varepsilon_A \colon a \mapsto 1 \otimes a$ for each $a \in A$. Thus we have a resolvent pair of categories and can apply the construction described above.

To begin, observe that the sequence

$$0 \longrightarrow \mathbb{Z} \hookrightarrow \mathbb{Z}G \longrightarrow \mathbb{Z}G/\mathbb{Z} \longrightarrow 0$$

is exact, and thus so is its tensor product with the G–module M:

$$\mathbb{Z} \otimes_{\mathbb{Z}} M \longrightarrow \mathbb{Z}G \otimes_{\mathbb{Z}} M \longrightarrow (\mathbb{Z}G/\mathbb{Z}) \otimes_{\mathbb{Z}} M \longrightarrow 0$$

or

$$M \longrightarrow F(M) \longrightarrow (\mathbb{Z}G/\mathbb{Z}) \otimes_{\mathbb{Z}} M \longrightarrow 0$$

Hence $R(M) \cong (\mathbb{Z}G/\mathbb{Z}) \otimes_{\mathbb{Z}} M$. Set $C = \mathbb{Z}$ to get

$$R^n(C) = R^n(\mathbb{Z}) = \otimes^n_{\mathbb{Z}}(\mathbb{Z}G/\mathbb{Z})$$

However, $\mathbb{Z}G/\mathbb{Z}$ is the free abelian group generated by the non-identity elements of G, and so $R^n(\mathbb{Z})$ is the free abelian group generated by symbols $[g_1|\ldots|g_n]$

where no $g_i \in G$ is the identity. The map s_n is defined by

$$s_n(g[g_1|\dots|g_n]) = [g|g_1|\dots|g_n]$$

which is zero if $g = 1$. This is exactly the usual contracting homotopy for the normalised bar resolution in group homology. The boundary maps are uniquely determined by the s maps and hence this resolution must agree with the usual normalised bar resolution.

3.1 Derivations

Recall that the group $\mathrm{Ext}(X, \mathcal{A})$ is the group of equivalence classes of (factor sets corresponding to) extensions of the rack X by the X–module $\mathcal{A}$. That is, the quotient of the group $Z(X, \mathcal{A})$ by the subgroup $B(X, \mathcal{A})$, where $Z(X, \mathcal{A})$ consists of factor sets satisfying the condition

$$\sigma_{x^y,z} + \phi_{x^y,z}\sigma_{x,y} = \psi_{y^z,x^z}\sigma_{y,z} + \sigma_{x^z,y^z} + \phi_{x^z,y^z}\sigma_{x,z}$$

and $B(X, \mathcal{A})$ consists of the subgroup of factor sets satisfying

$$\sigma_{x,y} = \psi_{y,x}u_y - u_{x^y} + \phi_{x,y}u_x$$

for all $x, y, z \in X$ and some class of elements $u = \{u_x \in A_x : x \in X\}$.

This looks very much like a more general form of the usual definition of $H^2(X; A)$ where A is an abelian group, which consists of the group of all homomorphisms $FA(X \times X) \longrightarrow A$ vanishing on elements of the form

$$(x^y, z) + (x, y) - (x^z, y^z) - (x, z)$$

modulo the subgroup of such homomorphisms f which may be expressed in the form

$$f(x, y) = g(x) - g(x^y)$$

for some homomorphism $g\colon FA(X) \longrightarrow A$.

The second twisted rack cohomology group similarly consists of homomorphisms

$FA(X \times X) \longrightarrow A$ which vanish on elements of the form

$$(x^y, z) + T(x, y) - (1 - T)(y, z) - (x^z, y^z) - T(x, z)$$

modulo homomorphisms $f\colon\ FA(X \times X) \longrightarrow A$ of the form

$$f(x, y) = Tg(x) - g(x^y) + (1 - T)(y)$$

for some homomorphism $g\colon\ FA(X) \longrightarrow A$.

We wish to formalise this connection and devise suitable generalisations of the higher cohomology groups $H^n(X; \mathcal{A})$ and the homology groups $H_n(X; \mathcal{A})$, where $\mathcal{A}$ is an arbitrary X–module.

We start by examining the construction of the group $B(X, \mathcal{A})$ of split factor sets, which should be analogous to the group $B^2(X; A)$ of 1–coboundaries in ordinary rack cohomology.

We may thus expect elements of $B(X; \mathcal{A})$ to be the image, under some suitable coboundary operator, of 'functions' $f\colon\ X \longrightarrow \mathcal{A}$. This concept is not yet well-defined, since it is not immediately clear what is meant by a 'function' from a rack (which is a set with some additional structure imposed on it) to a rack module (which is a special sort of trunk map).

Our model for this notion comes from condition (1.13) in theorem 1.4, and so we initially define a **1–coboundary** to be a family $u = \{u_x : x \in X, u_x \in A_x\}$ such that

$$u_{x^y} = \psi_{y,x}(u_y) + \phi_{x,y}(u_x)$$

It thus becomes necessary to find some way of describing X as some trunk map $\mathsf{D}(X) \longrightarrow \mathsf{Set}$, in order to define the notion of a '1–coboundary' in a more useful manner.

Let $\mathcal{S}_1$, then, denote the trunk map $\mathsf{D}(X) \longrightarrow \mathsf{Set}$ where $(\mathcal{S}_1)_x = \{x\}$ for all $x \in X$. Then a 1–coboundary may be regarded as a natural transformation $f\colon\ \mathcal{S}_1 \longrightarrow U\mathcal{A}$, with U denoting the forgetful functor $\mathsf{RMod}_X \longrightarrow \mathsf{Set}^{\mathsf{D}(X)}$, by setting $f_x(x) = u_x \in A_x = (U\mathcal{A})_x$ for all $x \in X$.

This set $\mathrm{Hom}_{\mathsf{Set}^{\mathsf{D}(X)}}(\mathcal{S}_1, U\mathcal{A})$ has an obvious abelian group structure defined by setting $(f+g)_x(x) = f_x(x) + g_x(x) \in A_x$.

Similarly, define $\mathcal{S}_2 \colon \mathsf{D}(X) \longrightarrow \mathsf{Set}$ by $(\mathcal{S}_2)_x = \{(p,q) \in X \times X : p^q = x\}$. A factor set σ may be regarded as a natural transformation $f \colon \mathcal{S}_2 \longrightarrow U\mathcal{A}$ by setting $f_x(p,q) = \sigma_{p,q} \in A_{p^q} = A_x = (U\mathcal{A})_x$ for all $x \in X$ and $(p,q) \in (\mathcal{S}_2)_x$.

In general, define the trunk map $\mathcal{S}_n \colon \mathsf{D}(X) \longrightarrow \mathsf{Set}$ so that

$$(\mathcal{S}_n)_x = \{(x_1, x_2, \ldots, x_n) \in X^n : x_1^{x_2 \cdots x_n} = x\}$$

for $n > 0$, and $(\mathcal{S}_0)_x = \{(*)\}$, for all $x \in X$.

The set $\mathrm{Hom}_{\mathsf{Set}^{\mathsf{D}(X)}}(\mathcal{S}_n, \mathcal{A})$ has an obvious abelian group structure given by setting

$$(f+g)_{x_1^{x_2 \cdots x_n}}(x_1, \ldots, x_n) = f_{x_1^{x_2 \cdots x_n}}(x_1, \ldots, x_n) + g_{x_1^{x_2 \cdots x_n}}(x_1, \ldots, x_n)$$

for all $x_1, \ldots, x_n \in X$.

If we now define a map $d^2 \colon \mathrm{Hom}_{\mathsf{Set}^{\mathsf{D}(X)}}(\mathcal{S}_1, \mathcal{A}) \longrightarrow \mathrm{Hom}_{\mathsf{Set}^{\mathsf{D}(X)}}(\mathcal{S}_2, \mathcal{A})$ by

$$(d^2 f)_{x^y}(x,y) = \psi_{y,x} f_y(y) - f_{x^y}(x^y) + \phi_{x,y} f_x(x)$$

then $B^2(X;\mathcal{A})$ may be seen to be exactly $\operatorname{im} d^2$. This map d^2 is an abelian group homomorphism.

We may also define a map $d^3 \colon \mathrm{Hom}_{\mathsf{Set}^{\mathsf{D}(X)}}(\mathcal{S}_2, \mathcal{A}) \longrightarrow \mathrm{Hom}_{\mathsf{Set}^{\mathsf{D}(X)}}(\mathcal{S}_3, \mathcal{A})$ by

$$\begin{aligned}(d^3 f)_{x^{yz}}(x,y,z) = \psi_{y^z, x^z} f_{y^z}(y,z) \\ + f_{x^z y^z}(x^z, y^z) - f_{x^{yz}}(x^y, z) \\ + \phi_{x^z, y^z} f_{x^z}(x,z) - \phi_{x^y, z} f_{x^y}(x,y)\end{aligned}$$

such that $Z^2(X;\mathcal{A}) = \ker d^3$.

A routine calculation confirms that $\operatorname{im} d^2 \subseteq \ker d^3$, so we now have a fragment of a chain complex

$$\mathrm{Hom}_{\mathsf{Set}^{\mathsf{D}(X)}}(\mathcal{S}_1, U\mathcal{A}) \xrightarrow{d^2} \mathrm{Hom}_{\mathsf{Set}^{\mathsf{D}(X)}}(\mathcal{S}_2, U\mathcal{A}) \xrightarrow{d^3} \mathrm{Hom}_{\mathsf{Set}^{\mathsf{D}(X)}}(\mathcal{S}_3, U\mathcal{A})$$

There is a bijective correspondence between trunk maps $\mathcal{S} \longrightarrow U\mathcal{A}$ and trunk maps $F\mathcal{S} \longrightarrow \mathcal{A}$, so $\mathrm{Hom}_{\mathsf{Set}^{\mathsf{D}(X)}}(\mathcal{S}, U\mathcal{A}) \cong \mathrm{Hom}_X(F\mathcal{S}, \mathcal{A})$ and hence the above complex is the same as

$$\mathrm{Hom}_X(\mathcal{F}_1, \mathcal{A}) \xrightarrow{d^2} \mathrm{Hom}_X(\mathcal{F}_2, \mathcal{A}) \xrightarrow{d^3} \mathrm{Hom}_X(\mathcal{F}_3, \mathcal{A})$$

where $\mathcal{F}_n$ is the free X–module with basis $\mathcal{S}_n$. As hoped, this is merely the application of the contravariant functor $\mathrm{Hom}_X(-, \mathcal{A})$ to a complex of free X–modules.

Seeking a similar perspective for the first cohomology $H^1(X; \mathcal{A})$, we define a **derivation** $f\colon X \longrightarrow \mathcal{A}$ to be a natural transformation $f\colon \mathcal{S}_1 \longrightarrow U\mathcal{A}$ such that $f_{x^y}(x^y) = \psi_{y,x} f_y(y) + \phi_{x,y} f_x(x)$. The set of such maps is, of course, $\ker d^2$, and we may denote it by $\mathrm{Der}(X, \mathcal{A})$. Note that if $\mathcal{A}$ is a trivial X–module, then $\mathrm{Der}(X, \mathcal{A}) = \mathrm{Hom}_{\mathsf{Set}^{\mathsf{D}(X)}}(\mathcal{S}_1, U\mathcal{A}) = \mathrm{Hom}_X(\mathcal{F}_1, \mathcal{A})$

Let $z \in X$ be an arbitrary rack element, and let a z–**principal derivation** be a natural transformation $f\colon \mathcal{S}_1 \longrightarrow \mathcal{A}$ such that $f_x(x) = \psi_{z,x^{\bar{z}}}(a)$ for each $x \in X$ and some fixed element $a \in A_z$. As the terminology suggests, such a map is itself a derivation. The set $\mathrm{PDer}_z(X, \mathcal{A})$ of all z–principal derivations has an abelian group structure, and may be regarded as the image of the X–map $d^1_z\colon \mathrm{Hom}_{\mathsf{Set}^{\mathsf{D}(X)}}(\mathcal{S}_0, U\mathcal{A}) \longrightarrow \mathrm{Hom}_{\mathsf{Set}^{\mathsf{D}(X)}}(\mathcal{S}_1, U\mathcal{A})$ given by

$$(d^1_z f)_x(x) = \psi_{z,x^{\bar{z}}} f_z(*).$$

Hence we may set $H^1(X; \mathcal{A})_z = \mathrm{Der}(X, \mathcal{A})/\mathrm{PDer}_z(X, \mathcal{A})$, and extend the above chain complex fragment by one dimension:

$$\mathrm{Hom}_X(\mathcal{F}_0, \mathcal{A}) \xrightarrow{d^1_z} \mathrm{Hom}_X(\mathcal{F}_1, \mathcal{A}) \xrightarrow{d^2} \mathrm{Hom}_X(\mathcal{F}_2, \mathcal{A}) \xrightarrow{d^3} \mathrm{Hom}_X(\mathcal{F}_3, \mathcal{A})$$

In many cases (when the coefficient module $\mathcal{A}$ is a homogeneous trivial, dihedral or Alexander module, for example) the first cohomology is independent of the choice of fixed element of X, and we may drop the z subscript. In particular, when $\mathcal{A}$ is a trivial module, the group $\mathrm{PDer}_z(X, \mathcal{A})$ is itself trivial.

With this discussion in mind, we now define the (z–)**standard complex** of a rack X to be:

$$\mathbf{F}_z \quad = \quad \ldots \xrightarrow{d_2} \mathcal{F}_1 \xrightarrow{d_1^z} \mathcal{F}_0 \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0 \tag{3.1}$$

where $\mathcal{F}_n$ is the free X–module on the trunk map $\mathcal{S}_n \colon \mathsf{D}(X) \to \mathsf{Set}$ given by

$$(\mathcal{S}_n)_x = \{(x_1, \ldots, x_n) : x_1, \ldots, x_n \in X; x_1^{x_2 \ldots x_n} = x\}$$

for all $x \in X$, and the boundary maps are given by

$$d_n = \sum_{i=1}^{n} (-1)^{i+1} d_n^i$$

where

$$(d_n^i)_{x_1^{x_2 \cdots x_n}}(x_1, \ldots, x_n) = \\ \phi_{x_1^{x_2 \cdots x_{\widehat{i+1}} \cdots x_n}, x_{i+1}^{x_{i+2} \cdots x_n}}(x_1, \ldots, \widehat{x_{i+1}}, \ldots, x_n) \\ - (x_1^{x_{i+1}}, \ldots, x_i^{x_{i+1}}, x_{i+2}, \ldots, x_n) \tag{3.2}$$

for $1 \leqslant i \leqslant n-1$,

$$(d_n^n)_{x_1^{x_2 \cdots x_n}}(x_1, \ldots, x_n) = (-1)^{n+1} \psi_{x_2^{x_3 \cdots x_n}, x_1^{x_3 \cdots x_n}}(x_2, \ldots, x_n) \tag{3.3}$$

for $n > 1$, and

$$(d_1^z)_x \colon (x) \mapsto \lambda_{z, x^{\bar{z}}}(*) \tag{3.4}$$

where $\widehat{\cdot}$ denotes elision of the marked symbol, and where z is an arbitrarily chosen, fixed element of X. In particular, both $\mathcal{F}_1$ and $\mathcal{F}_0$ are isomorphic to the rack algebra $\mathbb{Z}X$. We set the map $d_0 \colon \mathcal{F}_0 \to \mathbb{Z}$ to be the augmentation map $\varepsilon \colon \mathbb{Z}X \to \mathbb{Z}$.

Lemma 3.4

If $1 \leqslant i < j < n$ *then*

$$d_{n-1}^i d_n^j = d_{n-1}^{j-1} d_n^i,$$

if $1 \leqslant i < n-1$ *then*

$$d_{n-1}^i d_n^n = -d_{n-1}^{n-1} d_n^{i+1},$$

and

$$d_{n-1}^{n-1} d_n^n = (-1)^n d_{n-1}^{n-1} d_n^1$$

Proof

If $1 \leqslant i < j < n$,

$$\begin{aligned}
&(d_{n-1}^i)_{x_1^{x_2 \cdots x_n}} (d_n^j)_{x_1^{x_2 \cdots x_n}} (x_1, \ldots, x_n) = \\
&\qquad \phi_{x_1^{x_2 \cdots \widehat{x_{j+1}} \cdots x_n}, x_{j+1}^{x_{j+2} \cdots x_n}} \phi_{x_1^{x_2 \cdots x_{\widehat{i+1}} \cdots \widehat{x_{j+1}} \cdots x_n}, x_{i+1}^{x_{i+2} \cdots x_n}} \\
&\qquad (x_1, \ldots, \widehat{x_{i+1}}, \ldots, \widehat{x_{j+1}}, \ldots, x_n) \\
&\qquad - \phi_{x_1^{x_2 \cdots \widehat{x_{j+1}} \cdots x_n}, x_{j+1}^{x_{j+2} \cdots x_n}} (x_1^{x_{i+1}}, \ldots, x_i^{x_{i+1}}, x_{i+2}, \ldots, x_n) \\
&\qquad - \phi_{x_1^{x_2 \cdots x_{\widehat{i+1}} \cdots x_n}, x_{i+1}^{x_{i+2} \cdots x_n}} (x_1^{x_{j+1}}, \ldots, x_j^{x_{j+1}}, x_{j+2}, \ldots, x_n) \\
&\qquad + (x_1^{x_{i+1} x_{j+1}}, \ldots, x_i^{x_{i+1} x_{j+1}}, x_{i+2}^{x_{j+1}}, \ldots, x_j^{x_{j+1}}, x_{j+2}, \ldots, x_n) \\
&\qquad\qquad = (d_{n-1}^{j-1})_{x_1^{x_2 \cdots x_n}} (d_n^i)_{x_1^{x_2 \cdots x_n}} (x_1, \ldots, x_n).
\end{aligned}$$

If $1 \leqslant i < n-1$,

$$\begin{aligned}
&(d_{n-1}^i)_{x_1^{x_2 \cdots x_n}} (d_n^n)_{x_1^{x_2 \cdots x_n}} (x_1, \ldots, x_n) = \\
&\quad = (-1)^{n+1} \psi_{x_2^{x_3 \cdots x_n}, x_1^{x_3 \cdots x_n}} \phi_{x_2^{x_3 \cdots x_{\widehat{i+1}} \cdots x_n}, x_{i+1}^{x_{i+2} \cdots x_n}} (x_2, \ldots, \widehat{x_{i+1}}, \ldots, x_n) \\
&\qquad - (-1)^{n+1} \psi_{x_2^{x_3 \cdots x_n}, x_1^{x_3 \cdots x_n}} (x_2^{x_{i+1}}, \ldots, x_i^{x_{i+1}}, x_{i+2}, \ldots, x_n) \\
&\qquad\qquad = -(d_{n-1}^{n-1})_{x_1^{x_2 \cdots x_n}} (d_n^{i+1})_{x_1^{x_2 \cdots x_n}} (x_1, \ldots, x_n)
\end{aligned}$$

Finally,

$$\begin{aligned}
&(d_{n-1}^{n-1})_{x_1^{x_2 \cdots x_n}} (d_n^n)_{x_1^{x_2 \cdots x_n}} (x_1, \ldots, x_n) = \\
&\qquad (-1)^{2n+1} \psi_{x_2^{x_3 \cdots x_n}, x_1^{x_3 \cdots x_n}} \psi_{x_3^{x_4 \cdots x_n}, x_2^{x_4 \cdots x_n}} (x_3, \ldots, x_n) = \\
&\qquad (-1)^{2n} \phi_{x_1^{x_3 \cdots x_n}, x_2^{x_3 \cdots x_n}} \psi_{x_3^{x_4 \cdots x_n}, x_1^{x_4 \cdots x_n}} (x_3, \ldots, x_n) \\
&\qquad - (-1)^{2n} \psi_{x_3^{x_4 \cdots x_n}, x_1^{x_2 x_4 \cdots x_n}} (x_3, \ldots, x_n) = \\
&\qquad\qquad (-1)^n (d_{n-1}^{n-1})_{x_1^{x_2 \cdots x_n}} (d_n^1)_{x_1^{x_2 \cdots x_n}} (x_1, \ldots, x_n)
\end{aligned}$$

□

Theorem 3.5

The standard complex is indeed a chain complex of X–modules.

Proof

Using the above lemma, we find that

$$
\begin{aligned}
d_{n-1}d_n = \sum_{i=1}^{n-1}\sum_{j=1}^{n}(-1)^{i+j}d_{n-1}^i d_n^j = \\
\sum_{i=1}^{n-2}\sum_{j=1}^{i}(-1)^{i+j}d_{n-1}^i d_n^j + \sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}(-1)^{i+j}d_{n-1}^{j-1}d_n^i \\
+\sum_{i=1}^{n-2}(-1)^{i+n}d_{n-1}^i d_n^n + \sum_{i=1}^{n-2}(-1)^{i+n}d_{n-1}^{n-1}d_n^{i+1} \\
+(-1)^n d_{n-1}^{n-1}d_n^n + (-1)^{2n-1}d_{n-1}^{n-1}d_n^1 = 0
\end{aligned}
$$

□

This chain complex is a generalisation of the original definition of rack homology [17] [11], 'twisted' rack homology [10], and also generalises the theory of Andruskiewitsch and Graña [1] to the case where the coefficient module is heterogeneous.

Let $\mathcal{J}_zX$ be the free X–module generated by the trunk map $\mathcal{J}_z\colon \mathsf{D}(X) \longrightarrow \mathsf{Set}$ where $(\mathcal{J}_zX)_x = \{\lambda_{z,x^{\bar{z}}}(*)\}$.

Proposition 3.6

The module $\mathcal{J}_zX$ represents $\mathrm{Der}(X,-)$ in the sense that there is an isomorphism

$$\mathrm{Der}(X,\mathcal{A}) \cong \mathrm{Hom}_X(\mathcal{J}_zX,\mathcal{A})$$

for any X–module $\mathcal{A} = (A,\phi,\psi)$.

Proof

Given an arbitrary derivation $d\colon X \longrightarrow \mathcal{A}$, we can construct a unique X–map $\nu(d)\colon \mathcal{J}_zX \longrightarrow \mathcal{A}$ by setting $\nu(d)_x(\lambda_{z,x^{\bar{z}}}(*)) = d(x) \in A_x$. This is an X–map since

$$\nu(d)_{x^y}(\rho_{x,y}\lambda_{z,x^{\bar{z}}}(*)) = d_{x^y}\rho_{x,y}(x) = \phi_{x,y}d(x) = \phi_{x,y}\nu(d)_x(\lambda_{z,x^{\bar{z}}}(*))$$

and

$$\nu(d)_{x^y}(\lambda_{y,x}\lambda_{z,y^{\bar{z}}}(*)) = d_{x^y}\lambda_{y,x}(y) = \psi_{y,x}d(y) = \psi_{y,x}\nu(d)_y(\lambda_{z,y^{\bar{z}}}(*))$$

Conversely, given an X–map $f\colon \mathcal{J}_z X \longrightarrow \mathcal{A}$ we may construct a unique derivation $\eta(f)\colon X \longrightarrow \mathcal{A}$ by setting $\eta(f)(x) = f_x(\lambda_{z,x^{\bar{z}}}(*)) \in A_x$. This is a derivation, since

$$\begin{aligned}\eta(f)(x^y) &= f_{x^y}(\lambda_{z,x^{y\bar{z}}}(*)) = f_{x^y}(\rho_{x,y}\lambda_{z,x^{\bar{z}}}(*) + \lambda_{y,x}\lambda_{z,y^{\bar{z}}}(*)) \\ &= \phi_{x,y}f_x(\lambda_{z,x^{\bar{z}}}(*)) + \psi_{y,x}f_y(\lambda_{z,y^{\bar{z}}}(*)) = \phi_{x,y}\eta(f)(x) + \psi_{y,x}\eta(f)(y)\end{aligned}$$

It is evident that $\eta = \nu^{-1}$, and hence $\mathrm{Der}(X, \mathcal{A}) \cong \mathrm{Hom}_X(\mathcal{J}_z X, \mathcal{A})$ □

3.2 Projective modules

We begin by examining the exactness of the functor $\mathrm{Hom}_X(\mathcal{M}, -)$:

Proposition 3.7

The functor $\mathrm{Hom}_X(\mathcal{M}, -)$ *is left exact and covariant for any* X*–module* $\mathcal{M}$.

Proof

Given an exact sequence

$$0 \longrightarrow \mathcal{A} \xrightarrow{\alpha} \mathcal{B} \xrightarrow{\beta} \mathcal{C}$$

we must show that

$$0 \longrightarrow \mathrm{Hom}_X(\mathcal{M}, \mathcal{A}) \xrightarrow{\alpha_*} \mathrm{Hom}_X(\mathcal{M}, \mathcal{B}) \xrightarrow{\beta_*} \mathrm{Hom}_X(\mathcal{M}, \mathcal{C})$$

is exact, where $\alpha_*(f) = \alpha \circ f$ and $\beta_*(g) = \beta \circ g$.

Given an X–map $f\colon \mathcal{M} \longrightarrow \mathcal{A}$ such that $\alpha_*(f) = 0$, then $\alpha_x f_x(m) = 0$ for all $m \in M_x$. Since α_x is injective, $f_x(m) = 0$ and hence $\ker \alpha_*$ is trivial, and so α_* is a monomorphism of trivial X–modules.

Suppose $g \in \mathrm{im}\, \alpha_*$, so $g = \alpha f$ for some $f\colon \mathrm{Hom}_X(\mathcal{M}, \mathcal{A})$. Then $\beta_*(g) = \beta g = \beta\alpha f = 0$, since $\beta\alpha = 0$, and hence $\mathrm{im}\, \alpha_* \subset \ker \beta_*$.

Finally, consider an X–map $g\colon \mathcal{M} \longrightarrow \mathcal{B}$ such that $\beta_*(g) = 0$; that is, $\beta_x g_x(m) = 0$ for all $m \in M_x$. Then $g_x(m) \in \ker \beta_x = \operatorname{im} \alpha_x$, so there is a unique $a \in A_x$ such that $\alpha_x(a) = g_x(m)$ (since α_x is injective). Then define $f\colon \mathcal{M} \longrightarrow \mathcal{A}$ by $f_x(m) = a = \alpha_x^{-1} g_x(m)$. Then $\alpha f = g$, and so $\ker \beta_* \subset \operatorname{im} \alpha_*$. □

The functor $\operatorname{Hom}_X(\mathcal{M}, -)$ is thus left exact for any X–module $\mathcal{M}$, preserving injectivity of morphisms. It is, however, not necessarily right exact since, for example, applying $\operatorname{Hom}_X(\mathbb{Z}_2, -)$ to the exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\alpha} \mathbb{Q} \xrightarrow{\beta} \mathbb{Q}/\mathbb{Z} \longrightarrow 0$$

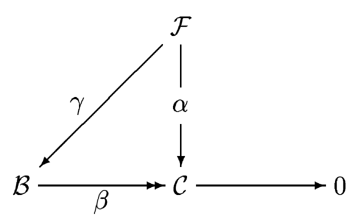
yields the sequence

$$0 \longrightarrow \operatorname{Hom}_X(\mathbb{Z}_2, \mathbb{Z}) \xrightarrow{\alpha_*} \operatorname{Hom}_X(\mathbb{Z}_2, \mathbb{Q}) \xrightarrow{\beta_*} \operatorname{Hom}_X(\mathbb{Z}_2, \mathbb{Q}/\mathbb{Z}) \longrightarrow 0.$$

But $\operatorname{Hom}_X(\mathbb{Z}_2, \mathbb{Q})$ is trivial, and $\operatorname{Hom}_X(\mathbb{Z}_2, \mathbb{Q}/\mathbb{Z})$ isn't, so the map β_* can't be surjective.

Proposition 3.8

Let $\mathcal{B}, \mathcal{C}$, and $\mathcal{F}$ be X–modules, with $\mathcal{F}$ free. If $\beta\colon \mathcal{B} \longrightarrow \mathcal{C}$ is surjective, and $\alpha\colon \mathcal{F} \longrightarrow \mathcal{C}$ any X–map, then there exists an X–map $\gamma\colon \mathcal{F} \longrightarrow \mathcal{B}$ such that $\alpha = \beta\gamma$:



Proof

Let $\mathcal{S}\colon \mathsf{D}(X) \longrightarrow \mathsf{Set}$ be a basis for $\mathcal{F}$. Since β is surjective, then for any $x \in X$, any i in some indexing set I_x, and $s_i \in S_x$, $\alpha_x(s_i) = \beta_x(b_i)$ for some $b_i \in B_x$.

There is a function $f_x\colon S_x \longrightarrow B_x$ with $f_x(s_i) = b_i$, for all $i \in I_x$, and hence an X–map $\gamma\colon \mathcal{F} \longrightarrow \mathcal{B}$ with $\gamma_x(s_i) = f_x(s_i)$ for all $i \in I_x$ and $x \in X$.

Finally, $\beta_x \gamma_x(s_i) = \beta_x f_x(s_i) = \beta_x(b_i) = \alpha_x(s_i)$, and hence $\alpha = \beta\gamma$. □

Corollary 3.9

If $\mathcal{F}$ is a free X–module, then $\operatorname{Hom}_X(\mathcal{F}, -)$ is exact.

Proof

Since $\mathrm{Hom}_X(\mathcal{F}, -)$ is left exact, then it is sufficient to show that it preserves surjectivity. If $\mathcal{B} \xrightarrow{\beta} \mathcal{C} \longrightarrow 0$ is exact, then by the previous proposition, the sequence $\mathrm{Hom}_X(\mathcal{F}, \mathcal{B}) \xrightarrow{\beta_*} \mathrm{Hom}_X(\mathcal{F}, \mathcal{C}) \longrightarrow 0$ is exact. For, given $f\colon \mathcal{F} \longrightarrow \mathcal{C}$, then there is an X–map $g\colon \mathcal{F} \longrightarrow \mathcal{B}$ such that $f = \beta_*(g) = \beta \circ g$. Hence, the induced map β_* is surjective. □

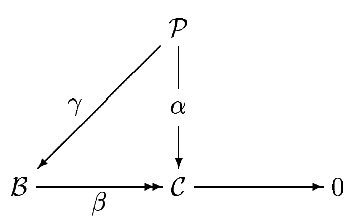
Proposition 3.10

Every X–module is a quotient of a free module.

Proof

Given an X–module $\mathcal{A}$, let $U\mathcal{A}$ denote the underlying trunk map $\mathsf{D}(X) \longrightarrow \mathsf{Set}$. Let f be the identity map $f = \mathrm{Id}\colon U\mathcal{A} \longrightarrow U\mathcal{A}$. By the definition of a free X–module, there is an X–map $\tilde{f}\colon F(U\mathcal{A}) \longrightarrow \mathcal{A}$ extending f. But $\tilde{f}$ is surjective, because f is. Hence $\mathcal{A}$ is a quotient of the X–module $F(U\mathcal{A})$. □

A **projective** X–module is one which satisfies the hypotheses of proposition 3.8: An X–module $\mathcal{P}$ such that for any two other X–modules $\mathcal{B}$ and $\mathcal{C}$, and X–maps $\beta\colon \mathcal{B} \longrightarrow \mathcal{C}$ and $\alpha\colon \mathcal{P} \longrightarrow \mathcal{C}$, with β surjective, there exists an X–map $\gamma\colon \mathcal{P} \longrightarrow \mathcal{C}$ such that $\alpha = \beta\gamma$.

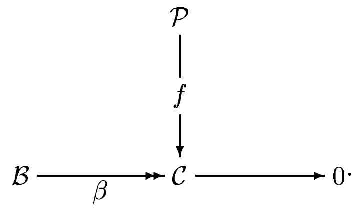


Proposition 3.11

An X–module $\mathcal{P}$ is projective iff the functor $\mathrm{Hom}_X(\mathcal{P}, -)$ is exact.

Proof

If $\mathcal{P}$ is projective, then the proof of corollary 3.9 shows that $\mathrm{Hom}_X(\mathcal{P}, -)$ is exact. If, conversely, $\mathrm{Hom}_X(\mathcal{P}, -)$ is exact, then consider the diagram



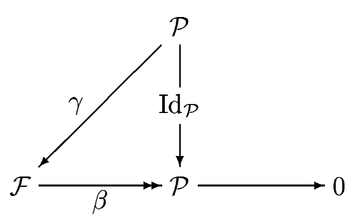
Since the map $\beta_*\colon \mathrm{Hom}_X(\mathcal{P}, \mathcal{B}) \longrightarrow \mathrm{Hom}_X(\mathcal{P}, \mathcal{C})$ is surjective, there is an X–map $g\colon \mathcal{P} \longrightarrow \mathcal{B}$ such that $f = \beta_*(g) = \beta \circ g$, which is the requirement for $\mathcal{P}$ to be projective. □

Proposition 3.12

An X–module $\mathcal{P}$ is projective iff it is a summand of a free module. Furthermore, any summand of a projective module is projective.

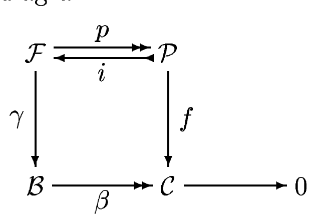
Proof

By proposition 3.10 there is a free module $\mathcal{F}$ which maps surjectively onto $\mathcal{P}$. We may thus construct the following diagram:



The map γ exists (since $\mathcal{P}$ is projective) and is injective (since its composition with the epimorphism β is the identity map). Any element $p \in P_x$ may be written as $\gamma_x\beta_x(p) + (p - \gamma_x\beta_x(p))$, the first term of which is in the image of γ_x, and the second in $\ker \beta_x$. The intersection in $\mathcal{F}$ of $\gamma(\mathcal{P})$ and $\ker \beta$ is trivial, and so $\mathcal{F} \cong \ker \beta \oplus \gamma(\mathcal{P})$ where $\gamma(\mathcal{P}) \cong \mathcal{P}$.

Conversely, consider the diagram



where $pi = \mathrm{Id}_{\mathcal{P}}$. Since $\mathcal{F}$ is projective, there is a map γ making the diagram commute, and so we may construct a map $g = \gamma i\colon \mathcal{P} \longrightarrow \mathcal{B}$, such that the composition $\beta g = \beta\gamma i = fpi = f$. Thus $\mathcal{P}$ is projective. □

A **projective resolution** of a module $\mathcal{A}$ is an exact sequence

$$\ldots \longrightarrow \mathcal{P}_n \xrightarrow{d_n} \mathcal{P}_{n-1} \longrightarrow \ldots \longrightarrow \mathcal{P}_1 \xrightarrow{d_1} \mathcal{P}_0 \xrightarrow{\varepsilon} \mathcal{A} \longrightarrow 0$$

such that each module $\mathcal{P}_i$ is projective.

An abelian category A is said to have **enough projectives** if there exists a projective resolution for any object in A.

Proposition 3.13

The category RMod_X *has enough projectives.*

Proof

By proposition 3.10, for any X–module $\mathcal{A}$, there is a free module $\mathcal{F}_0$ and a short exact sequence

$$0 \longrightarrow \mathcal{R}_0 \xrightarrow{i_0} \mathcal{F}_0 \xrightarrow{\varepsilon} \mathcal{A} \longrightarrow 0$$

where $\mathcal{R}_0$ is the kernel of $\varepsilon\colon \mathcal{F}_0 \longrightarrow \mathcal{A}$. Similarly, there is a free module $\mathcal{F}_1$ and an exact sequence

$$0 \longrightarrow \mathcal{R}_1 \xrightarrow{i_1} \mathcal{F}_1 \xrightarrow{p_1} \mathcal{R}_0 \longrightarrow 0$$

and, by induction, exact sequences

$$0 \longrightarrow \mathcal{R}_n \xrightarrow{i_n} \mathcal{F}_n \xrightarrow{p_n} \mathcal{R}_{n-1} \longrightarrow 0$$

for all $n > 0$. We may assemble these into a long exact sequence

$$\ldots \mathcal{F}_3 \xrightarrow{d_3} \mathcal{F}_2 \xrightarrow{d_2} \mathcal{F}_1 \xrightarrow{d_1} \mathcal{F}_0 \xrightarrow{\varepsilon} \mathcal{A} \longrightarrow 0$$

by defining $d_n = i_{n-1}p_n\colon \mathcal{F}_n \longrightarrow \mathcal{F}_{n-1}$, which gives the required free (and hence projective) resolution, □

3.3 Injective modules

We now seek to dualise the notion of a projective module and define a class of X–modules related to the exactness of the functor $\mathrm{Hom}_X(-, \mathcal{M})$

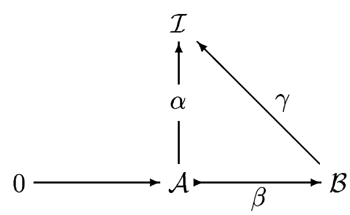
Proposition 3.14

The functor $\mathrm{Hom}_X(-, \mathcal{M})$ *is left exact and contravariant for any X–module $\mathcal{M}$.*

Proof

This proof is dual to that of proposition 3.7. □

An X–module $\mathcal{I}$ is **injective** if, for any two other X–modules $\mathcal{A}$ and $\mathcal{B}$, any X–map $\alpha\colon \mathcal{A} \longrightarrow \mathcal{I}$, and any injective X–map $\beta\colon \mathcal{A} \longrightarrow \mathcal{B}$, there exists an X–map $\gamma\colon \mathcal{B} \longrightarrow \mathcal{I}$ such that $\gamma\beta = \alpha$:

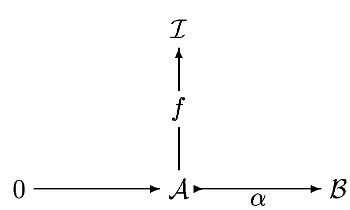


Proposition 3.15

An X–module $\mathcal{I}$ is injective iff the functor $\mathrm{Hom}_X(-, \mathcal{I})$ *is exact.*

Proof

If $\mathcal{I}$ is injective, then to show exactness of $\mathrm{Hom}_X(-, \mathcal{I})$ we need only show that α^* is surjective in the diagram



That is, that there exists an X–map $g\colon \mathcal{B} \longrightarrow \mathcal{M}$ with $\alpha^*(g) = f$. But $\alpha^*(g) = g\alpha$ and so g exists by the injectivity of $\mathcal{I}$. The converse follows almost immediately, since exactness of $\mathrm{Hom}_X(-, \mathcal{I})$ implies the surjectivity of α^* which in turn implies the injectivity of $\mathcal{I}$. □

An **injective resolution** of a module $\mathcal{A}$ is an exact sequence

$$0 \longrightarrow \mathcal{A} \xrightarrow{\varepsilon} \mathcal{I}_0 \xrightarrow{d_0} \mathcal{I}_1 \longrightarrow \ldots \longrightarrow \mathcal{I}_n \xrightarrow{d_n} \mathcal{I}_{n+1} \longrightarrow \ldots$$

such that each module $\mathcal{I}_i$ is injective.

An abelian category A has **enough injectives** if there exists an injective resolution for any object in A.

3.4 Flat modules

The previous discussions on projective and injective modules arose from consideration of the exactness of the functors $\mathrm{Hom}_X(-, \mathcal{M})$ and $\mathrm{Hom}_X(\mathcal{M}, -)$. We now turn our attention to the exactness of the covariant functors $- \otimes_X \mathcal{M}$ and $\mathcal{M} \otimes_X -$.

These two questions are effectively the same, since $\mathcal{A} \otimes_X \mathcal{B} \cong \mathcal{B} \otimes_{X^*} \mathcal{A}$ for any X–modules $\mathcal{A}$ and $\mathcal{B}$, by the isomorphism $a \otimes b \mapsto b \otimes a$.

Proposition 3.16

The functor $- \otimes_X \mathcal{M}$ is right exact and covariant for any X–module $\mathcal{M}$.

Proof

If

$$\mathcal{A} \xrightarrow{\alpha} \mathcal{B} \xrightarrow{\beta} \mathcal{C} \longrightarrow 0$$

is exact, then we must show that

$$\mathcal{A} \otimes_X \mathcal{M} \xrightarrow{\alpha \otimes_X \mathrm{Id}} \mathcal{B} \otimes_X \mathcal{M} \xrightarrow{\beta \otimes_X \mathrm{Id}} \mathcal{C} \otimes_X \mathcal{M} \longrightarrow 0$$

is too.

Firstly, $(\beta \otimes_X \mathrm{Id})(\alpha \otimes_X \mathrm{Id}) = \beta\alpha \otimes_X \mathrm{Id} = 0 \otimes_X \mathrm{Id} = 0$, and so $\mathrm{im}(\alpha \otimes_X \mathrm{Id}) \subseteq \ker(\beta \otimes_X \mathrm{Id})$.

Secondly, $\beta \otimes_X \mathrm{Id}$ induces a map $\beta' \colon (\mathcal{B} \otimes_X \mathcal{M}) / \mathrm{im}(\alpha \otimes_X \mathrm{Id})$ defined such that $\beta'_x([b_x \otimes_X m_x]) = \beta_x(b_x) \otimes_X m_x$, so that $\beta'\nu = \beta \otimes_X \mathrm{Id}$ where ν is the canonical natural map $\nu \colon \mathcal{B} \otimes_X \mathcal{M} \to (\mathcal{B} \otimes_X \mathcal{M}) / \mathrm{im}(\alpha \otimes_X \mathrm{Id})$.

Consider the map $f\colon \mathcal{M} \times \mathcal{C} \longrightarrow (\mathcal{B} \otimes_X \mathcal{M})/\operatorname{im}(\alpha \otimes_X \mathrm{Id})$ such that

$$f_x\colon (m_x, c_x) \mapsto [m_x \otimes_X b_x]$$

where $\beta_x(b_x) = c_x$. This map is well-defined, and since f is X–biadditive there is an X–map $f'\colon \mathcal{C} \otimes_X \mathcal{M} \longrightarrow (\mathcal{B} \otimes_X \mathcal{M})/\operatorname{im}(\alpha \otimes_X \mathrm{Id})$ with $f'_x(c_x \otimes_X m_x) = [b_x \otimes_X m_x]$, which is the inverse of β'. Then

$$\ker(\beta \otimes_X \mathrm{Id}) = \ker \beta'\nu = \ker \nu = \operatorname{im}(\alpha \otimes_X \mathrm{Id})$$

Finally, let $\sum_i c_i \otimes_X m_i \in (\mathcal{C} \otimes_X \mathcal{M})_x$. Since β is surjective, there are $b_i \in B_x$ with $\beta_x(b_i) = c_i$ for all i. Then $(\beta \otimes_X \mathrm{Id})_x(\sum_i b_i \otimes_X m_i) = \sum_i c_i \otimes_X m_i$. and so $\beta \otimes_X \mathrm{Id}$ is surjective.

Covariance follows by the observation that $(g \otimes_X \mathrm{Id})(f \otimes_X \mathrm{Id}) = (gf) \otimes_X \mathrm{Id}$ for any X–maps $f\colon \mathcal{A} \longrightarrow \mathcal{B}$ and $g\colon \mathcal{B} \longrightarrow \mathcal{C}$. □

An X–module $\mathcal{M}$ is **flat** if the functor $- \otimes_X \mathcal{M}$ is exact. Since $- \otimes_X \mathcal{M}$ is always right exact (preserves injectivity) for any $\mathcal{M}$, flatness is equivalent to the requirement that $- \otimes_X \mathcal{M}$ be left exact.

Proposition 3.17

Let $\mathcal{V}_x$ be the trunk map $\mathsf{D}(X) \longrightarrow \mathsf{Set}$ where

$$\mathcal{V}_x(y) = \begin{cases} \{(*)\} & \text{if } x = y \\ \emptyset & \text{otherwise} \end{cases}$$

for some fixed $x \in X$. Then the free X–module $F\mathcal{V}_x$ is flat.

Proof

Let $f\colon \mathcal{A} \longrightarrow \mathcal{B}$ be an X–monomorphism. Then the map $f \otimes_X F\mathcal{V}_x$ is simply $f_x\colon A_x \longrightarrow B_x$, which is a monomorphism of abelian groups. Hence $- \otimes_X F\mathcal{V}_x$ preserves injectivity, and so $F\mathcal{V}_x$ is flat. □

Proposition 3.18

A direct sum of X–modules is flat iff each of the summands are flat.

Proof

Let $\{\mathcal{A}_s\}$ and $\{\mathcal{B}_s\}$ be collections of X–modules indexed by some set S. Then given a family $\{f_s\colon \mathcal{A}_s \to \mathcal{B}_s\}$ of X–maps, it follows that there is a unique X–map $(\oplus_S f_s)\colon \oplus_S \mathcal{A}_s \to \oplus_S \mathcal{B}_s$, with $(\oplus_S f_s)$ a monomorphism iff each map f_s is an X–monomorphism.

Now let $f\colon \mathcal{A} \to \mathcal{B}$ be an X–monomorphism and consider the commutative diagram

$$\begin{array}{ccc} \mathcal{A} \otimes_X (\bigoplus_S \mathcal{M}_s) & \xrightarrow{f \otimes_X \mathrm{Id}} & \mathcal{B} \otimes_X (\bigoplus_S \mathcal{M}_s) \\ \downarrow & & \downarrow \\ \bigoplus_S (\mathcal{A} \otimes_X \mathcal{M}_s) & \xrightarrow[\bigoplus_S (f \otimes_X \mathrm{Id}_{\mathcal{M}_s})]{} & \bigoplus_S (\mathcal{B} \otimes_X \mathcal{M}_s) \end{array}$$

where the vertical maps are the canonical isomorphisms. Then by the initial remarks, $f \otimes_X \mathrm{Id}$ is injective (and hence $\oplus_S \mathcal{M}_s$ is flat) iff each $f \otimes_X \mathrm{Id}_{\mathcal{M}_s}$ is injective (and hence each summand of $\oplus_S \mathcal{M}_s$ is flat). □

Proposition 3.19

All projective modules are flat.

Proof

As discussed in section 2.3, any free X–module may be regarded as a direct sum of copies of free modules of the form $F\mathcal{V}_x$, which are all themselves flat. As just shown, any direct sum of flat modules is also flat, so all free X–modules are flat. Any projective module, though, may be regarded as a summand of a free module, and since all summands of flat modules are also flat, it follows that all projective X–modules are flat. □

A **flat resolution** of a module A is an exact sequence

$$\ldots \longrightarrow \mathcal{M}_n \xrightarrow{d_n} \mathcal{M}_{n-1} \longrightarrow \ldots \longrightarrow \mathcal{M}_1 \xrightarrow{d_1} \mathcal{M}_0 \xrightarrow{\varepsilon} \mathcal{A} \longrightarrow 0$$

such that each module $\mathcal{M}_i$ is flat. An abelian category A is said to have **enough flats** if there exists a projective resolution for any object in A.

Corollary 3.20

The category RMod_X *has enough flats.*

3.5 The bar resolution

Adapting the categorical construction, described at the beginning of the chapter, to the question of rack homology proves to be a slightly more complicated endeavour than that for groups, since the rack algebra $\mathbb{Z}X$ is a more complicated object than the group ring $\mathbb{Z}G$.

First of all, given a fixed rack X, we consider the 'underlying trivial X–module' functor $U\colon \mathsf{RMod}_X \longrightarrow \mathsf{TMod}_X$. This forms a relative abelian category, since TMod_X is an abelian category if RMod_X is.

Furthermore, we can make a resolvent pair of categories, since the 'underlying trivial X–module' functor $U\colon \mathsf{RMod}_X \longrightarrow \mathsf{TMod}_X$ has a left adjoint given by $F = \mathbb{Z}X \otimes -\colon \mathsf{TMod}_X \longrightarrow \mathsf{RMod}_X$, and the corresponding adjunction has counit $\varepsilon\colon \mathrm{Id}_{\mathsf{TMod}_X} \longrightarrow UF$ given by $(\varepsilon_M)_x(m) = (*) \otimes m \in (\mathbb{Z}X \otimes \mathcal{M})_x$.

The sequence

$$0 \longrightarrow \mathbb{Z} \hookrightarrow \mathbb{Z}X \longrightarrow \mathbb{Z}X/\mathbb{Z} \longrightarrow 0$$

is exact, and hence so is

$$\mathcal{M} \xrightarrow{\varepsilon_{\mathcal{M}}} \mathbb{Z}X \otimes \mathcal{M} \xrightarrow{\pi_{\mathcal{M}}} (\mathbb{Z}X/\mathbb{Z}) \otimes \mathcal{M} \longrightarrow 0$$

as a sequence of trivial X–modules. Thus, $R(\mathcal{M}) = (\mathbb{Z}X/\mathbb{Z}) \otimes \mathcal{M}$ and, setting C to be $\mathbb{Z}$ considered as a trivial homogeneous X–module, we obtain

$$R^n(\mathbb{Z}) = \otimes^n(\mathbb{Z}X/\mathbb{Z})$$

But $\mathcal{R} = \otimes^n(\mathbb{Z}X/\mathbb{Z})$ is the free trivial X–module with R_x the free abelian group generated by all symbols $[\chi_1|\ldots|\chi_n]$ where $\chi_i \in (\mathbb{Z}X)_x$ is not $(*)$.

The chain modules $\mathcal{B}_n = B_n(\mathbb{Z})$, then, are the X–modules $\mathbb{Z}X \otimes R^n(\mathbb{Z})$, and so $(\mathcal{B}_n)_x$ is the free abelian group generated by symbols $\chi[\chi_1|\ldots|\chi_n]$ where $\chi, \chi_1, \ldots, \chi_n \in (\mathbb{Z}X)_x$ and, additionally, χ might be $(*)$. This object has an obvious free X–module structure derived from the multiplicative structure in $\mathbb{Z}X$.

As before, there are maps $s_n\colon \mathcal{B}_n \longrightarrow \mathcal{B}_{n+1}$ defined as

$$(s_n)_x(\chi[\chi_1|\dots|\chi_n]) = [\chi|\chi_1|\dots|\chi_n]$$

which is zero if $\chi = (*)$.

We may now recursively define the boundary operators $\partial_n\colon \mathcal{B}_n \longrightarrow \mathcal{B}_{n-1}$. Set $s_{-1}\colon \mathbb{Z} \longrightarrow \mathcal{B}_0 = \mathbb{Z}X$ to be the X–map such that

$$(s_{-1})_x(1) = [\,].$$

This satisfies the requirement $\varepsilon s_{-1} = \mathrm{Id}_{\mathbb{Z}}$.

Next we define the map $\partial_1\colon \mathcal{B}_1 \longrightarrow \mathbb{Z}X$ using the formula $\partial_1 s_0 = \mathrm{Id}_{\mathbb{Z}X} - s_{-1}\varepsilon$ to get:

$$(\partial_1)_x\colon \begin{cases} [\rho_{x^{\bar{u}},u}(*)] & \mapsto \quad \rho_{x^{\bar{u}},u}(*) - (*) \\ [\rho_{x^{\bar{u}},u}\lambda_{y,x^{\bar{u}\bar{y}}}(*)] & \mapsto \quad \rho_{x^{\bar{u}},u}\lambda_{y,x^{\bar{u}\bar{y}}}(*) \end{cases}$$

Observe that $\operatorname{im}\partial_1$ is the augmentation module $\mathcal{I}X$, which in turn is $\ker\varepsilon$.

The second boundary map $\partial_2\colon \mathcal{B}_2 \longrightarrow \mathcal{B}_1$ is calculated similarly, and we thus obtain

$$(\partial_2)_x\colon \begin{cases} [\rho_{x^{\bar{u}},u}(*)|\rho_{x^{\bar{u}},u}(*)] & \mapsto \quad \rho_{x^{\bar{u}},u}(*)[\rho_{x^{\bar{v}},v}(*)] \\ & \qquad -[\rho_{x^{\bar{v}\bar{u}},uv}(*)] + [\rho_{x^{\bar{u}},u}(*)] \\ [\rho_{x^{\bar{u}},u}(*)|\rho_{x^{\bar{v}},v}\lambda_{y,x^{\bar{v}\bar{y}}}(*)] & \mapsto \quad \rho_{x^{\bar{u}},u}[\rho_{x^{\bar{v}},v}\lambda_{y,x^{\bar{v}\bar{y}}}(*)] \\ & \qquad -[\rho_{x^{\bar{v}\bar{u}},v}\lambda_{y,x^{\bar{v}\bar{u}\bar{y}}}(*)] \\ [\rho_{x^{\bar{u}},u}\lambda_{y,x^{\bar{u}\bar{y}}}(*)|\rho_{x^{\bar{v}},v}(*)] & \mapsto \quad \rho_{x^{\bar{u}},u}\lambda_{y,x^{\bar{u}\bar{y}}}(*)[\rho_{x^{\bar{v}},v}(*)] \\ & \qquad -[\rho_{x^{\bar{v}\bar{u}},uv}\lambda_{y^{\bar{u}},x^{\bar{v}\bar{y}\bar{u}}}(*)] \\ & \qquad +[\rho_{x^{\bar{u}},u}\lambda_{y,x^{\bar{u}\bar{y}}}(*)] \\ [\rho_{x^{\bar{u}},u}\lambda_{y,x^{\bar{u}\bar{y}}}(*)|\rho_{x^{\bar{v}},v}\lambda_{z,x^{\bar{v}\bar{z}}}(*)] & \mapsto \quad \rho_{x^{\bar{u}},u}\lambda_{y,x^{\bar{u}\bar{y}}}(*)[\rho_{x^{\bar{v}},v}\lambda_{z,x^{\bar{v}\bar{z}}}(*)] \\ & \qquad -[\rho_{x^{\bar{v}\bar{u}},uv}\lambda_{z,x^{\bar{v}\bar{u}\bar{z}}}(*)] \\ & \qquad +[\rho_{x^{\bar{v}\bar{y}\bar{u}},uyv}\lambda_{z,x^{\bar{v}\bar{y}\bar{u}\bar{z}}}(*)] \end{cases}$$

Call this resolution the (**categorical**) (**normalised**) **bar resolution** of X. Even by $\mathcal{B}_2$ it bears little resemblance to the standard complex $\mathbf{F}$ constructed earlier in this chapter, though, and it is not immediately obvious whether or not the homology groups of $\mathbf{F}$ will be isomorphic to those derived from this resolution. In fact, as we shall see later, this is not in general the case: Chapter 6 provides a striking counterexample in the form of the trivial racks.

This presents us with a particularly curious situation. We saw in chapter 2 that the rack modules therein described are exactly the Beck modules over Rack. In chapter 1 we saw that extensions of racks by such objects are classified by the second cohomology group of the standard complex **F** described in the first section of this chapter. And yet the corresponding derived functors (constructed from projective resolutions of rack modules) give different results in at least some important cases.

Given an X–module $\mathcal{A} = (A, \phi, \psi)$, define the **module of invariants** or **fixed points** to be the X–submodule $\mathcal{A}^X$ of $\mathcal{A}$ such that

$$(\mathcal{A}^X)_x = \{a \in A_x : \phi_{x^{\bar{w}},w}(a) = a, \psi_{x,y}(a) = 0 \text{ for all } y \in X\}.$$

This is the largest submodule of $\mathcal{A}$ which is trivial under the action of X.

Proposition 3.21

Let $\mathbb{Z}$ denote the additive group of integers considered as a trivial homogeneous X–module. Then there is a natural isomorphism $\mathrm{Hom}_X(\mathbb{Z}, \mathcal{A}) \cong \mathcal{A}^X$.

Proof

Let $\nu\colon \mathrm{Hom}_X(\mathbb{Z}, \mathcal{A}) \longrightarrow \mathcal{A}^X; f \mapsto f(1)$. Then since $\mathbb{Z}$ is a trivial (homogeneous) X–module, the naturality of f implies

$$\begin{aligned}\phi_{x,y} f_x(1) &= f_{x^y}(1) \\ \psi_{y,x} f_y(1) &= 0\end{aligned}$$

and hence

$$\operatorname{im}\nu_x = \{f_x(1) \in A_x : \phi_{x,y} f_x(1) = f_{x^y}(1) = \mathrm{Id}_{x,x^y} f_x(1), \text{ and } \psi_{x,y} f_x(1) = 0 \text{ for all } y \in X\}$$

which is simply the definition of $(\mathcal{A}^X)_x$.

Conversely, for any $\{a_x \in (\mathcal{A}^X)_x : x \in X\}$ there is a unique collection of abelian group homomorphisms $f_x\colon \mathbb{Z} \longrightarrow (\mathcal{A}^X)_x$ such that $f_x(1) = a_x$. This correspondence gives a natural transformation $\mu\colon \mathcal{A}^X \longrightarrow \mathrm{Hom}_X(\mathbb{Z}, \mathcal{A})$ which is the inverse of ν. □

Dually, we define the **module of coinvariants** to be the X–module $\mathcal{A}_X$ where

$$(\mathcal{A}_X)_x = A_x / \langle \phi_{x^{\bar{w}},w}(a_x) - a_x, \psi_{y,x^{\bar{y}}}(a_y) \rangle$$

This is the largest quotient module of $\mathcal{A}$ which is trivial under the action of X. Both $-_X$ and $-^X$ are functors $\mathsf{RMod}_X \longrightarrow \mathsf{TMod}_X$. As noted above, $-^X = \mathrm{Hom}_X(\mathbb{Z}, -)$.

Theorem 3.22

The functor $-^X\colon \mathsf{RMod}_X \longrightarrow \mathsf{TMod}_X$ *is right adjoint to the inclusion functor* $\mathsf{TMod}_X \hookrightarrow \mathsf{RMod}_X$. *Dually,* $-_X$ *is left adjoint to this inclusion. Furthermore,* $-_X$ *is equivalent to* $\mathbb{Z} \otimes_X -$.

Proof

We begin by observing the equivalence of $-_X$ and $\mathbb{Z} \otimes_X -$. Given an X–module $\mathcal{A} = (A, \phi, \psi)$, note that $\mathcal{A}_X$ is the quotient module of $\mathcal{A}$ subject to relations $\phi_{x,y}(a) = a$ and $\psi_{y,x}(b) = 0$ for all $x, y \in X$, $a \in A_x$ and $b \in A_y$. This is isomorphic to $\mathbb{Z} \otimes_X \mathcal{A}$ by the map $a \mapsto 1 \otimes a$. The adjointness results follow immediately from the relevant adjointness results on Hom_X and $\otimes_X$. □

3.6 Derived functors

We now have all the ingredients for the definition of right- and left-derived functors for racks: An abelian category of 'modules' which has enough projectives, enough injectives, and enough flats, together with a normalised free resolution.

Traditionally, the derived functors of Hom and $\otimes$ are called, respectively, Ext and Tor. This places us in something of a quandary, given that as discussed earlier, they are unlikely to be as strongly related to extensions and torsion as the corresponding functors in the homology and cohomology of groups and other related objects.

More appropriate terminology having yet to come to light, and in accordance with tradition, we define

$$\mathrm{Ext}^n_X(\mathcal{A},\mathcal{B}) = (R^n\mathrm{Hom}_X(-,\mathcal{B}))(\mathcal{A}) = H^n(\mathrm{Hom}_X(\mathbf{P}_\mathcal{A},\mathcal{B})) \tag{3.5}$$

$$\mathrm{Tor}^X_n(\mathcal{A},\mathcal{B}) = (R^n - \otimes_X\mathcal{B})(\mathcal{A}) = H_n(\mathbf{P}_\mathcal{A}\otimes_X \mathcal{B}) \tag{3.6}$$

where $\mathbf{P}_\mathcal{A}$ is a deleted projective resolution of $\mathcal{A}$.

Given a rack X and an X-module $\mathcal{A}$, the homology and cohomology of X with coefficients in $\mathcal{A}$ are

$$H^n(X;\mathcal{A}) = \mathrm{Ext}^n(X,\mathcal{A}) = \mathrm{Ext}^n_X(\mathbb{Z},\mathcal{A}) \tag{3.7}$$

$$H_n(X;\mathcal{A}) = \mathrm{Tor}_n(X,\mathcal{A}) = \mathrm{Tor}^X_n(\mathbb{Z},\mathcal{A}) \tag{3.8}$$

This, again, introduces potential ambiguity and confusion. Where the context is not sufficient to discern which homology or cohomology theory (or which Ext or Tor) is meant, we shall use $\overline{H}_n, \overline{H}^n, \overline{\mathrm{Ext}}$, and $\overline{\mathrm{Tor}}$ for the derived functors, and $\widehat{H}_n, \widehat{H}^n, \widehat{\mathrm{Ext}}$, and $\widehat{\mathrm{Tor}}$ for the versions constructed from the standard complex.

3.7 Quandle homology and cohomology

We now wish to specialise our formulation of rack homology and cohomology to derive a new, similarly general homology and cohomology theory for quandles.

In this section X will be a quandle, the functors Hom_X and $\otimes_X$ should be understood to be defined on the relevant categories of quandle modules over X, and unless indicated to the contrary, the term 'X-module' will be synonymous with the term 'quandle module over X'.

The proofs of the following results are identical to the corresponding results in the category of rack modules:

Theorem 3.23

(i) $\mathrm{Hom}_X(\mathcal{M},-)$ *is a left exact covariant functor for any* X*-module* $\mathcal{M}$.

(ii) If $\mathcal{F}$ *is a free quandle module over* X*, then* $\mathrm{Hom}_X(\mathcal{F},-)$ *is exact.*

(iii) Every X*-module is a quotient of a free* X*-module.*

(iv) *An X-module $\mathcal{P}$ is projective iff* $\mathrm{Hom}_X(\mathcal{P}, -)$ *is exact.*

(v) *An X-module $\mathcal{P}$ is projective iff it is a summand of a free module, and any summand of a projective module is itself projective.*

(vi) QMod_X *has enough projectives.*

(vii) $\mathrm{Hom}_X(-, \mathcal{M})$ *is a left exact contravariant functor for any X-module $\mathcal{M}$.*

(viii) *An X-module $\mathcal{I}$ is injective iff* $\mathrm{Hom}_X(-, \mathcal{I})$ *is exact.*

(ix) $- \otimes_X \mathcal{M}$ *is a right exact covariant functor for any X-module $\mathcal{M}$.*

(x) *The quandle algebra $\mathbb{Z}X$ is flat.*

(xi) *A direct sum of X-modules is flat iff each summand is flat.*

(xii) *All projectives are flat.*

(xiii) QMod_X *has enough flats.*

We obtain a similar standard complex $\mathbf{F}$ for quandle homology and cohomology. Let $\mathcal{F}_n$ be the free quandle module on the trunk map $\mathcal{S}_n\colon\ \mathsf{D}(X) \longrightarrow \mathsf{Set}$ defined by

$$(\mathcal{S}_n)_x = \{(x_1, \ldots, x_n) : x_1 \in [x]; x_2, \ldots, x_n \in X; x_i \neq x_{i+1} \text{ for } 1 \leqslant i \leqslant n-1\}$$

The boundary maps $d_n\colon\ F_n \longrightarrow F_{n-1}$ are defined as for rack homology and cohomology.

We also obtain a bar resolution $\mathbf{B}$ where $B_n = \mathbb{Z}X \otimes^n (\mathbb{Z}X/Z)$. So $(B_n)_x$ is the free abelian group generated by symbols of the form $\chi[\chi_1|\ldots|\chi_n]$ for $\chi, \chi_1, \ldots, \chi_n \in (\mathbb{Z}X)_x$ with none of $\chi_1, \ldots, \chi_n$ equal to $(*)$.

As before, there are also maps $s_n\colon\ B_n \longrightarrow B_{n+1}$ such that

$$(s_n)_x(\chi[\chi_1|\ldots|\chi_n]) = [\chi|\chi_1|\ldots|\chi_n]$$

which is zero if $\chi = (*) \in (\mathbb{Z}X)_x$.

Starting with $(s_{-1})_x\colon\ 1 \mapsto [\,]$, we may recursively define the boundary maps to construct a free resolution of X-modules.

We may construct derived functors from this (or some other) resolution:

$$(\overline{\mathrm{Ext}_Q})^n_X(A,B) = (R^n\mathrm{Hom}_X(-,B))(A) = H^n(\mathrm{Hom}_X(\mathbf{P}_A,B)) \quad (3.9)$$

$$(\overline{\mathrm{Tor}_Q})^X_n(A,B) = (R^n - \otimes_X B)(A) = H_n(\mathbf{P}_A \otimes_X B) \quad (3.10)$$

and corresponding homology and cohomology theories for quandles:

$$\overline{H}^n_Q(X;A) = (\overline{\mathrm{Ext}_Q})^n(X,A) = (\overline{\mathrm{Ext}_Q})^n_X(\mathbb{Z},A) \quad (3.11)$$

$$\overline{H}^Q_n(X;A) = (\overline{\mathrm{Tor}_Q})_n(X,A) = (\overline{\mathrm{Tor}_Q})^X_n(\mathbb{Z},A) \quad (3.12)$$

Homology and cohomology groups, and Ext and Tor groups constructed from the standard complex will be denoted $\widehat{H^Q}_n$, $\widehat{H_Q}^n$, $\widehat{\mathrm{Ext}}_Q$, and $\widehat{\mathrm{Tor}}_Q$ where there might be any ambiguity.

3.8 Involutory homology and cohomology

There is a similar specialisation of rack homology and cohomology to the subcategories of involutory racks and quandles.

Let X be an involutory rack, and consider the functors Hom_X and $\otimes_X$ to be defined on the category IRMod_X. Unless indicated otherwise, the term 'X-module' is to be understood as shorthand for 'involutory rack module over X' for the rest of this section.

The proofs of the following results are identical to the corresponding results in the category of rack modules:

Theorem 3.24

(i) $\mathrm{Hom}_X(M,-)$ *is a left exact covariant functor for any X-module M.*

(ii) If F is a free involutory rack module over X, then $\mathrm{Hom}_X(F,-)$ *is exact.*

(iii) Every X-module is a quotient of a free X-module.

(iv) An X-module P is projective iff $\mathrm{Hom}_X(P,-)$ *is exact.*

(v) An X-module P is projective iff it is a summand of a free module, and any summand of a projective module is itself projective.

(vi) IRMod_X *has enough projectives.*

(vii) $\operatorname{Hom}_X(-, M)$ *is a left exact contravariant functor for any* X*-module* M*.*

(viii) *An* X*-module* I *is injective iff* $\operatorname{Hom}_X(-, I)$ *is exact.*

(ix) $- \otimes_X M$ *is a right exact covariant functor for any* X*-module* M*.*

(x) *The involutory rack algebra* $\mathbb{Z}X$ *is flat.*

(xi) *A direct sum of* X*-modules is flat iff each summand is flat.*

(xii) *All projectives are flat.*

(xiii) IRMod_X *has enough flats.*

There is a standard complex $\mathbf{F}$ which is analogous to that constructed for rack and quandle (co)homology:

Let F_n be the free involutory rack module on the trunk map $S_n \colon \mathsf{D}(X) \longrightarrow \mathsf{Set}$ defined by

$$(S_n)_{[x]} = \{(x_1, \ldots, x_n) : x_1 \in [x]; x_2, \ldots, x_n \in X\}$$

modulo the submodule G such that

$$G_{x_1^{x_2 \cdots x_n}} = \langle (x_1^{x_i}, \ldots, x_{i-1}^{x_i}, x_i, \ldots | x_n) - \rho_{x_1^{x_2 \cdots x_n}, x_i^{x_{i+1} \cdots x_n}} (x_1, \ldots, x_n) \rangle$$

for all $x_1, \ldots, x_n \in X$ and $1 \leqslant i \leqslant n$. This is the n-dimensional generalisation of condition 1.21 on involutory extensions.

The boundary maps $d_n \colon F_n \longrightarrow F_{n-1}$ are defined as for rack and quandle homology and cohomology.

There is also a well-defined bar resolution $\mathbf{B}$ with $B_n = \mathbb{Z}X \otimes^n (\mathbb{Z}X/Z)$, which has boundary maps $\partial_n \colon B_n \longrightarrow B_{n-1}$ recursively defined from the contracting chain homotopy s, with

$$(s_n)_x(\chi[\chi_1 | \ldots | \chi_n]) = [\chi | \chi_1 | \ldots | \chi_n]$$

for any $\chi \in (\mathbb{Z}X)_x$ and $\chi_1, \ldots, \chi_n \in (\mathbb{Z}X)_x \setminus \{(*)\}$.

Derived functors

$$(\overline{\text{Ext}_I})^n_X(A,B) = (R^n\text{Hom}_X(-,B))(A) = H^n(\text{Hom}_X(\mathbf{P}_A,B)) \quad (3.13)$$

$$(\overline{\text{Tor}_I})^X_n(A,B) = (R^n - \otimes_X B)(A) = H_n(\mathbf{P}_A \otimes_X B) \quad (3.14)$$

and corresponding homology and cohomology theories

$$\overline{H}^n_I(X;A) = (\overline{\text{Ext}_I})^n(X,A) = (\overline{\text{Ext}_I})^n_X(\mathbb{Z},A) \quad (3.15)$$

$$\overline{H}^I_n(X;A) = (\overline{\text{Tor}_I})_n(X,A) = (\overline{\text{Tor}_I})^X_n(\mathbb{Z},A) \quad (3.16)$$

may be defined from this resolution.

Homology and cohomology groups, and Ext and Tor groups constructed from the standard complex will be denoted $\widehat{H}^I_n, \widehat{H}^n_I, \widehat{\text{Ext}_I}$, and $\widehat{\text{Tor}_I}$ where the precise meaning cannot be discerned from the context.

Involutory quandle homology and cohomology theories may be defined similarly. The standard complex $\mathbf{F}$ is the obvious intersection of the standard complexes defined for quandle (co)homology and involutory rack (co)homology. A bar resolution $\mathbf{B}$ may be constructed similarly, and derived functors

$$(\overline{\text{Ext}_{IQ}})^n_X(A,B) = (R^n\text{Hom}_X(-,B))(A) = H^n(\text{Hom}_X(\mathbf{P}_A,B)) \quad (3.17)$$

$$(\overline{\text{Tor}_{IQ}})^X_n(A,B) = (R^n - \otimes_X B)(A) = H_n(\mathbf{P}_A \otimes_X B) \quad (3.18)$$

and corresponding homology and cohomology theories

$$\overline{H}^n_{IQ}(X;A) = (\overline{\text{Ext}_{IQ}})^n(X,A) = (\overline{\text{Ext}_{IQ}})^n_X(\mathbb{Z},A) \quad (3.19)$$

$$\overline{H}^{IQ}_n(X;A) = (\overline{\text{Tor}_{IQ}})_n(X,A) = (\overline{\text{Tor}_{IQ}})^X_n(\mathbb{Z},A) \quad (3.20)$$

defined therefrom.

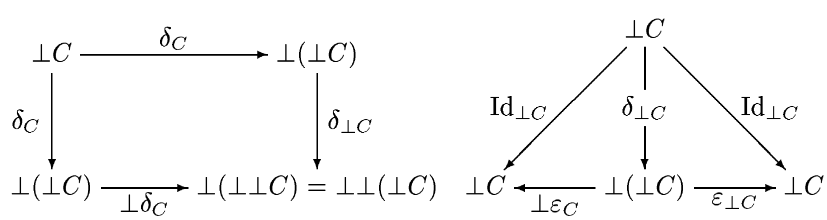
Homology, cohomology, Ext and Tor groups constructed from the standard complex $\mathbf{F}$ will be denoted $\widehat{H}^{IQ}_n, \widehat{H}^n_{IQ}, \widehat{\text{Ext}_{IQ}}$, and $\widehat{\text{Tor}_{IQ}}$ where necessary to avoid any ambiguity.

Chapter 4

Triples

The work of Barr and Beck [3] presents an alternative method for defining homology and cohomology theories, by means of a construction known variously as a 'cotriple', 'standard construction', 'comonad', or 'cotriad'. The reader is directed to their work (especially Beck's doctoral thesis [5] and the book by Barr and Wells [4]) for detailed proofs of the assertions and theory in this expository section.

A **cotriple** $(\perp, \varepsilon, \delta)$ in a category C is an endofunctor $\perp\colon \mathsf{C} \longrightarrow \mathsf{C}$ together with natural transformations $\varepsilon\colon \perp \longrightarrow \mathrm{Id}_{\mathsf{C}}$ and $\delta\colon \perp \longrightarrow \perp\perp$ such that the following diagrams commute, for every object C of C:



There is a corresponding dualisation: the 'triple', 'monad', 'triad', or 'dual standard construction', consisting of an endofunctor $\top\colon \mathsf{C} \longrightarrow \mathsf{C}$, and natural transformations $\eta\colon \mathrm{Id}_{\mathsf{C}} \longrightarrow \top$ and $\mu\colon \top\top \longrightarrow \top$ satisfying appropriate dualisations of the above two diagrams.

The first known appearance of the cotriple construction was in the work of Godement, concerning sheaves, although the application to be discussed in this chapter was pioneered by Barr and Beck.

The study of adjoint pairs of functors gives rise to a particularly rich source of cotriples. Consider an adjoint pair (L, R) of functors between categories C and D, with $\mathrm{Hom}_{\mathsf{C}}(C, RD) \cong \mathrm{Hom}_{\mathsf{D}}(LC, D)$ for any objects C in C and D in D. Such an adjunction gives rise to two natural transformations: the **unit** $\eta\colon \mathrm{Id}_{\mathsf{C}} \longrightarrow RL$ and the **counit** $\varepsilon\colon LR \longrightarrow \mathrm{Id}_{\mathsf{D}}$. The unit η_B is the morphism in $\mathrm{Hom}_{\mathsf{C}}(C, RLC)$ corresponding to $\mathrm{Id}_{LC} \in \mathrm{Hom}_{\mathsf{D}}(LC, LC)$, and the counit ε_D is the morphism in $\mathrm{Hom}_{\mathsf{D}}(LRD, D)$ corresponding to the identity morphism $\mathrm{Id}_{RD} \in \mathrm{Hom}_{\mathsf{C}}(RD, RD)$.

This gives a cotriple $(\perp, \varepsilon, \delta)$ on C by defining

$$\begin{array}{rcl} \perp & = & RL\colon \mathsf{C} \longrightarrow \mathsf{C} \\ \delta_C & = & R(\eta_{LC})\colon R(LC) \longrightarrow R(LR(LC)) \end{array}$$

It is routine to check that this satisfies the axioms for a cotriple.

Hilton conjectured that every triple arises from an adjoint pair of functors. In 1965 Kleisli, and Eilenberg and Moore, independently proved that this was the case, by means of two different constructions.

The construction devised by Eilenberg and Moore is as follows: Let $\top$ be a triple on a category C. Then a $\top$–**algebra** is a pair (C, γ) where $\top C \xrightarrow{\gamma} C$ is a morphism (the '**structure map**') in C, such that the diagrams

$$\begin{array}{ccc} C & \xrightarrow{\eta_C} & \top C \\ & \searrow^{\mathrm{Id}_C} & \downarrow \gamma \\ & & C \end{array} \qquad \begin{array}{ccc} \top^2 C & \xrightarrow{\top\gamma} & \top C \\ \mu_C \downarrow & & \downarrow \gamma \\ \top C & \xrightarrow[\gamma]{} & C \end{array}$$

commute.

A **morphism** of $\top$–algebras is a map $f\colon C \longrightarrow D$ in C such that

$$\begin{array}{ccc} \top C & \xrightarrow{\top f} & \top D \\ \gamma \downarrow & & \downarrow \delta \\ C & \xrightarrow[f]{} & D \end{array}$$

commutes. There is thus a category of $\top$–algebras over C, which we denote by $\mathsf{C}^\top$.

Now define a functor $U^\top\colon \mathsf{C}^\top \longrightarrow \mathsf{C}$ by $(C,\gamma) \mapsto C$ and $f \mapsto f$, and another $F^\top\colon \mathsf{C} \longrightarrow \mathsf{C}^\top$ by $C \mapsto (\top C, \mu_C))$ and $f \mapsto \top f$. These two functors are the required adjunction.

Kleisli's approach constructs a different category $\mathcal{K}(\top)$ which embeds in $\mathsf{C}^\top$. The objects of $\mathcal{K}(\top)$ are the objects of C, and for any two objects A, B of C, $\mathrm{Hom}_{\mathcal{K}(\top)}(A,B) = \mathrm{Hom}_{\mathsf{C}}(A, \top B)$. In fact, amongst all the ways of factoring $\top$ as an adjoint pair, Kleisli's category $\mathcal{K}(\top)$ is initial, and Eilenberg and Moore's category $\mathsf{C}^\top$ is terminal.

Now, given an arbitrary adjoint pair of functors $(L,R)\colon \mathsf{C} \longrightarrow \mathsf{D}$, this (as noted above) determines a triple $\top = RL$ on C. The Eilenberg-Moore construction determines a (possibly different) adjunction $(U^\top, F^\top)\colon \mathsf{C} \longrightarrow \mathsf{C}^\top$. The (**Eilenberg-Moore**) **comparison functor** $\Phi\colon \mathsf{D} \longrightarrow \mathsf{C}^\top$ maps an object D of D to a $\top$–algebra $(LD, L\varepsilon_D)$ and a morphism f in D to the $\top$–algebra morphism Lf.

There is, of course, an obvious dual framework of $\perp$–**coalgebras**. When a functor has both a left and a right adjoint, the related categories of algebras and coalgebras are isomorphic.

If a functor $R\colon \mathsf{D} \longrightarrow \mathsf{C}$ has a left adjoint for which Φ is an equivalence of categories, then R is said to be **tripleable** (or **monadic**).

Example 4.1 (Beck [5])

The category Group *is tripleable over the category* Set *with respect to the usual forgetful functor.*

If X is a set, then the elements of the free group FX are words in generators (x) where $x \in X$, with the empty word denoted $(\ )$.

A triple $\top = (UF, \eta, \mu)$ may be formed from the usual free/forgetful adjunction as discussed above. The unit of this adjunction is a natural transformation $\eta\colon \mathrm{Id} \longrightarrow UF$. The function η_X maps an element x to (x) (strictly the image of the generator (x) in the underlying set of FX).

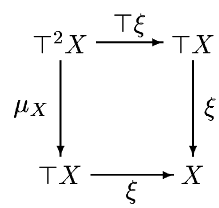
The transformation $\mu\colon \top\top \longrightarrow \top$ essentially corresponds to the operation of multiplying out one level of parentheses in the free group FX. More precisely, an element of $\top\top X$ is a word in generators (w), where w is an element of the underlying set of FX; that is, a word in generators (x) for $x \in X$. Then μ_X maps $(w) \mapsto w$ and is extended to other elements by multiplication.

For example, let $w_1 = (x_1)(x_2)$ and $w_2 = (w_2)^{-1}$. Then $w = (w_1)(w_2) = ((x_1)(x_2))((x_2)^{-1})$ is an element of $\top\top X$, and $\mu_X(w) = (x_1)(x_2)(x_2)^{-1} = (x_1) \in \top X$. This map μ is actually $U\varepsilon_F$, where ε is the counit of the adjunction.

To show that U is tripleable, we must demonstrate that the category $\mathsf{Set}^{\top}$ is equivalent to the category Group.

Given a $\top$–algebra $\xi\colon \top X \to X$, then, we may define a group structure on X by defining $x \cdot y = \xi[(x)(y)]$, for all $x, y \in X$. Similarly, we define the identity $1 = \xi[(\ )]$, the image of the empty word, and inverses $x^{-1} = \xi[(x)^{-1}]$ for all $x \in X$.

This multiplication operation is associative: Let $w_1 = ((x)(y))((z))$ and $w_2 = ((x))((y)(z))$ be words in $\top\top X$, and consider the commutative diagram



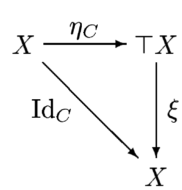
to see that

$$\begin{aligned} (x \cdot y) \cdot z &= \xi[(x \cdot y)(z)] = \xi\top(\xi)(w_1) \\ x \cdot (y \cdot z) &= \xi[(x)(y \cdot z)] = \xi\top(\xi)(w_2) \end{aligned}$$

and that

$$\xi\mu_X(w_1) = \xi[(x)(y)(z)] = \xi\mu_X(w_2)$$

The other diagram



ensures that $\xi[(x)] = x$.

Now let $w_3 = ((x))((x)^{-1})$ *and* $w_4 = ((x)^{-1})((x))$ *be two further words in* $\top\top X$. *Consideration of the same commutative diagram yields:*

$$\begin{aligned} x \cdot x^{-1} &= \xi[(x)(x^{-1})] = \xi\top(\xi)(w_3) \\ x^{-1} \cdot x &= \xi[(x^{-1})(x)] = \xi\top(\xi)(w_4) \end{aligned}$$

and

$$\xi\mu_X(w_3) = \xi\mu_X(w_4) = \xi[(\ )] = 1$$

Finally, let $w_5 = ((x))((\ ))$ *and* $w_6 = ((\ ))((x))$. *Then*

$$\begin{aligned} x \cdot 1 &= \xi[(x)(\ )] = \xi\top(\xi)(w_5) \\ 1 \cdot x &= \xi[(\ )(x)] = \xi\top(\xi)(w_6) \end{aligned}$$

and

$$\xi\mu_X(w_5) = \xi\mu_X(w_6) = \xi[(x)] = x$$

Hence any $\top$*–algebra determines a unique, well-defined group. Conversely, a group determines a unique* $\top$*–algebra* $\xi\colon \top X \longrightarrow X$ *in* Set *by* $(x) \mapsto x$ *and* $(x)(y) \mapsto x \cdot y$. *This map is precisely the Eilenberg-Moore comparison functor* $\Phi\colon \mathsf{Group} \longrightarrow \mathsf{Set}^\top$, *the preceding construction its inverse, and so* Φ *is an isomorphism of categories.*

The comparison functor leaves the underlying sets unchanged and simply interchanges two equivalent formulations of a given group structure.

The relevance of triples and cotriples to homology in general (and the current discussion in particular) arises from the fact that given a cotriple $\perp$ in C and an object C of C, we may construct a simplicial object $\perp_* C$ as follows:

Set $\perp_n C = \perp^{n+1} C$ and define the face and degeneracy operators

$$\begin{aligned} \partial_i &= \perp^i \varepsilon_{\perp^{n-i}A}\colon \perp^{n+1}A \longrightarrow \perp^n A \\ \sigma_i &= \perp^i \delta_{\perp^{n-i}A}\colon \perp^{n+1}A \longrightarrow \perp^{n+2} A \end{aligned}$$

Now, suppose we have a covariant functor $S\colon \mathsf{C} \longrightarrow \mathsf{A}$ where A is an abelian category, we define the **cotriple homology** by forming the augmented simplicial object $\perp_* \longrightarrow C$, applying the functor S to obtain an augmented simplicial object $S(\perp_* C) \longrightarrow S(C)$ in A, and taking the homology of the resulting complex. The homology theory thus defined is the **cotriple homology of C with coefficients S (relative to the cotriple $\perp$)**, and is denoted $H_*(C;S)_\perp$.

Dually, given a contravariant functor $T\colon \mathsf{C} \longrightarrow \mathsf{A}$ we construct the **cotriple cohomology** theory $H^*(C;T)_\perp$. In practice, the cotriple is usually omitted where no ambiguity would arise, so the homology and cohomology are often denoted $H_*(C;S)$ and $H^*(C;T)$.

Example 4.2

Let G be a group, $\perp$ be given by the adjoint pair (F,U) where $F\colon \mathsf{Set} \longrightarrow \mathsf{Group}$ is the free group functor, and U is the corresponding forgetful functor, and let A be a G–module. Then $H^n(G;A) = H^n(G;\mathrm{Der}(-,A))_\perp$ is a formulation of the usual cohomology of G (with coefficients in A) in terms of cotriple cohomology.

Similarly, the contravariant functor $\mathrm{Der}(-,A)$ has a left adjoint which is usually written as $\mathrm{Diff}(-) \otimes_G A$. Then the usual definition of group homology may be considered as cotriple homology thus: $H_n(G;A) = H_n(G;\mathrm{Diff}(-) \otimes_G A)_\perp$.

In this case, the underlying functor U happens to be tripleable, which is an important criterion for deciding whether or not the cotriple (co)homology theory has the usual algebraic description in terms of torsion and extension.

Barr [2] describes a (**standard**) **Cartan-Eilenberg** (or **CE**) **setting**:

Given a regular category A, denote (for each object A) the category of abelian group objects in the slice category A/A by $\mathsf{Ab}(\mathsf{A}/A)$. There is an obvious inclusion functor $I_A\colon \mathsf{Ab}(\mathsf{A}/A) \hookrightarrow \mathsf{A}/A)$, and we assume that it has a left adjoint which we denote by Diff^A.

Given a morphism $f\colon B \longrightarrow A$ in A, we obtain a functor $f_!\colon \mathsf{A}/B \longrightarrow \mathsf{A}/A$ given by composition with f. Assume further that this has a right adjoint $f^* = B \times_A -$ given by pulling back along f. This right adjoint f^* induces a functor $f^*\colon \mathsf{Ab}(\mathsf{A}/A) \longrightarrow \mathsf{Ab}(\mathsf{A}/A)$, which we will assume has, in turn, a left adjoint denoted $f_\sharp$:

$$\begin{array}{ccc} \mathsf{A}/A & \underset{I_A}{\overset{\mathrm{Diff}^A}{\rightleftarrows}} & \mathsf{Ab}(\mathsf{A}/A) \\ {\scriptstyle f_!}\uparrow\downarrow{\scriptstyle f^*} & & {\scriptstyle f_\sharp}\uparrow\downarrow{\scriptstyle f^*} \\ \mathsf{A}/B & \underset{I_B}{\overset{\mathrm{Diff}^B}{\rightleftarrows}} & \mathsf{Ab}(\mathsf{A}/B) \end{array}$$

The upper functors are adjoint to the lower functors, and the left functors are adjoint to the right functors. The diagram of the right adjoints commutes, and hence so does that of the left adjoints. The functor $f_\sharp$ is in general $A^e \otimes_{B^e} -$ where A^e is the appropriate enveloping object.

We require further that given a base category B, there is an underlying functor $U\colon \mathsf{A} \longrightarrow \mathsf{B}$ which preserves regular epimorphisms and has as left adjoint a functor $F\colon \mathsf{B} \longrightarrow \mathsf{A}$. Let $\perp = (FU, \varepsilon, \delta)$ be the resulting cotriple on A.

Suppose, also, that for each object A of A there is a chain complex functor $C_*^A\colon \mathsf{A}/A \longrightarrow \mathsf{ChComp}(\mathsf{Ab}(\mathsf{A}/A))$, the category of chain complexes in $\mathsf{Ab}(\mathsf{A}/A)$. Further suppose that for a morphism $f\colon B \longrightarrow A$ the diagram

$$\begin{array}{ccc} \mathsf{A}/A & \xrightarrow{C_*^A} & \mathsf{ChComp}(\mathsf{Ab}(\mathsf{A}/A)) \\ {\scriptstyle f_!}\downarrow & & \downarrow{\scriptstyle \mathsf{ChComp} f_\sharp} \\ \mathsf{A}/B & \xrightarrow[C_*^B]{} & \mathsf{ChComp}(\mathsf{Ab}(\mathsf{A}/B)) \end{array}$$

commutes.

Now define an object A of A to be U–**projective** if UA is projective in B with respect to the class of regular epimorphisms.

Then the following theorem of Barr gives sufficient conditions for a Cartan-Eilenberg (co)homology theory to be equivalent to a cotriple (co)homology theory:

Theorem 4.1

Suppose that A is U–projective, and that in a Cartan-Eilenberg setting,

(i) $\perp A$ *is U–projective;*

(ii) $C_*(A)$ *is a projective resolution of* $\mathrm{Diff}(A)$*;*

(iii) For each $n \geqslant 0$ there is a functor $\tilde{C}_n\colon \mathsf{B}/UA \longrightarrow \mathsf{Ab}(\mathsf{A}/A)$ such that the

diagram

$$\begin{array}{ccc} \mathsf{A}/A & \xrightarrow{C_n} & \mathsf{Ab}(\mathsf{A}/A) \\ {\scriptstyle U/A}\downarrow & \swarrow {\scriptstyle \tilde{C}_n} & \\ \mathsf{B}/UA & & \end{array}$$

commutes.

Then the complexes $C_*(A)$ *and* $\mathrm{Diff}(\bot^{*+1}A)$ *are chain equivalent.*

4.1 Tripleability

We begin by investigating the tripleability of Rack over Set. The **free rack** $FR(X)$ on a set X [15, Example 9] is defined to be the set $X \times F(X)$ equipped with the rack operation $(x^w)^{(y^v)} = x^{wv^{-1}yv}$ where $x, y \in X$ and v, w are words in the free group $F(X)$. There is an obvious forgetful functor which maps each rack to its underlying set; this is adjoint to the free rack functor.

For ease of notation, then, let $F\colon \mathsf{Set} \longrightarrow \mathsf{Rack}$ denote the free rack functor just described, and in this instance let $G\colon \mathsf{Set} \longrightarrow \mathsf{Group}$ denote the free group functor. Further, let $U\colon \mathsf{Rack} \longrightarrow \mathsf{Set}$ be the indicated forgetful functor.

Let $\top$ be the triple constructed from the endofunctor UF on Set. We seek to verify that the Eilenberg-MacLane comparison functor $\Phi\colon \mathsf{Rack} \longrightarrow \mathsf{Set}^{\top}$ is an equivalence of categories.

Given a set X, then, the set $\top X$ consists of elements of the form $(x)^w$ where $x \in X$ and w is a word in GX spelled in generators (y) for $y \in X$. As before, w may be the empty word $(\;)$. A typical element of $\top X$ might be $(x)^{(y)(z)^{-1}(y)^2}$.

The set $\top^2 X$ consists of elements of the form $(u)^v$ where $u \in \top X$ and v is a word spelled in generators (w) where $w \in \top X$. A typical element of $\top^2 X$ might be $a = \left((x)^{(y)(z)^{-1}(y)^2}\right)^{\left((y)^{(z)(x)^2(y)}\right)\left((y)^{(\;)}\right)^{-1}}$.

The unit $\eta\colon \mathrm{Id} \longrightarrow UF$ maps an element x to the element $(x)^{(\;)}$. The other natural transformation μ is formed from the counit $\varepsilon\colon FU \longrightarrow \mathrm{Id}$, and requires a little more explanation. Analogously to the case of Group, μ may be thought of as the operation of ‘removing the outermost parentheses’, thus:

$$\mu_X(a) = (x)^{(y)(z)^{-1}(y)^2(y)^{(z)(x)^2(y)}(y)^{-1}} = (x)^{(y)(z)^{-1}(y)(x)^{-2}(z)^{-1}(y)(z)(x)^2}$$

We are now in a position to prove

Theorem 4.2

The canonical forgetful functor $U\colon \mathsf{Rack} \longrightarrow \mathsf{Set}$ *is tripleable.*

Proof

Let $\top X \xrightarrow{\xi} X$ be a $\top$–algebra. We may define a rack operation on X by setting $x^y = \xi[(x)^{(y)}]$.

Then let $w_1 = \left((x)^{(y)}\right)^{((z))}$ and $w_2 = \left((x)^{(z)}\right)^{\left((y)^{(z)}\right)}$ and observe that

$$\begin{aligned}(x^y)^z = \xi\left[(x^y)^{(z)}\right] &= \xi\top(\xi)(w_1) = \xi\mu_X(w_1)\\ &= \xi\left[(x)^{(y)(z)}\right] = \xi\left[(x)^{(z)(y)^{(z)}}\right]\\ &= \xi\mu_X(w_2) = \xi\top(\xi)(w_2) = \xi\left[(x^z)^{(y^z)}\right] = (x^z)^{(y^z)}.\end{aligned}$$

The equalities in the first and third lines follow from the commutativity of the diagram

$$\begin{array}{ccc} \top^2 X & \xrightarrow{\top\xi} & \top X \\ {\scriptstyle \mu_X}\downarrow & & \downarrow{\scriptstyle \xi} \\ \top X & \xrightarrow[\xi]{} & X \end{array}$$

and that in the second line follows from the rack identity in the (underlying set of) the free rack on X.

Note that $\xi[(x)] = x$ by the commutativity of

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & \top X \\ & {\scriptstyle \mathrm{Id}_X}\searrow & \downarrow{\scriptstyle \xi} \\ & & X \end{array}$$

Verification of the first rack axiom follows similarly to that of the second. Let

$w_3 = \left((x)^{(y)^{-1}}\right)^{((y))} \in \top^2 X$. Then by commutativity of the above diagrams:

$$(x^{\overline{y}})^y = \xi\left[(x^{y^{-1}})^{(y)}\right] = \xi\top(\xi)(w_3) = \xi\mu_X(w_3) = \xi\left[(x)^{(y)^{-1}(y)}\right] = \xi\left[(x)\right] = x$$

Hence a $\top$–algebra determines a unique rack. This assignment is functorial, any map of $\top$–algebras determining a unique rack homomorphism.

Conversely, given a rack R, the Eilenberg-Maclane functor Φ gives a $\top$–algebra $\top U(R) = UFU(R) \xrightarrow{U\varepsilon_R} U(R)$. This is exactly the inverse of the above construction, for the map $\varepsilon_R\colon FU(R) \longrightarrow R$ is the 'evaluation' map which takes a formal element $(x)^w$ of $FU(R)$ and maps it to $x^w \in R$. □

4.2 Differentials

In chapter 2 it was shown that, given a fixed rack X, the category of abelian group objects in the slice category Rack/X is equivalent to the category RMod_X of rack modules over X. The equivalence is given by mapping an X–module $\mathcal{A}$ to the abelian group object $\mathcal{A} \rtimes X \longrightarrow X$ in the slice category.

We begin by examining the inclusion functor $I_X\colon \mathsf{Ab}(\mathsf{Rack}/X) \longrightarrow \mathsf{Rack}/X$.

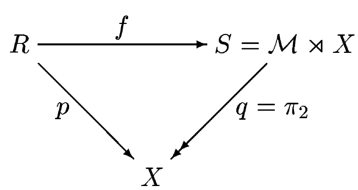
Proposition 4.3

Let $S \longrightarrow X$ be an abelian group object in Rack/X (so that $S = \mathcal{M} \rtimes X$ for some X–module $\mathcal{M} = (M, \phi, \psi)$), and let R be a rack over X. Then

$$\mathrm{Hom}_{\mathsf{Rack}/X}(R, I_X S) \cong \mathrm{Der}_X(R, \mathcal{M}).$$

Proof

By commutativity of the diagram



the composition $qf = p$ and so f must be of the form (d, p) where $d\colon R \longrightarrow \mathcal{M}$.

The map f is a rack homomorphism, and so $f(a^b) = f(a)^{f(b)}$ for any $a, b \in R$ such that $p(a) = x$ and $p(b) = y$ where $x, y \in X$.

Thus $(d(a^b), p(a^b)) = (d(a)^{d(b)}, p(a)^{p(b)})$ and hence

$$d(a^b) = \phi_{x,y} d(a) + \psi_{y,x} d(b)$$

so d is a derivation from R into the X–module $\mathcal{M}$.

Conversely, an X–derivation $d\colon R \longrightarrow \mathcal{M}$ yields a unique map $f\colon R \longrightarrow S$ over X for any rack R over X. □

This inclusion functor I_X has a left adjoint, which we denote by Diff^X. We call $\mathrm{Diff}^X(R)$ the X–module of **differentials** on R.

Proposition 4.4

The functor $\mathrm{Diff}^X\colon \mathsf{Rack}/X \longrightarrow \mathsf{Ab}(\mathsf{Rack}/X)$ *has the form* $\mathbb{Z}X \otimes_{-} \mathcal{J}-$. *That is,*

$$\mathrm{Hom}_{\mathsf{Rack}/X}(R, I_X S) \cong \mathrm{Hom}_{\mathsf{Ab}(\mathsf{Rack}/X)}(\mathbb{Z}X \otimes_R \mathcal{J}R, S)$$

for any racks $p\colon R \longrightarrow X$ *and* $q = \pi_2\colon S = \mathcal{M} \rtimes X \twoheadrightarrow X$ *over* X.

Proof

We may regard $\mathcal{M}$ as an R–module by means of the change of racks functor $p^*\colon \mathsf{RMod}_X \longrightarrow \mathsf{RMod}_R$. So

$$\mathrm{Hom}_{\mathsf{Rack}/X}(R, I_X S) = \mathrm{Der}_X(R, \mathcal{M}) = \mathrm{Der}_R(R, p^*\mathcal{M}).$$

But $\mathrm{Der}_R(R, -)$ is represented by the module $\mathcal{J}R$ (defined in section 3.1) and so

$$\mathrm{Der}_R(R, p^*\mathcal{M}) \cong \mathrm{Hom}_R(\mathcal{J}R, p^*\mathcal{M}).$$

Finally we apply the left adjoint to p^* to obtain

$$\mathrm{Hom}_R(\mathcal{J}R, p^*\mathcal{M}) \cong \mathrm{Hom}_X(\mathbb{Z}X \otimes_R \mathcal{J}R, \mathcal{M}) \cong \mathrm{Hom}_{\mathsf{Ab}(\mathsf{Rack}/X)}(\mathbb{Z}X \otimes_R \mathcal{J}R, S).$$

□

Analogous results hold in the categories $\mathsf{Quandle}$, $\mathsf{InvRack}$, and $\mathsf{InvQuandle}$.

4.3 The free cotriple in Rack

In this section we investigate whether either of the homology theories discussed in chapter 3 is equivalent to the cotriple homology theory derived from the free cotriple in the category Rack.

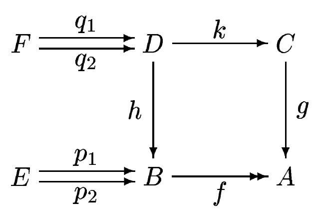
To start with, we must first check that the hypotheses of theorem 4.1 are satisfied.

Lemma 4.5

The category Rack *is regular, in the sense that regular epimorphisms are stable over pullbacks.*

Proof

Given a pullback diagram



where $f\colon B \longrightarrow A$ is a regular epimorphism (that is, a coequaliser of two rack homomorphisms $p_1, p_2\colon E \longrightarrow B$), we must show that $k\colon D \longrightarrow C$ is also a regular epimorphism – that is, there exist $q_1, q_2\colon F \longrightarrow D$ with coequaliser k.

For any $c \in C$, consider $a = g(c) \in A$. There is a $b \in B$ with $f(b) = a = g(c)$, since f is an epimorphism.

Furthermore, since $D = B \times_A C = \{(b, c) \in B \times C : f(b) = g(c)\}$, we may see that the pair $(b, c) \in D$, and thus k (which is projection onto the second coordinate) is an epimorphism.

Every epimorphism in Rack is regular, though. For, given $r\colon X \longrightarrow Y$, we form the **kernel pair** of r – the pullback of r along itself. Concretely, let $K = X \times_Y X = \{(x, w) \in X \times X : r(x) = r(w)\}$. It then follows that r is the coequaliser of the obvious projection maps $s_1, s_2\colon K \longrightarrow X$. □

Proposition 4.6

The derived-functor (co)homology theory for racks, as defined in the previous chapter, is a standard Cartan-Eilenberg setting.

Proof

The previous lemma shows that Rack is a regular category. The discussion in the previous section provided a concrete description of the differential functor $\mathrm{Diff}_X \colon \mathsf{Rack} \to \mathsf{RMod}_X$, which is shown to be left adjoint to the inclusion functor $I_X \colon \mathsf{Ab}(\mathsf{Rack}/X) \to \mathsf{Rack}/X$.

Now given an arbitrary rack homomorphism $f \colon X \to Y$, the induced functor $f_! \colon \mathsf{Rack}/Y \to \mathsf{Rack}/X$ is the composition of the slice morphism with f, and has a right adjoint $Y \times_X -$ determined by pulling back along $f \colon Y \to X$.

This right adjoint induces a functor $f^* \colon \mathsf{Ab}(\mathsf{Rack}/X) \to \mathsf{Ab}(\mathsf{Rack}/Y)$ which has a left adjoint $f_\sharp \colon \mathsf{Ab}(\mathsf{Rack}/Y) \to \mathsf{Ab}(\mathsf{Rack}/X)$ given by $\mathbb{Z}X \otimes_Y -$.

Our base category is Set and the functor $U \colon \mathsf{Rack} \to \mathsf{Set}$ is the usual 'underlying set' functor, whose left adjoint $F \colon \mathsf{Set} \to \mathsf{Rack}$ is the 'free rack' functor. We thus have a cotriple $\perp$ on Rack.

Finally, we have a functor $C_*^X \colon \mathsf{Rack}/X \to \mathsf{ChComp}(\mathsf{Ab}(\mathsf{Rack}/X))$ which assigns the bar resolution $\mathbf{B}(X)$ (strictly, its semidirect product with X) to X. $\square$

We now investigate whether this theory satisfies the hypotheses of theorem 4.1. To begin with, we note that any rack X is U–projective since any morphism in Set can be uniquely factorised into a composition of an epimorphism and a monomorphism.

By deleting the final term $\mathbb{Z}$ in the bar resolution $\mathbf{B}$ we obtain a deleted projective resolution of the augmentation module $\mathcal{I}X$, which is in general not isomorphic to $\mathrm{Diff}^X(X)$. This contrasts with many other cases of cotriple homology, where $\mathrm{Diff}^X(X) = \mathcal{I}X$.

Deleting the final term $\mathbb{Z}$ in the standard complex $\mathbf{F}$ also fails to provide a projective resolution of $\mathrm{Diff}^X(X)$, which leads us to suspect that neither rack homology theory discussed in chapter 3 is equivalent to the cotriple theory constructed from the free cotriple in Rack.

4.4 The free cotriple in Quandle

Following the discussion in the previous section it now seems likely that neither the existing quandle (co)homology theory, nor the derived functor theory, are equivalent to the theory derived from the free cotriple on Quandle.

Given a set X, the **free quandle** $FQ(X)$ on X is the free rack $FR(X)$ factored by the relation $x^x = x$ for all $x \in X$.

As one might reasonably expect, the free quandle functor $FQ\colon$ Set $\longrightarrow$ Quandle has an adjoint forgetful functor $U\colon$ Quandle $\longrightarrow$ Set.

Theorem 4.7

The forgetful functor $U\colon$ Quandle $\longrightarrow$ Set is tripleable.

Proof

The proof follows mostly from the corresponding theorem for the supercategory of racks; we need only check that the quandle identity is satisfied.

Let $w_1 = ((x)^{(x)})$ and $w_2 = ((x)^{(\ )})$ be two elements of $\top^2 X$. Then

$$\begin{aligned} x^x = \xi[(x^x)] = \xi\top(\xi)(w_1) = \xi\mu_X(w_1) \\ = \xi[(x)^{(x)}] = \xi[(x)^{(\ )}] \\ = \xi\top\xi)(w_2) = \xi[(x)] = x \end{aligned}$$

□

Derived functor (co)homology in Quandle is a standard Cartan-Eilenberg theory by an analogous argument to proposition 4.6. The definition of Diff^X is also analogous, and hence the bar resolution is not a projective resolution of $\mathrm{Diff}^X(X)$ in general. Neither is the standard complex $\mathbf{F}$, since it fails, in general, to be acyclic. This seems to suggest that neither the theory derived from the standard complex $\mathbf{F}$, nor the derived-functor theory, is equivalent to the theory constructed from the free cotriple on Quandle.

4.5 The free cotriple in InvRack and InvQuandle

Given a set X, the **free involutory rack** $FI(X)$ on X is the free rack $FR(X)$ factored by the relation $x^{yy} = x$ for all $x, y \in X$.

This functor $FI\colon \mathsf{InvRack} \longrightarrow \mathsf{Set}$ has an obvious adjoint 'underlying set' functor $U\colon \mathsf{InvRack} \longrightarrow \mathsf{Set}$.

The **free involutory quandle** $FIQ\colon \mathsf{Set} \longrightarrow \mathsf{InvQuandle}$ and its adjoint forgetful functor are similarly defined.

Theorem 4.8

The forgetful functor $U\colon \mathsf{InvRack} \longrightarrow \mathsf{Set}$ *is tripleable.*

Proof

As in the quandle case, this result follows mostly from the analogous result in Rack. It remains only to verify that the involutory condition is satisfied

Let $w_1 = ((x)^{(y)})^{((y))}$ and $w_2 = ((x)^{(\ )})$ be two elements of $\top^2 X$. Then

$$
\begin{aligned}
x^{yy} = \xi[(x^y)^{(y)}] = \xi\top(\xi)(w_1) = \xi\mu_X(w_1) \\
= \xi[(x)^{(y)(y)}] = \xi[(x)^{(\ )}] \\
= \xi\mu_X(w_2) = \xi\top(\xi)(w_2) = \xi[(x)] = x
\end{aligned}
$$

□

Corollary 4.9

The forgetful functor $U\colon \mathsf{InvQuandle} \longrightarrow \mathsf{Set}$ *is tripleable.*

The derived functor (co)homology theories in InvRack and InvQuandle are standard Cartan-Eilenberg theories by analogous arguments to proposition 4.6. The definitions of Diff^X are similarly analogous, and hence the respective bar resolutions are not projective resolutions of $\mathrm{Diff}^X(X)$ in general, in either case. Neither are the standard complexes $\mathbf{F}$ since they are not in general resolutions. This seems to indicate that neither of the involutory (co)homology theories derived from the standard complexes $\mathbf{B}$, nor the derived-functor theories, are equivalent to the cotriple theories constructed from the relevant free cotriples on the categories InvRack and InvQuandle.

4.6 The conjugation cotriple

As noted in the introductory remarks of chapter 1, the operator group provides a map $\mathrm{Op}\colon \mathsf{Rack} \longrightarrow \mathsf{Group}$ which is not functorial. The conjugation functor $\mathrm{Conj}\colon \mathsf{Group} \longrightarrow \mathsf{Rack}$, however, has as left adjoint the associated group functor $\mathrm{As}\colon \mathsf{Rack} \longrightarrow \mathsf{Group}$.

Hence we have a cotriple $\perp = \mathrm{As}\,\mathrm{Conj}\colon \mathsf{Group} \longrightarrow \mathsf{Group}$, and thus we may form a (possibly nonstandard) cotriple (co)homology theory for groups.

The question arises as to whether Group is tripleable over Rack with respect to the conjugation functor Conj.

We begin by examining $\top X$ for a given rack X. The group $\mathrm{As}\,X$ is the free group generated by symbols (x) modulo relations $(x^y) = (y)^{-1}(x)(y)$ for all $x, y \in X$. Then $\top X = \mathrm{Conj}\,\mathrm{As}\,X$ is the rack whose elements are words in those generators, with rack operation given by $w^z = z^{-1}wz$ for all $w, z \in \mathrm{As}\,X$.

Then $\mathrm{As}\top X$ is the free group generated by symbols of the form (w) where $w \in \top X$, modulo relations of the form $(w^z) = (z)^{-1}(w)(z)$ for all $w, z \in \top X$. But $(w^z) = (z^{-1}wz)$, by the rack operation in $\top X$, and hence $(z^{-1}wz) = (z)^{-1}(w)(z)$ in $\top^2 X$.

As before, the structure map $\mu_X\colon \top^2 X \longrightarrow \top X$ corresponds to the operation of 'removing the outermost level of parentheses' followed by any necessary simplification. The map $\eta_X\colon X \longrightarrow \top X$ is defined by $x \mapsto (x)$.

Let $\top X \xrightarrow{\xi} X$ be a $\top$–algebra and define a group structure on X by

$$x \cdot y = \xi[(x)(y)]$$

for all $x, y \in X$, and let the identity element in X be

$$1 = \xi[(\;)],$$

the image of the empty word in $\mathrm{As}\,X$, and the inverse of x be

$$x^{-1} = \xi[(x)^{-1}]$$

for all $x \in X$.

This is very similar to the case of the free triple in Group, and an argument almost identical to that in example 4.1 shows that Group is indeed tripleable over Rack with respect to the functor Conj.

Chapter 5

Sequences

This chapter is concerned with the construction of various exact sequences, for use in computing homology and cohomology groups of racks and quandles. The constructions of the long exact (co)homology sequence, the Künneth formulæ, and the universal coefficient sequences follow the standard methods for results of this type.

Of the results in this chapter, the long exact sequence is valid for both the original 'rack space' (co)homology theory, the 'derived functor' theory and the 'cotriple' theory; while the formulations of the Künneth formulæ and the universal coefficient theorems are stated and proved in terms of derived functors, although analogous results hold for the original theories [12].

5.1 Long exact sequences

It is well known that, given an abelian category A, the category $\mathsf{ChComp}(\mathsf{A})$, of chain complexes and chain maps in A, is also abelian, and hence it makes sense to speak of exact sequences of (co)chain complexes. A sequence

$$0 \rightarrow \mathbf{A} \xrightarrow{i} \mathbf{B} \xrightarrow{p} \mathbf{C} \rightarrow 0$$

is short exact if and only if each sequence

$$0 \longrightarrow A_n \xrightarrow{i_n} B_n \xrightarrow{p_n} C_n \longrightarrow 0$$

is short exact, for each $n \in \mathbb{Z}$.

Lemma 5.1

For a short exact sequence

$$0 \longrightarrow \mathbf{A} \xrightarrow{i} \mathbf{B} \xrightarrow{p} \mathbf{C} \longrightarrow 0$$

of chain complexes of rack modules, there is a homomorphism

$$\partial_n \colon H_n(\mathbf{C}) \longrightarrow H_{n-1}(\mathbf{A}).$$

Proof

This is an adaptation of the usual diagram-chasing argument. Consider the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{D}_{n+1} & \xrightarrow{i_{n+1}} & \mathcal{E}_{n+1} & \xrightarrow{p_{n+1}} & \mathcal{F}_{n+1} & \longrightarrow & 0 \\
 & & \downarrow d_{n+1} & & \downarrow e_{n+1} & & \downarrow f_{n+1} & & \\
0 & \longrightarrow & \mathcal{D}_n & \xrightarrow[i_n]{} & \mathcal{E}_n & \xrightarrow[p_n]{} & \mathcal{F}_n & \longrightarrow & 0 \\
 & & \downarrow d_n & & \downarrow e_n & & \downarrow f_n & & \\
0 & \longrightarrow & \mathcal{D}_{n-1} & \xrightarrow[i_{n-1}]{} & \mathcal{E}_{n-1} & \xrightarrow[p_{n-1}]{} & \mathcal{F}_{n-1} & \longrightarrow & 0
\end{array}$$

where each row is exact, and each column is a chain complex.

Suppose $z_x \in (\mathcal{F}_n)_x$ such that $f_n(z_x) = 0$. Then since p_n is an epimorphism, there is a $w_x \in (\mathcal{E}_n)_x$ such that $(p_n)_x(w_x) = z_x$. By the commutativity of the bottom right-hand square, $(e_n)_x(w_x) \in \ker(p_{n-1})_x = \operatorname{im}(i_{n-1})_x$. Since i_{n-1} is a monomorphism of X–modules, there is a unique $v_x \in (\mathcal{D}_{n-1})_x$ such that $(i_{n-1})_x(v_x) = (e_n)_x(w_x)$.

Now suppose that z_x is lifted to a different element w'_x of $(\mathcal{E}_n)_x$. Then this construction yields a (possibly) different element v'_x of $(\mathcal{D}_{n-1})_x$.

But $(z_x - z'_x) \in \ker(p_n)_x = \operatorname{im}(i_n)_x$, so there is an element u_x of $(\mathcal{D}_n)_x$ such that $(d_n)_x(u_x) = (v_x - v'_x) \in (B_{n-1}(\mathbf{D}))_x \subset (\mathcal{D}_{n-1})_x$.

This construction thus determines a well-defined abelian group homomorphism $(\partial_n)_x\colon\ (Z_n(\mathbf{F}))_x \longrightarrow (\mathcal{D}_{n-1})_x/(B_{n-1}(\mathbf{D}))_x$ for each $x \in X$.

It is relatively simple to see that $(\partial_n)_x(z_x) \in \ker(d_{n-1})_x$, and thus that $(\partial_n)_x$ maps cycles in $\mathcal{F}_n$ to cycles in $\mathcal{D}_{n-1}$. For, $(e_{n-1})_x(e_n)_x(w_x) = 0$, and hence $(d_{n-1})_x(v_x) = 0$, by commutativity and by the injectivity of i_{n-2}.

If $b_x \in B_n(\mathbf{F})$, that is, if there is a $c_x \in (\mathcal{F}_{n+1})_x$ such that $(f_{n+1})_x(c_x) = b_x$, then there exists (by the surjectivity of p_{n+1}) an element a_x of $(\mathcal{E}_{n+1})_x$ such that $(p_{n+1})_x(a_x) = c_x$. By the commutativity of the top right-hand square, there is an element k_x of $(\mathcal{E}_n)_x$ such that $(p_n)_x((e_{n+1})_x(a_x)) = b_x$. There is also a unique element g_x of $(\mathcal{D}_{n+1})_x$ such that $(i_{n+1})_x(g_x) = a_x$, and hence (by commutativity of the top left-hand square) the unique preimage, in $(\mathcal{D}_n)_x$, of k_x, is in the image of $(d_{n+1})_x$. Thus, $(\partial_n)_x(b_x) = 0 \in (\mathcal{D}_{n-1})_x$.

These homomorphisms fit together to form an X–module map

$$\partial_n\colon\ Z_n(\mathbf{F}) \longrightarrow \mathcal{D}_{n-1}/B_{n-1}(\mathbf{D}).$$

Since ∂_n maps boundaries in $\mathcal{F}_n$ to $0 \in \mathcal{D}_{n-1}$ and cycles in $\mathcal{F}_n$ to cycles in $\mathcal{D}_{n-1}$, it is the desired map $H_n(\mathbf{F}) \longrightarrow H_{n-1}(\mathbf{D})$. □

Theorem 5.2

Let

$$0 \longrightarrow \mathbf{D} \xrightarrow{i} \mathbf{E} \xrightarrow{p} \mathbf{F} \longrightarrow 0$$

be a short exact sequence of chain complexes of rack modules. Then there exists a long exact sequence

$$\ldots \longrightarrow H_n(\mathbf{D}) \xrightarrow{(i_n)_*} H_n(\mathbf{E}) \xrightarrow{(p_n)_*} H_n(\mathbf{F}) \xrightarrow{\partial_n} H_{n-1}(\mathbf{D}) \xrightarrow{(i_{n-1})_*} H_{n-1}(\mathbf{E}) \longrightarrow \ldots$$

of homology modules.

Proof

This is an adaptation of the standard proof, and is in three parts.

Firstly, $\mathrm{im}(i_n)_* \subseteq \ker(p_n)_*$ since $(p_n)_*(i_n)_* = (p_n i_n)_* = 0_* = 0$ by the exactness of the original short sequence.

Now, let $[z_x] \in H_n(\mathbf{E})_x$ be the homology class of a cycle $z_x \in (\mathcal{E}_n)_x$ such that $((p_n)_*)_x([z_x]) = 0$. Then $(p_n)_x(z_x) = (f_{n+1})_x(a_x)$ for some $a_x \in (\mathcal{F}_{n+1})_x$. Since p_{n+1} is surjective, there is an element $b_x \in (\mathcal{E}_{n+1})_x$ with $(p_{n+1})_x(b_x) = a_x$. Hence $(p_n)_x(z_x) = (p_n)_x(e_{n+1})_x(b_x)$ and so $(p_n)_x(z_x - (e_{n+1})_x(b_x)) = 0$. By exactness of the short sequence, there is thus a cycle $c_x \in (\mathcal{D}_n)_x$ such that $z_x - (e_{n+1})_x(b_x) = (i_n)_x(c_x)$, and therefore $[z_x] = [z_x - (e_{n+1})_x(b_x)] = ((i_n)_*)_x([c_x]) \in \operatorname{im}((i_n)_*)_x$. Thus $\ker(p_n)_* = \operatorname{im}(i_n)_*$, so the long sequence is exact at $H_n(\mathbf{E})$.

Let $[z_x]$ be a cycle in $\operatorname{im}((p_n)_*)_x \subseteq H_n(\mathbf{F})_x$. Then $(e_n)_x(z_x) = 0$ and since $(i_{n-1})_x$ is a monomorphism, $(i_{n-1})_x^{-1}(0) = \partial_x([z_x]) = 0$. Hence the composition $\partial_n(p_n)_*$ is the zero map and so $\operatorname{im}(p_n)_* \subseteq \ker \partial_n$.

Given $z_x \in (\mathcal{F}_n)_x$, then $(\partial_n)_x([z_x]) = [w_x] \in H_{n-1}(\mathbf{D})$, and there exists a $v_x \in (\mathcal{E}_n)_x$ such that $(i_{n-1})_x(w_x) = (e_n)_x(v_x)$ and $(p_n)_x(v_x) = z_x$. If $[w_x]$ is the zero class (that is, if $[z_x] \in \ker((p_n)_*)_x$) then there is a $u_x \in (\mathcal{D}_n)_x$ with $(d_n)_x(u_x) = w_x$. Then $(i_n)_x(d_n)_x(u_x) = (e_n)_x(v_x)$ and so $(e_n)_x(v_x) = (e_n)_x(i_n)_x(u_x)$. Hence $v_x - (i_n)_x(u_x)$ is a cycle in $(\mathcal{E}_n)_x$, and $(p_n)_x(v_x - (i_n)_x(u_x)) = (p_n)_x(v_x) = z_x$, so $[z_x] \in \operatorname{im}((p_n)_*)_x$. Thus $\ker \partial_n = \operatorname{im}(p_n)_*$ and so the long sequence is exact at $H_n(\mathbf{F})$.

Let $z_x \in (\mathcal{F}_n)_x$, then $((i_{n-1})_*)_x(\partial_n)_x([z_x]) = [(e_n)_x(v_x)]$ for some $v_x \in (\mathcal{E}_n)_x$. But $(e_n)_x(v_x)$ is a boundary in $(\mathcal{E}_{n-1})_x$ and so its homology class is zero. Hence $\operatorname{im} \partial_n \subseteq \ker(i_{n-1})_*$.

Finally, given a cycle $w_x \in (\mathcal{D}_{n-1})_x$ such that $(i_{n-1})_x(w_x) = (e_{n-1})_x(v_x)$ for some $v_x \in (\mathcal{E}_n)_x$, then $(p_n)_x(v_x)$ is a cycle in $(\mathcal{F}_n)_x$ since $(f_n)_x(p_n)_x(v_x) = (p_{n-1})_x(e_n)_x(v_x) = (p_{n-1})_x(i_{n-1})_x(w_x) = 0$, and $(\partial_n)_x([(p_n)_x(v_x)]) = [w_x]$. Hence $\ker(i_{n-1})_* \subseteq \operatorname{im} \partial_n$, and so the long sequence is exact at $H_{n-1}(\mathbf{D})$. □

There is a cohomological version of this theorem, the proof of which is dual to the above argument, and omitted.

Theorem 5.3

Let

$$0 \longrightarrow \mathbf{D} \xrightarrow{i} \mathbf{E} \xrightarrow{p} \mathbf{F} \longrightarrow 0$$

be a short exact sequence of chain complexes of rack modules. Then there exists *a long exact sequence*

$$\ldots \to H^n(\mathbf{D}) \xrightarrow{(i_n)^*} H^n(\mathbf{E}) \xrightarrow{(p_n)^*} H^n(\mathbf{F}) \xrightarrow{\partial^n} H^{n+1}(\mathbf{D}) \xrightarrow{(i_{n+1})^*} H^{n+1}(\mathbf{E}) \to \ldots$$

of cohomology modules.

There remains one more important result concerning the connecting homomorphism:

Theorem 5.4

The connecting homomorphism is natural. That is, given a commutative diagram of chain complexes of X–modules

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{D} & \xrightarrow{i} & \mathbf{E} & \xrightarrow{p} & \mathbf{F} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\ 0 & \longrightarrow & \mathbf{D}' & \xrightarrow[j]{} & \mathbf{E}' & \xrightarrow[q]{} & \mathbf{F}' & \longrightarrow & 0 \end{array}$$

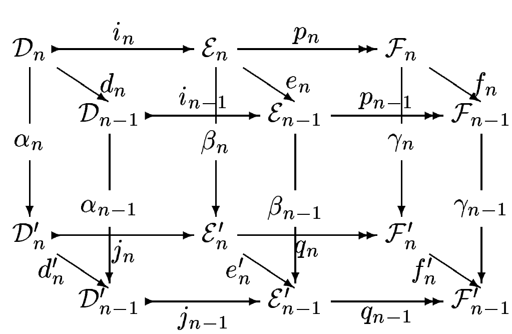
where the rows are exact, the diagram

$$\begin{array}{ccc} H_n(\mathbf{F}) & \xrightarrow{\partial_n} & H_{n-1}(\mathbf{D}) \\ {\scriptstyle (\gamma_n)_*}\downarrow & & \downarrow{\scriptstyle (\alpha_{n-1})_*} \\ H_n(\mathbf{F}') & \xrightarrow[\partial'_n]{} & H_{n-1}(\mathbf{D}') \end{array}$$

commutes.

Proof

The proof of this theorem is, again, a slight modification of the standard diagram chasing argument.



Recall that, given a cycle $c_x \in (\mathcal{F}_n)_x$, the connecting homomorphism maps the homology class $[c_x] \in H_n(\mathbf{F})_x$ to the class $[a_x] \in H_{n-1}(\mathbf{D})_x$ where $c_x = (p_n)_x(b_x)$ for some $b_x \in (\mathcal{E}_n)_x$ such that $(i_{n-1})_x(a_x) = (e_n)_x(b_x)$.

Then $(\gamma_n)_x(c_x) = (\gamma_n)_x(p_n)_x(b_x) = (q_n)_x(\beta_n)_x(b_x)$. But $(j_{n-1})_x(\alpha_{n-1})_x(a_x) = (\beta_{n-1})_x(i_{n-1})_x(a_x) = (\beta_{n-1})_x(e_n)_x(b_x) = (e'_n)_x(\beta_n)_x(b_x)$.

So $(\partial'_{n-1})_x((\gamma_n)_*)_x[c_x] = ((\alpha_{n-1})_*)_x(a_x) = ((\alpha_{n-1})_*)_x(\partial_n)_x[c_x]$. □

The dual result in cohomology also holds. The proof, a diagram chase dual to the above argument, is omitted.

Theorem 5.5

The cohomological connecting homomorphism is natural. That is, given a commutative diagram of chain complexes of X–modules

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{D} & \xrightarrow{i} & \mathbf{E} & \xrightarrow{p} & \mathbf{F} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\ 0 & \longrightarrow & \mathbf{D}' & \xrightarrow[j]{} & \mathbf{E}' & \xrightarrow[q]{} & \mathbf{F}' & \longrightarrow & 0 \end{array}$$

where the rows are exact, the diagram

$$\begin{array}{ccc} H^n(\mathbf{F}) & \xrightarrow{\partial^n} & H^{n+1}(\mathbf{D}) \\ {\scriptstyle (\gamma_n)^*}\downarrow & & \downarrow{\scriptstyle (\alpha_{n+1})^*} \\ H^n(\mathbf{F}') & \xrightarrow[\partial^{n\prime}]{} & H^{n+1}(\mathbf{D}') \end{array}$$

commutes.

5.2 Künneth formulæ

A **double (chain) complex** is a collection $\{\mathcal{B}_{p,q} : p, q \in \mathbb{Z}\}$ of X–modules, together with two collections $\{\partial^{\mathrm{h}}_{p,q} \colon \mathcal{B}_{p,q} \to \mathcal{B}_{p-1,q}\}$ and $\{\partial^{\mathrm{v}}_{p,q} \colon \mathcal{B}_{p,q} \to \mathcal{B}_{p,q-1}\}$

of X–maps, called the **differentials**, such that

$$\begin{aligned}
\partial^{\mathrm{h}}_{p-1,q}\partial^{\mathrm{h}}_{p,q} &= 0\\
\partial^{\mathrm{v}}_{p,q-1}\partial^{\mathrm{v}}_{p,q} &= 0\\
\partial^{\mathrm{h}}_{p,q-1}\partial^{\mathrm{v}}_{p,q} + \partial^{\mathrm{v}}_{p-1,q}\partial^{\mathrm{h}}_{p,q} &= 0
\end{aligned}$$

for all $p, q \in \mathbb{Z}$.

Given a double complex $\mathbf{B}$, we may construct a one-dimensional chain complex $\mathrm{Tot}\mathbf{B}$ as follows:

$$\begin{aligned}
(\mathrm{Tot}\mathbf{B})_n &= \bigoplus_{p+q=n} B_{p,q}\\
\partial_n &= \bigoplus_{p+q=n} (\partial^{\mathrm{h}}_{p,q} + \partial^{\mathrm{v}}_{p,q})\colon\ (\mathrm{Tot}\mathbf{B})_n \longrightarrow (\mathrm{Tot}\mathbf{B})_{n-1}
\end{aligned}$$

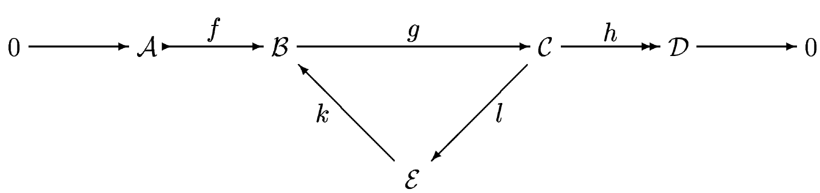
For any two chain complexes $\mathbf{C}$ and $\mathbf{D}$ we can form the **tensor product** complex by first forming the double complex $\mathbf{B}$, where $\mathcal{B}_{p,q} = \mathcal{C}_p \otimes_X \mathcal{D}_q$, and forming its total complex, which we denote by $\mathbf{C} \otimes_X \mathbf{D}$. The differential map is defined as

$$(\partial^{\otimes}_n)_x(c \otimes d) = (\partial^{\mathbf{C}}_p)_x(c) \otimes d + (-1)^p c \otimes (\partial^{\mathbf{D}}_q)_x(d)$$

Before stating and proving the Künneth theorem, we require the following short lemma:

Lemma 5.6

If



is a commutative diagram of rack modules, in which both the row and the triangle are exact, then there exist unique maps $\alpha\colon \mathcal{D} \longrightarrow \mathcal{E}$ *and* $\beta\colon \mathcal{E} \longrightarrow \mathcal{A}$ *making commuting outer triangles and a short exact sequence*

$$0 \longrightarrow \mathcal{D} \xrightarrow{\alpha} \mathcal{E} \xrightarrow{\beta} \mathcal{A} \longrightarrow 0$$

Proof

The map α exists because $\mathrm{Hom}_X(-,\mathcal{E})$ is a right-exact contravariant functor for any X–module $\mathcal{E}$. Similarly, β exists because $\mathrm{Hom}_X(\mathcal{A},-)$ is left-exact and covariant for any X–module $\mathcal{A}$. Recall that any morphism in an abelian category has a unique expression as a composition of a monomorphism and an epimorphism. Since $f\beta = l$, and f is injective, then β must be unique and surjective. Similarly, $\alpha h = k$ with h injective, and so α must be unique and injective. We thus have a short exact sequence

$$0 \longrightarrow \mathcal{D} \xrightarrow{\alpha} \mathcal{E} \xrightarrow{\beta} \mathcal{A} \longrightarrow 0$$

□

The Künneth formula, then, expresses the homology of $\mathbf{C} \otimes_X \mathbf{D}$ in terms of the homology of $\mathbf{C}$ and $\mathbf{D}$:

Theorem 5.7

Let $\mathbf{C}$ and $\mathbf{D}$ be chain complexes of X–modules, at least one of which is composed entirely of flat modules. Then there is a natural short exact sequence

$$0 \to \bigoplus_{p+q=n} H_p(\mathbf{C}) \otimes_X H_q(\mathbf{D}) \to H_n(\mathbf{C} \otimes_X \mathbf{D}) \to \bigoplus_{p+q=n-1} \mathrm{Tor}_1^X(H_p(\mathbf{C}), H_q(\mathbf{D})) \to 0$$

Proof

This proof is a slightly modified version of the one in [22].

Without loss of generality, we may assume that $\mathbf{C}$ is flat, since there is a natural isomorphism

$$\mathbf{C} \otimes_X \mathbf{D} \cong \mathbf{D} \otimes_{X^*} \mathbf{C}$$

given by

$$c \otimes d \mapsto (-1)^{pq} d \otimes c$$

for $c \in (\mathcal{C}_p)_x, d \in (\mathcal{D}_q)_x, x \in X$.

Let $\mathbf{Z}^{\mathbf{C}}$ and $\mathbf{Z}^{\mathbf{D}}$ denote the chain complexes formed from the cycle submodules

$Z_n(\mathbf{C})$ and $Z_n(\mathbf{D})$ with trivial differentials, and let $\mathbf{B}^{\mathbf{C}}$ and $\mathbf{B}^{\mathbf{D}}$ denote the corresponding boundary complexes, also with trivial differential maps. Denote by $\overline{\mathbf{B}^{\mathbf{C}}}$ the shifted complex $\{B_{n-1}(\mathbf{C})\}$, so that the differential in $\mathbf{C}$ may be regarded as a chain map $\partial^{\mathbf{C}}\colon\ \mathbf{C} \to \overline{\mathbf{B}^{\mathbf{C}}}$. We thus have a short exact sequence of chain complexes

$$\mathbf{0} \to \mathbf{Z}^{\mathbf{C}} \xrightarrow{\iota^{\mathbf{C}}} \mathbf{C} \xrightarrow{\partial^{\mathbf{C}}} \overline{\mathbf{B}^{\mathbf{C}}} \to \mathbf{0}$$

Since $\mathbf{C}$ is flat, it follows that $\mathbf{Z}^{\mathbf{C}}$, $\mathbf{B}^{\mathbf{C}}$ and $\overline{\mathbf{B}^{\mathbf{C}}}$ are flat, too, and hence the sequence

$$\mathbf{0} \to \mathbf{Z}^{\mathbf{C}} \otimes_X \mathbf{D} \xrightarrow{\iota^{\mathbf{C}} \otimes_X \mathrm{Id}} \mathbf{C} \otimes_X \mathbf{D} \xrightarrow{\partial^{\mathbf{C}} \otimes_X \mathrm{Id}} \overline{\mathbf{B}^{\mathbf{C}}} \otimes_X \mathbf{D} \to \mathbf{0} \tag{5.1}$$

is exact. Applying theorem 5.2 we obtain the long exact sequence

$$\begin{aligned} \cdots \to H_n(\mathbf{Z}^{\mathbf{C}} \otimes_X \mathbf{D}) \xrightarrow{(\iota_n^C \otimes_X \mathrm{Id})_*} H_n(\mathbf{C} \otimes_X \mathbf{D}) \xrightarrow{(\partial_n^C \otimes_X \mathrm{Id})_*} \\ H_n(\overline{\mathbf{B}^{\mathbf{C}}} \otimes_X \mathbf{D}) \xrightarrow{\partial_n} H_{n-1}(\mathbf{Z}^{\mathbf{C}} \otimes_X \mathbf{D}) \to \cdots \end{aligned} \tag{5.2}$$

The differential in $\overline{\mathbf{B}^{\mathbf{C}}}$ is trivial, and so the differential map in $\overline{\mathbf{B}^{\mathbf{C}}} \otimes_X \mathbf{D}$ is $\pm\mathrm{Id}\otimes_X\partial^{\mathbf{D}}$ and we may thus calculate the homology $H_*(\overline{\mathbf{B}^{\mathbf{C}}}\otimes_X\mathbf{D})$ using $\mathrm{Id}\otimes_X\partial^{\mathbf{D}}$, considering the complex

$$\begin{aligned} \cdots \to (\overline{\mathbf{B}^{\mathbf{C}}} \otimes_X \mathbf{D})_{n+1} \xrightarrow{(\mathrm{Id}\otimes_X\partial^{\mathbf{D}})_{n+1}} (\overline{\mathbf{B}^{\mathbf{C}}} \otimes_X \mathbf{D})_n \\ \xrightarrow{(\mathrm{Id}\otimes_X\partial^{\mathbf{D}})_n} (\overline{\mathbf{B}^{\mathbf{C}}} \otimes_X \mathbf{D})_{n-1} \to \cdots \end{aligned}$$

But $\overline{\mathbf{B}^{\mathbf{C}}}$ is flat, so

$$\begin{aligned} \ker(\mathrm{Id} \otimes_X \partial^{\mathbf{D}})_n &= (\overline{\mathbf{B}^{\mathbf{C}}} \otimes_X \mathbf{Z}^{\mathbf{D}})_n = (\mathbf{B}^{\mathbf{C}} \otimes_X \mathbf{Z}^{\mathbf{D}})_{n-1} \\ \mathrm{im}(\mathrm{Id} \otimes_X \partial^{\mathbf{D}})_{n-1} &= (\overline{\mathbf{B}^{\mathbf{C}}} \otimes_X \mathbf{B}^{\mathbf{D}})_n = (\mathbf{B}^{\mathbf{C}} \otimes_X \mathbf{B}^{\mathbf{D}})_{n-1} \end{aligned}$$

so

$$H_n(\mathbf{B}^{\mathbf{C}} \otimes_X \mathbf{D}) = (\mathbf{B}^{\mathbf{C}} \otimes_X H(\mathbf{D}))_{n-1}.$$

Since $\mathbf{Z}^{\mathbf{C}}$ is also flat,

$$H_n(\mathbf{Z}^{\mathbf{C}} \otimes_X \mathbf{D}) = (\mathbf{Z}^{\mathbf{C}} \otimes_X H(\mathbf{D}))_{n-1}$$

and so the long exact homology sequence becomes

$$\cdots \longrightarrow (\mathbf{Z^C} \otimes_X H(\mathbf{D}))_n \xrightarrow{(\iota_n^C \otimes_X \mathrm{Id})_*} H_n(\mathbf{C} \otimes_X \mathbf{D}) \xrightarrow{(\partial_n^C \otimes_X \mathrm{Id})_*}$$
$$(\mathbf{B^C} \otimes_X H(\mathbf{D}))_{n-1} \xrightarrow{\partial_n} (\mathbf{Z^C} \otimes_X H(\mathbf{D}))_{n-1} \longrightarrow \cdots \quad (5.3)$$

Now, we consider the connecting homomorphism ∂ in the long exact sequence (5.2). Pick a representative $(\partial_n^{\mathbf{C}})_x(c_x) \otimes_X z_x$ of a generator $(\partial_n^{\mathbf{C}})_x(c_x) \otimes_X [z_x]$ of $H_n(\mathbf{B^C} \otimes_X \mathbf{D})_x = (B_n^{\mathbf{C}} \otimes_X H_n(\mathbf{D}))_x$.

Then $(\partial_n^{\mathbf{C}})_x(c_x) \otimes_X z_x = ((\partial_n^{\mathbf{C}})_x \otimes_X \mathrm{Id}_x)(c_x \otimes_X z_x)$ and $(\partial_n^{\mathbf{C}\otimes_X\mathbf{D}})_x(c_x \otimes_X z_x) = (\partial_n^{\mathbf{C}})_x(c_x) \otimes_X z_x$.

Thus $(\partial_n)_x((\partial_n^{\mathbf{C}})_x(c_x) \otimes_X [z_x])$ is the homology class in $H_{n-1}(\mathbf{Z^C} \otimes_X \mathbf{D})_x = (\mathbf{Z^C} \otimes_X H_{n-1}(\mathbf{D}))_x$, of $(\partial_n^{\mathbf{C}})_x(c_x) \otimes_X z_x$, and so the connecting map ∂_n in the long exact sequence (5.2) is induced by the inclusion $B_n^{\mathbf{C}} \hookrightarrow Z_n^{\mathbf{C}}$.

Since $\mathbf{Z^C}$ is flat, the sequence

$$\mathbf{0} \longrightarrow \mathrm{Tor}_1^X(H(\mathbf{C}), H(\mathbf{D})) \longrightarrow \mathbf{B^C} \otimes_X H(\mathbf{D}) \xrightarrow{\partial}$$
$$\mathbf{Z^C} \otimes_X H(\mathbf{D}) \longrightarrow H(\mathbf{C}) \otimes_X H(\mathbf{D}) \longrightarrow \mathbf{0}$$

is exact. By lemma 5.6 we thus have a short exact sequence

$$\mathbf{0} \longrightarrow H(\mathbf{C}) \otimes_X H(\mathbf{D}) \longrightarrow H(\mathbf{C} \otimes_X \mathbf{D}) \longrightarrow \mathrm{Tor}_1^X(H(\mathbf{C}), H(\mathbf{D})) \longrightarrow \mathbf{0}$$

which is merely an alternative form of the sequence in the statement of the theorem. Every step in this argument is natural, hence the Künneth sequence is natural. □

5.3 Universal coefficient theorems

In the case where one of the complexes in the Künneth formula is simply an X–module – that is, a complex concentrated in dimension 0.

Theorem 5.8

If $\mathbf{C}$ is a chain complex of flat X–modules and $\mathcal{A}$ is an X–module, then there *is a short exact sequence*

$$0 \longrightarrow H_n(\mathbf{C}) \otimes_X \mathcal{A} \longrightarrow H_n(\mathbf{C} \otimes_X \mathcal{A}) \longrightarrow \mathrm{Tor}_1^X(H_{n-1}(\mathbf{C}), \mathcal{A}) \longrightarrow 0$$

Proof

Let $\mathbf{A}$ be the complex with $\mathcal{A}_0 = \mathcal{A}$ and $\mathcal{A}_i = 0$ for $i \geqslant 1$. Then the result follows from the Künneth theorem. □

Corollary 5.9

For any rack X, and any X–module $\mathcal{A}$, there is a short exact sequence

$$0 \longrightarrow H_n(X;\mathbb{Z}) \otimes_X \mathcal{A} \longrightarrow H_n(X;\mathcal{A}) \longrightarrow \mathrm{Tor}_1^X(H_{n-1}(X;\mathbb{Z}), \mathcal{A}) \longrightarrow 0$$

There are dual versions of these results:

Theorem 5.10

If $\mathbf{C}$ is a chain complex of free X–modules and $\mathcal{A}$ is an X–module, then there is a short exact sequence

$$0 \longrightarrow \mathrm{Ext}_X^1(H_{n-1}(\mathbf{C}), \mathcal{A}) \longrightarrow H^n(\mathrm{Hom}_X(\mathbf{C}, \mathcal{A})) \longrightarrow \mathrm{Hom}_X(H_n(\mathbf{C}, \mathcal{A})) \longrightarrow 0$$

Corollary 5.11

For any rack X and any X–module $\mathcal{A}$ there is a short exact sequence

$$0 \longrightarrow \mathrm{Ext}_X^1(H_{n-1}(X;\mathbb{Z}), \mathcal{A}) \longrightarrow H^n(X;\mathcal{A}) \longrightarrow \mathrm{Hom}_X(H_n(X;\mathbb{Z}), \mathcal{A}) \longrightarrow 0$$

Chapter 6

Computations

This penultimate chapter is concerned with the derivation and discussion of various general results about homology and cohomology of certain classes of racks.

6.1 Trivial racks

In this section we obtain a somewhat surprising result which shows that the definition of rack homology and cohomology due to Fenn, Rourke, Sanderson, Carter, Saito, Andruskiewitsch, Graña, and others, is not equivalent to the Cartan-Eilenberg style, derived functor theory developed in chapter 3.

Let $T_m = \{0, \ldots, m-1\}$ be the trivial rack of order m, with rack operation being given by $p^q = p$ for all $p, q \in T_m$.

Theorem 6.1

The higher derived-functor homology and cohomology groups $\overline{H}_n(T_m; \mathcal{A})$ and $\overline{H}^n(T_m; \mathcal{A})$ of T_m are trivial for $n \geqslant 2$.

Proof

The augmentation module $\mathcal{I}T_m$ is free, since $\operatorname{As} T_m = \mathbb{Z}^m = FA_m$, the free abelian group of rank m. So

$$0 \longrightarrow \mathcal{I}T_m \longrightarrow \mathbb{Z}T_m \longrightarrow \mathbb{Z} \longrightarrow 0$$

is a T_m–free resolution of $\mathbb{Z}$.

Thus $\overline{H}_n(T_m;\mathcal{A}) = \overline{H}^n(T_m;\mathcal{A}) = 0$ for $n \geqslant 2$. □

This contrasts with the following result regarding the homotopy type of the rack space BT_m:

Theorem 6.2 ([17])

The rack space BT_m has the homotopy type of $\Omega\left(\bigvee_m S^2\right)$, the loop space on a wedge of m copies of S^2.

Corollary 6.3

In general $\widehat{H}_n(T_m;\mathcal{A})$ and $\widehat{H}^n(T_m;\mathcal{A})$ are nontrivial for $n \geqslant 2$.

The **projective dimension** of an X–module $\mathcal{M}$, denoted $\mathrm{pd}(\mathcal{M})$, is the smallest n such that

$$0 \longrightarrow \mathcal{P}_n \longrightarrow \mathcal{P}_{n-1} \longrightarrow \ldots \longrightarrow \mathcal{P}_1 \longrightarrow \mathcal{P}_0 \longrightarrow \mathcal{M} \longrightarrow 0$$

is a projective resolution of $\mathcal{M}$. If no such finite resolution exists, then we set $\mathrm{pd}(\mathcal{M}) = \infty$.

Similarly, the **flat dimension** of $\mathcal{M}$, denoted $\mathrm{fd}(\mathcal{M})$, is the smallest n such that a finite flat resolution

$$0 \longrightarrow \mathcal{F}_n \longrightarrow \mathcal{F}_{n-1} \longrightarrow \ldots \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_0 \longrightarrow \mathcal{M} \longrightarrow 0$$

exists. Set $\mathrm{fd}(\mathcal{M}) = \infty$ if there is no such finite resolution.

For a rack X, define its **cohomological dimension**, $\mathrm{cd}(X)$, to be the smallest n such that $H^i(X;\mathcal{A}) = 0$ for all $i > n$ and all X–modules $\mathcal{A}$, or ∞ if no such integer exists. Analogously, its **homological dimension** $\mathrm{hd}(X)$ is the smallest n such that $H_i(X;\mathcal{A}) = 0$ for all $i > n$ and all X–modules $\mathcal{A}$, or ∞ if there is no such integer.

While the projective and flat dimensions are defined only in the context of derived functors, the homological and cohomological dimensions are defined for both forms of rack and quandle (co)homology. Where the meaning may not be discerned from context, we shall denote the dimensions pertaining to the original (co)homology theories by $\widehat{\mathrm{hd}}$ and $\widehat{\mathrm{cd}}$, and those relating to the derived functor approach by $\overline{\mathrm{hd}}$ and $\overline{\mathrm{cd}}$.

The above results show that, as a T_m–module, $\mathrm{pd}(\mathbb{Z}) = \mathrm{fd}(\mathbb{Z}) = 1$, and further that $\widehat{\mathrm{hd}}(T_m) = \widehat{\mathrm{cd}}(T_m) = \infty$ whereas $\overline{\mathrm{hd}}(T_m) = \overline{\mathrm{cd}}(T_m) = 1$.

6.2 Free racks

We now consider the case of F_m, the free rack on m elements:

Theorem 6.4 ([17])

The rack space BF_m has the homotopy type of $\bigvee_m S^1$, a wedge of m circles. Hence

$$\widehat{H}_n(F_m; \mathcal{A}) = \widehat{H}^n(F_m; \mathcal{A}) = 0$$

for any F_m–module $\mathcal{A}$ and any $n \geqslant 2$. Thus

$$\widehat{\mathrm{hd}}(F_m) = \widehat{\mathrm{cd}}(F_m) = 1.$$

In this case the derived functors agree, at least in higher dimensions:

Theorem 6.5

The derived functor homology and cohomology groups of the free rack F_m are trivial in dimensions 2 and above:

$$\overline{H}_n(F_m; \mathcal{A}) = \overline{H}^n(F_m; \mathcal{A}) = 0$$

for any F_m–module $\mathcal{A}$ and any $n \geqslant 2$. Hence

$$\overline{\mathrm{hd}}(F_m) = \overline{\mathrm{cd}}(F_m) = 1.$$

Proof

Consider the augmentation module $\mathcal{I}F_m$. The orbit groups of this module are generated as free abelian groups by symbols of the forms $(\rho_{x^{\bar{u}},u} - 1)$ and $\rho_{x^{\bar{u}},u}\lambda_{y^v,x^{\bar{u}v\bar{y}v}}$ for all $x, y \in F_m$, and all words u, v in the associated group $\mathrm{As}\, F_m$ (which is the free group FG_m on m symbols).

As an F_m–module, $\mathcal{I}F_m$ is free on the basis $\mathcal{S}\colon \mathsf{D}(X) \longrightarrow \mathsf{Set}$ where

$$S_x = \{(\rho_{x^{\bar{u}},u} - 1), \rho_{x^{\bar{u}},u}\lambda_{y^v,x^{\bar{u}\bar{v}\bar{y}v}} : x \in F_m; u, v \in \mathrm{As}\, F_m = FG_m\}$$

so

$$0 \longrightarrow \mathcal{I}F_m \longrightarrow \mathbb{Z}F_m \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

is an F_m–free resolution of $\mathbb{Z}$. □

There are similar results for free quandles and for free involutory racks and quandles:

Theorem 6.6

Let FQ_m, FI_m, and FIQ_m be, respectively, the free quandle, the free involutory rack, and the free involutory quandle, on m symbols. Then

$$\begin{aligned} & \overline{\mathrm{hd}}_Q(FQ_m) = \overline{\mathrm{cd}}_Q(FQ_m) \\ = \; & \overline{\mathrm{hd}}_I(FI_m) = \overline{\mathrm{cd}}_I(FI_m) \\ = \; & \overline{\mathrm{hd}}_{IQ}(FIQ_m) = \overline{\mathrm{cd}}_{IQ}(FIQ_m) = 1. \end{aligned}$$

Chapter 7

Applications

In this final chapter, we examine a few applications, to knot theory, of the cohomology theories developed in the preceding chapters. We shall restrict our consideration to the theories $\widehat{H}^*$, $\widehat{H}^*_Q$, $\widehat{H}^*_I$, and $\widehat{H}^*_{IQ}$ derived from the standard complex **F** rather than the derived-functor theories $\overline{H}^*$, $\overline{H}^*_Q$, $\overline{H}^*_I$, and $\overline{H}^*_{IQ}$.

Carter, Saito and their colleagues (Jelsovsky, Kamada, Langford, Elhamdadi, and others) have derived a family of invariants of codimension–2 embedded submanifolds by constructing 'state sums' from cocycles in quandle cohomology [11] [13] [10]. A particularly elegant application of these new invariants was the proof [11] that the twice-twist-spun trefoil, considered as an oriented 2–sphere embedded in $\mathbb{R}^4$, is not isotopic to its orientation-inverse.

The state-sum invariants for the classical case $L\colon \coprod_i S^1_i \hookrightarrow S^3$ are defined by taking a diagram D for L and colouring it with elements of a given quandle X so that the colours agree at crossing points as shown in figure 7.1. This is the

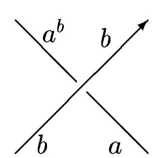


Figure 7.1: Colouring a link

same as choosing a homomorphism $c\colon \Gamma_Q(L) \to X$, where $\Gamma_Q(L)$ denotes the fundamental quandle of the link L.

Next pick a cocycle $\phi \in H^2_Q(X;A)$, the second quandle cohomology group of X with coefficients in the abelian group A (which we write multiplicatively).

Apply ϕ to each crossing τ of D by taking the labels of the incoming arcs as the arguments of ϕ, as shown in figure 7.2. The (**Boltzmann**) **weight** of the

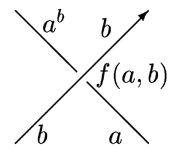


Figure 7.2: Applying a 2-cocycle f to a link

crossing τ is defined to be $\phi(a,b)^{\varepsilon(\tau)}$, where $\varepsilon(\tau)$ denotes the sign of τ. The **state sum** of L corresponding to the cocycle f is

$$\Phi_\phi(L) = \sum_c \prod_\tau \phi(a,b)^{\varepsilon(\tau)}$$

In other words, calculate the Boltzmann weight for each crossing relative to the colouring c, and then compute the product, in A, of the weights of all the crossings. Finally, sum over all possible colourings c of L with X. The state sum is an element of the group ring $\mathbb{Z}A$ and is an ambient isotopy invariant of L.

This construction may be generalised to higher dimensions – for the case of an n–manifold L embedded in an $(n+2)$–manifold, we immerse L in $\mathbb{R}^{n+1}$ and colour the components of the image with elements of our chosen quandle X. We then select a suitable cocycle $\phi \in H^{n+1}(X;A)$ and calculate the weight of each $(n+1)$–tuple point by applying ϕ to the labels of the incoming sheets in vertically-ascending order.

In this chapter, we shall be concerned solely with the classical case.

7.1 Ambient isotopy invariants of classical links

The work of Carter, Saito and their associates has been chiefly concerned with state sum invariants derived from quandle cohomology. The definition of quandle cohomology ensures that the relevant state sums are invariant under the first Reidemeister move. In the framework of the general theory presented in this thesis, the basic Carter/Saito theory is concerned with the case where the coefficient module is homogeneous and trivial. Twisted quandle cohomology [10] is concerned with the case where the coefficients are in a homogeneous Alexander module.

There are, then, two generalisations to consider – namely the case where the coefficients are in a heterogeneous module, and the case where the coefficients are in a nontrivial module.

Considering the first of these scenarios, we take a trivial heterogeneous module $\mathcal{A} = (A, \mathrm{Id}, 0)$ where each A_x is a multiplicative abelian group.

As before, we choose a colouring $c\colon \Gamma_Q(L) \longrightarrow X$ for the link and consider the weight of a given crossing τ with sign $\varepsilon(\tau)$, and whose incoming strands are labelled x and y.

Recall that a quandle 2–cocycle $\phi \in H^2_Q(X; \mathcal{A})$ is a trunk map $\mathcal{F}_2(X) \longrightarrow \mathcal{A}$ and hence determines an element of A_{x^y} for each ordered pair (x, y) in $X \times X$.

Applying ϕ to a crossing τ of L thus yields an element $\mathbf{a} \in \oplus_x A_x$ where

$$a_z = \begin{cases} \phi(x,y)^{\varepsilon(\tau)} & \text{if } z = x^y \\ 1 & \text{otherwise} \end{cases}$$

This element $\mathbf{a}$ is the weight of the crossing τ. We now calculate the weight for every crossing in the diagram and take their pointwise product, obtaining an element of $\oplus_x A_x$ which is in some sense the total weight of L relative to the colouring c. Finally we repeat this procedure for each possible X–colouring of the link L and sum the resulting total weights to obtain an element $\Phi_\phi \in \oplus_x \mathbb{Z} A_x$. In this way we obtain a polynomial invariant of L for each element of the colouring rack X.

This rather complex procedure is best illustrated by a simple example:

Example 7.1

Let $\mathcal{A}$ be the trivial heterogeneous quandle module over T_2 with $A_0 = \mathbb{Z}_2$ and $A_1 = \mathbb{Z}$. Then $H^2_Q(T_2; \mathcal{A}) \cong \mathbb{Z}_2 \oplus \mathbb{Z}$. The link depicted in figure 7.3 is colourable with T_2 in four obvious ways, as each component may be assigned either colour. Let $f \in H^2_Q(T_2; \mathcal{A})$ where $f(0,0) = 1 \in \mathbb{Z}_2, f(0,1) = s \in \mathbb{Z}_2, f(1,0) = t \in \mathbb{Z}$, and $f(1,1) = 1 \in \mathbb{Z}$. This is a quandle 2–cocycle.

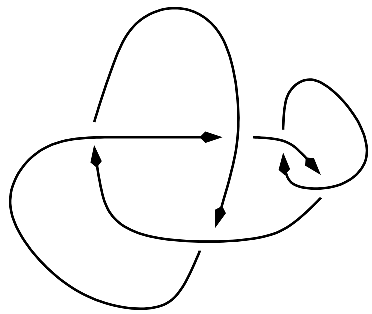


Figure 7.3: A two-component link

Colouring both components 0 yields the identity in both orbit groups, as does colouring both components 1. Colouring one component 0 and the other 1 yields $s \in A_0 = \mathbb{Z}_2$ *and* $t \in A_1 = \mathbb{Z}$*. The values of the invariants, then, are* $1 + s^2 = 2$ *(an element of* $\mathbb{Z}A_0$*) and* $1 + t^2$ *(in* $\mathbb{Z}A_1$*).*

For our second, more complicated, example, we consider the oriented link depicted in figure 7.4, which is 6^2_3 in the table in Rolfsen's book [29].

Example 7.2

Let $\mathcal{A}$ *denote the trivial heterogeneous quandle module over the dihedral quandle* D_4 *such that* $A_0 = A_2 = \mathbb{Z}_5 = \langle s : s^5 = 1\rangle$ *and* $A_1 = A_3 = \mathbb{Z}_6 = \langle t : t^6 = 1\rangle$*. The link* 6^2_3 *may be coloured with this quandle in sixteen ways, as depicted in*

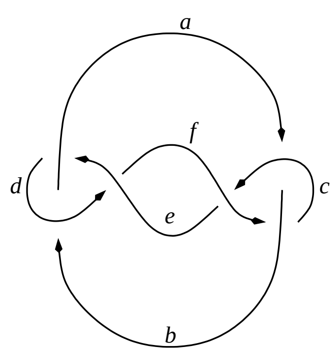


Figure 7.4: The link 6^2_3

table 7.1.

	a	b	c	d	e	f
1	0	0	0	0	0	0
2	0	2	1	3	1	3
3	0	0	2	2	2	2
4	0	2	3	1	3	1
5	1	3	0	2	0	2
6	1	1	1	1	1	1
7	1	3	2	0	2	0
8	1	1	3	3	3	3

	a	b	c	d	e	f
9	2	2	0	0	0	0
10	2	0	1	3	1	3
11	2	2	2	2	2	2
12	2	0	3	1	3	1
13	3	1	0	2	0	2
14	3	3	1	1	1	1
15	3	1	2	0	2	0
16	3	3	3	3	3	3

Table 7.1: Colourings of the link 6^2_3 with D_4

Let $\phi \in H^2_Q(D_4; \mathcal{A})$ be the cocycle defined by

$$\phi(a,b) = \begin{cases} s \in A_0 = \mathbb{Z}_5 & \textit{if } a = 0 \text{ and } b = 1,3 \\ t^3 \in A_1 = \mathbb{Z}_6 & \textit{if } a = 1 \text{ and } b = 0,3 \\ s^2 \in A_2 = \mathbb{Z}_5 & \textit{if } a = 2 \text{ and } b = 1,3 \\ t^3 \in A_3 = \mathbb{Z}_6 & \textit{if } a = 3 \text{ and } b = 0,1 \\ 1 \in A_{a^b} & \textit{otherwise} \end{cases}$$

Then the state sum Φ_ϕ is given by summing

$$\phi(a,c)\phi(f,b)\phi(b,d)\phi(e,a)\phi(e,d)^{-1}\phi(f,c)^{-1}$$

for each colouring in table 7.1.

This yields the products displayed in table 7.2 which, when summed, give values

	A_0	A_1	A_2	A_3
1	1	1	1	1
2	s^2	t^3	s	1
3	1	1	1	1
4	s^2	1	s	t^3
5	s^2	1	s	t^3
6	1	1	1	1
7	s^2	t^3	s	1
8	1	1	1	1

	A_0	A_1	A_2	A_3
9	1	1	1	1
10	s^2	1	s	t^3
11	1	1	1	1
12	s^2	t^3	s	1
13	s^2	t^3	s	1
14	1	1	1	1
15	s^2	1	s	t^3
16	1	1	1	1

Table 7.2: Products in $\Phi_\phi(6^2_3)$

of $8 + 8s^2$, $12 + 4t^3$, $8 + 8s$, and $12 + 4t^3$ for the state sum polynomials.

Reversing the orientation of one component of this link results in the situation depicted in figure 7.5. It is known that this new oriented link (which we shall *denote -6^2_3) is not ambient isotopic to the original link. The state sum for this*

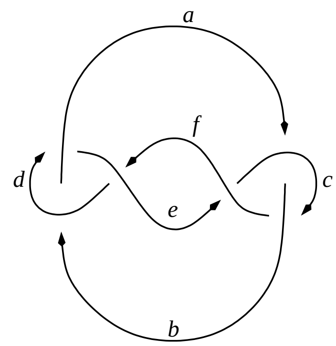


Figure 7.5: The link -6^2_3

link is given by summing

$$\phi(a,c)^{-1}\phi(f,b)^{-1}\phi(b,d)^{-1}\phi(e,a)^{-1}\phi(e,d)\phi(f,c)$$

for each of the sixteen colourings.

This yields the products displayed in table 7.3 which, when summed, give values

	A_0	A_1	A_2	A_3		A_0	A_1	A_2	A_3
1	1	1	1	1	9	1	1	1	1
2	s^3	t^3	s^4	1	10	s^3	1	s^4	t^3
3	1	1	1	1	11	1	1	1	1
4	s^3	1	s^4	t^3	12	s^3	t^3	s^4	1
5	s^3	1	s^4	t^3	13	s^3	t^3	s^4	1
6	1	1	1	1	14	1	1	1	1
7	s^3	t^3	s^4	1	15	s^3	1	s^4	t^3
8	1	1	1	1	16	1	1	1	1

Table 7.3: Products in $\Phi_\phi(-6^2_3)$

of $8+8s^3$, $12+4t^3$, $8+8s^4$, and $12+4t^3$ for the state sum polynomials, thus confirming that 6^2_3 and -6^2_3 are not ambient isotopic as oriented links.

This somewhat baroque example illustrates both the power of these invariants and the subtleties involved when heterogeneous coefficient modules are employed.

The second generalisation to consider is the case where the coefficient module is nontrivial. This is the case in the 'twisted quandle cohomology' invariants of Carter, Saito and Elhamdadi [10]. As remarked earlier, twisted quandle (co)homology is quandle cohomology with coefficients in a homogeneous Alexander module. The usual state-sum definition is modified slightly to take account of the nontrivial nature of the coefficient modules, using the concept of the 'Alexander numbering' of a link diagram.

Given a diagram of an oriented link L, give each strand a normal co-orientation such that the normal, the orientation, and the upward vertical vector out of the plane, form a right-handed triple. The immersed link divides the plane into a number of distinct regions. Let the **Alexander number** $\mathcal{L}(\infty)$ of the outermost (infinite) region be 0, and number all the other regions so that $\mathcal{L}(R_2) = \mathcal{L}(R_1)+1$ if R_1 and R_2 are adjacent regions with the co-orientation vector of the dividing arc pointing from R_1 to R_2, as depicted in figure 7.6. The Alexander number

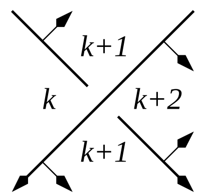


Figure 7.6: Alexander numbering

$\mathcal{L}(\tau)$ of a crossing τ is defined to be the Alexander number of the 'source region' of the crossing, so in the above diagram, the crossing has Alexander number k.

Now, given a colouring $c\colon \Gamma_Q(L) \to X$ of L by a quandle X, and a chosen cocycle $f \in H^2_Q(X;\mathcal{A})$, Carter, Saito and Elhamdadi define the **twisted (Boltzmann) weight** to be

$$B_T(\tau, c) = \left(f(x,y)^{\varepsilon(\tau)}\right)^{T^{-\mathcal{L}(\tau)}},$$

namely the ordinary weight raised to the power $T^{-\mathcal{L}(\tau)}$. This imposes a $\mathbb{Z}$–action on the orbit groups of the coefficient Alexander modules. The twisted state-sum invariant is then defined as

$$\Phi^T_f(L) = \sum_c \prod_\tau B_T(\tau, c)$$

It is not obvious how to extend this construction to arbitrary nontrivial coefficient modules, but by analogy we can devise a suitable generalisation for the case where the coefficients are in a dihedral X–module.

Let $f \in H^2_Q(X; \mathcal{D})$ where $\mathcal{D}$ is a dihedral X–module. Then the corresponding weight is

$$B_{\mathcal{D}}(\tau, c) = \left(f(x,y)^{\varepsilon(\tau)}\right)^{(-1)^{\mathcal{L}(\tau)}}$$

and the (dihedral) state-sum invariant is

$$\Phi^{\mathcal{D}}_f(L) = \sum_c \prod_\tau B_{\mathcal{D}}(\tau, c).$$

Consider an arc α which starts at ∞ and ends in the region R, intersecting only transversely, finitely many times with, and missing the double-points of, the link diagram of L. Trace along α from ∞ to the end in the region R, and construct a word in $\mathrm{As}\,X$ from the labels of the intersected arcs by appending a label y if an arc labelled y is crossed in the same direction as the co-orientation vector, or a $\bar{y}$ if the co-orientation vector points the other way. This process is illustrated by figure 7.7, which path gives the word $yz\bar{z}z\bar{u}u\bar{y} = yz\bar{y}$. This construction is

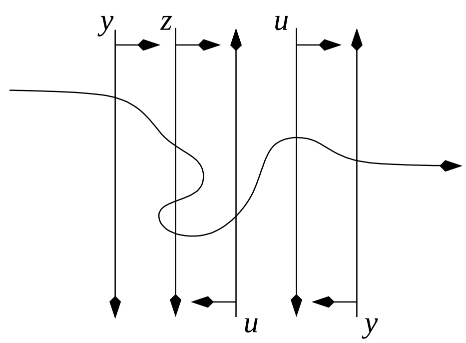


Figure 7.7: Construction of a word in $\mathrm{As}\,X$

well-defined, as figure 7.8 shows. The word defined by the left-hand diagram is xy, while the right-hand diagram gives the word $yx^y = y\bar{y}xy = xy$. Hence the resulting word is independent of the choice of path. This procedure determines a well-defined element of $\mathrm{As}\,X$ for each region R, which we denote by $w(R)$. As before, assign to each crossing τ the word $w(\tau) = w(R_\tau)$ where R_τ is the destination region of τ.

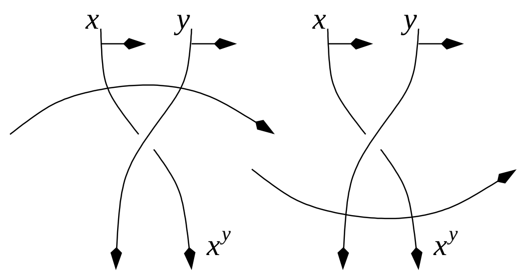


Figure 7.8: Path independence of the operator word

Now, given a coefficient module $\mathcal{A} = (A, \phi, \psi)$ let the weight of τ be

$$B_{\mathcal{A}}(\tau, c) = \phi_{x^y, \overline{w(\tau)}} \left(f(x, y)^{\varepsilon(\tau)} \right)$$

The state-sum invariant is then

$$\Phi_f^{\mathcal{A}}(L) = \sum_c \prod_\tau B_{\mathcal{A}}(\tau, c).$$

This construction only works for certain classes of X–modules, as shown by figure 7.9, in which the boxed words indicate the operator word for that region.

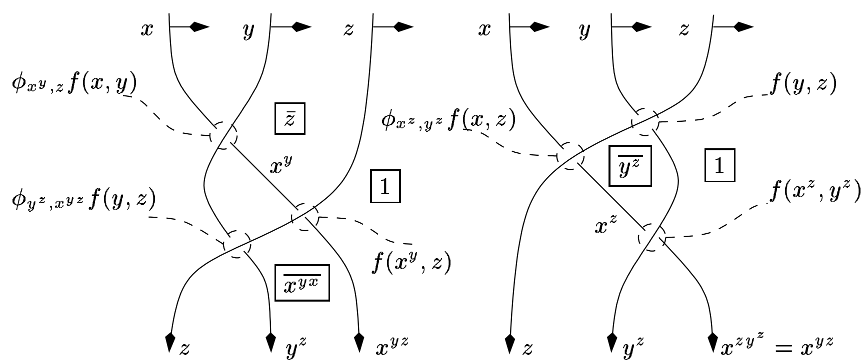


Figure 7.9: R_3 and cocycles

For invariance under the third Reidemeister move, we require that

$$\phi_{x^y,z} f(x, y) \cdot f(x^y, z) \cdot \phi_{y^z,x^{yz}} = \phi_{x^z,y^z} f(x, z) \cdot f(x^z, y^z) \cdot f(y, z)$$

for all $x, y, z \in X$. This is *almost* the definition of a 2–cocycle in $H^2_Q(X; \mathcal{A})$, and if we restrict ourselves to modules such that

$$\phi_{y^z,x^{yz}} f(y,z) \cdot f(y^z)^{-1} = \psi_{y^z,x^z} f(y,z)$$

for all $x, y, z \in X$, and all 2–cocycles $f \in H^2_Q(X; \mathcal{A})$ then we are assured of invariance under the third Reidemeister move. It is, however, not immediately clear what this condition means in the general case where $\mathcal{A}$ is heterogeneous. The above generalisation of the weight function can be applied in the case where the coefficient module is homogeneous and satisfies this additional condition. In particular, this holds where $\mathcal{A}$ is a homogeneous trivial, dihedral, or Alexander module.

The generalised construction may be seen to encompass the case where $\mathcal{A}$ is trivial homogeneous. The dihedral case, and that of twisted cohomology, need some slight modification.

The homogeneous dihedral case described earlier gives the same answer as this new formulation, since we are only concerned with the parity of the Alexander numbering for each region, and both the destination and source regions of a crossing have the same parity.

This new construction gives different, but isomorphic, results to Carter, Saito and Elhamdadi's twisted cocycle construction. The Alexander numbering of the source region of a crossing differs from that of the destination region by two, and hence the invariants derived from the new formulation may be expected to differ by factors of T^2 from those derived from the usual version of twisted cohomology.

7.2 Isotopy invariants of framed classical links

We now turn our attention to cocycle invariants derived from rack cohomology. It seems reasonable to expect that these will not in general be invariant under the first Reidemeister move.

Example 7.3

As an example, consider the unknot U_n with framing n, as depicted in figure 7.10 in the case $n = 4$. This has fundamental rack $\Gamma(U_n) = C_n = \{0, \ldots, n-1\}$, the

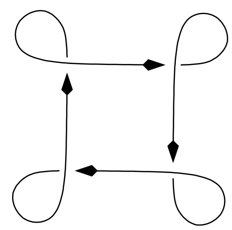


Figure 7.10: The unknot with framing 4

cyclic rack of order n, and has a single colouring with the trivial rack $T_1 = \{0\}$ given by $c\colon i \mapsto 0$.

Considering the second cohomology of T_1 with coefficients in the trivial T_1–module $\mathbb{Z}$, a trivial calculation reveals that $H^2_R(T_1;\mathbb{Z}) \cong \mathbb{Z}$ generated by the 2–cocycle f where $f(0,0) = t$, with t being the cyclic generator of $\mathbb{Z}$.

Then the state-sum invariant of U_n evaluates to

$$\Phi_f(U_n) = \sum_c \prod_\tau B(\tau, c)^{\varepsilon(\tau)} = \prod_\tau t = t^n$$

The corresponding invariant derived from quandle cohomology is trivial, suggesting that the difference between state sum invariants derived from rack cohomology, and those derived from quandle cohomology is precisely the mechanism which measures and preserves framing.

Example 7.4

Consider the (m,n)–framed Hopf link $H_{m,n}$ as depicted in figure 7.11 in the case where $m = -4$ and $n = 3$. This may be coloured in four distinct ways by

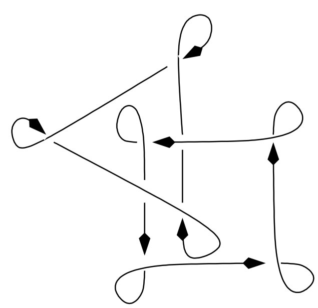


Figure 7.11: The $(-4,3)$-framed Hopf link

the trivial rack T_2. Let $\mathcal{A}$ denote the trivial heterogeneous T_2–module where

$A_0 = \mathbb{Z} = \langle s \rangle$ and $A_1 = \mathbb{Z} = \langle t \rangle$. Then let $f \in H^2_R(T_2; \mathcal{A})$ be the 2–cocycle where

$$f(x,y) = \begin{cases} s & \text{if } x = 0 \text{ and } y = 0 \\ s & \text{if } x = 0 \text{ and } y = 1 \\ t & \text{if } x = 1 \text{ and } y = 0 \\ t & \text{if } x = 1 \text{ and } y = 1 \end{cases}$$

Then the state-sum invariant derived from f is

$$\Phi_f(H_{m,n}) = \sum_c \prod_\tau B(\tau, c)^{\varepsilon(\tau)}$$

which gives polynomials $t^{m+n-2} + t^{m-1} + t^{n-1} + 1$ and $s^{m+n-2} + s^{m-1} + s^{n-1} + 1$. This invariant thus classifies framed Hopf links, in the sense that $\Phi_f(H_{m,n}) = \Phi_f(H_{p,q})$ iff either $m = p$ and $n = q$ or $m = q$ and $n = p$.

The construction of the state-sum may be extended, as in the previous section, to cocycles in certain classes of nontrivial coefficient module, among them dihedral and Alexander modules.

It remains to formally prove that state sums derived from rack cocycles are indeed framed isotopy invariants:

Theorem 7.1

If L_1 and L_2 are two framed links which are isotopic, then given any cocycle $f \in H^2_R(X; \mathcal{A})$ (where $\mathcal{A}$ is a trivial, Alexander, or dihedral X–module), it follows that $\Phi_f(L_1) = \Phi_f(L_2)$.

Proof

The links L_1 and L_2 are framed isotopic iff they differ by a finite sequence of Reidemeister moves of types R_2 and R_3. But the state-sum is invariant under R_2–moves, since the weights of both crossings are the same, but with opposite exponents, and thus cancel out. Invariance under R_3–moves follows from the definition of a rack cocycle. □

7.3 Isotopy invariants of unoriented classical links

The final application we shall examine is the construction of cocycle invariants from involutory rack and quandle cohomology.

Example 7.5

Consider the links 6^2_3 and -6^2_3 examined earlier. Colour each with the dihedral quandle D_4 and use the same coefficient module as before. Let $\psi \in H^2_{IQ}(D_4; \mathcal{A})$ be the involutory quandle cocycle defined thus:

$$\psi(a,b) = \begin{cases} t^3 & \text{if } a = 1 \text{ and } b = 0,3 \\ t^3 & \text{if } a = 3 \text{ and } b = 0,1 \\ 1 & \text{otherwise} \end{cases}$$

This involutory cocycle is a modified version of the one used in the earlier example. Calculating the products in the state sum gives the values listed in table 7.4, which sum to give values of 16, $12 + 4t^3$, 16, and $12 + 4t^3$ for the

	A_0	A_1	A_2	A_3		A_0	A_1	A_2	A_3
1	1	1	1	1	9	1	1	1	1
2	1	t^3	1	1	10	1	1	1	t^3
3	1	1	1	1	11	1	1	1	1
4	1	1	1	t^3	12	1	t^3	1	1
5	1	1	1	t^3	13	1	t^3	1	1
6	1	1	1	1	14	1	1	1	1
7	1	t^3	1	1	15	1	1	1	t^3
8	1	1	1	1	16	1	1	1	1

Table 7.4: Products in $\Phi_\psi(6^2_3)$

state-sum polynomials.

Calculating the products for the link -6^2_3 gives exactly the same polynomials, demonstrating that this invariant (which is the earlier invariant 'made involutory') cannot distinguish the two oriented links.

This example seems to suggest that the difference between ordinary quandle cohomology and involutory quandle cohomology is precisely the mechanism which can detect orientation and reversibility in links. This intuition is formalised by the following theorem:

Theorem 7.2

Let L_1 and L_2 be isotopic as unoriented links. Then for any involutory rack or quandle cocycle $\phi \in H^2_{I*}(X;\mathcal{A})$, for X an involutory rack (or quandle) and $\mathcal{A}$ a trivial, dihedral or Alexander involutory rack (or quandle) module, it follows that $\Phi_\phi(L_1) = \Phi_\phi(L_2)$.

Proof

Invariance under the second and third Reidemeister moves (and for involutory quandle cocycles, the first as well) is assured by earlier results, so we need only show that reversing the orientation of one of the strands at a crossing leaves the weight for that crossing unaltered.

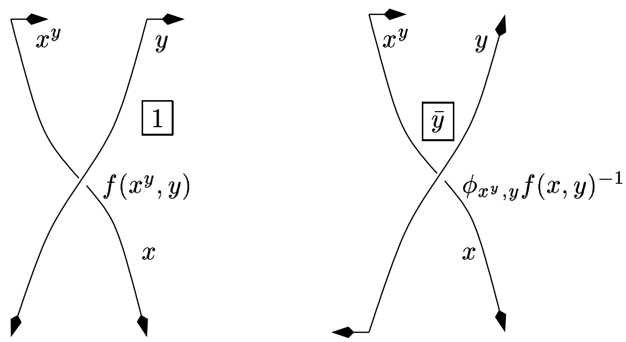


Figure 7.12: Reversing orientation in an involutory cocycle invariant

This situation is depicted in figure 7.12, in which both crossings are required to contribute the same weight to the state-sum. But as indicated this is exactly the same as requiring that $f(x^y, y) = \phi_{x^y,y} f(x,y)^{-1}$, which is one of the axioms for an involutory rack (or quandle) module. □

The final example shows that these involutory invariants are capable of discerning links with homeomorphic complements.

Example 7.6

The links in figure 7.13, the first of which is the Whitehead link, and which are denoted 5^2_1 and 7^2_7 in Rolfsen's tables, have homeomorphic complements in S^3 and hence the link groups alone are incapable of distinguishing them. The homeomorphism is given by performing a Dehn twist, in the appropriate direction, on the small circle in either link.

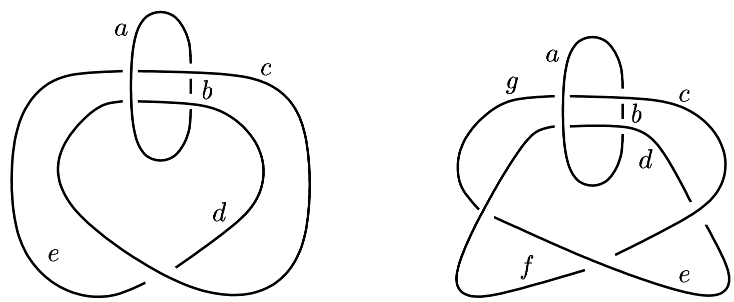


Figure 7.13: The links 5^2_1 and 7^2_7

Each of these links is colourable with the dihedral rack D_4, and so we consider state sums derived from cocycles in $H^2(D_4;\mathcal{A})$ where $\mathcal{A}$ is the trivial D_4–module such that $A_0 = A_2 = \mathbb{Z}_2 = \langle s : s^2 = 1\rangle$, and $A_1 = A_3 = \mathbb{Z}_4 = \langle t : t^4 = 1\rangle$. Let ϕ be the 2–cocycle such that

$$\phi(x,y) = \begin{cases} s & \text{if } x = 0 \text{ and } y = 2,3 \\ & \text{or } x = 2 \text{ and } y = 0,3 \\ t^2 & \text{if } x = 1 \text{ and } y = 2,3 \\ & \text{or } x = 3 \text{ and } y = 1,2 \\ 1 & \text{otherwise} \end{cases}$$

The Whitehead link 5^2_1 may be coloured in eight different ways by D_4, as indicated in table 7.5

	a	b	c	d	e
1	0	0	0	0	0
2	0	0	2	2	2
3	1	1	1	1	1
4	1	1	3	3	3
5	2	2	2	2	2
6	2	2	0	0	0
7	3	3	3	3	3
8	3	3	1	1	1

Table 7.5: Colourings of the Whitehead link 5^2_1 with D_4

The state sum is calculated by summing

$$\phi(b,c)\phi(a,d)\phi(e,a)\phi(c,a)\phi(d,c)$$

over each of these eight colourings, which yields trivial values for all four polynomials.

The link 7^2_7 may also be coloured in eight different ways by D_4, as indicated in table 7.6.

	a	b	c	d	e	f	g
1	0	0	0	0	0	0	0
2	0	0	2	2	2	2	2
3	1	1	0	2	2	0	2
4	1	1	2	0	0	2	0
5	2	2	0	0	0	0	0
6	2	2	2	2	2	2	2
7	3	3	0	2	2	0	2
8	3	3	2	0	0	2	0

Table 7.6: Colourings of the link 7^2_7 with D_4

The state sum may be calculated by summing

$$\phi(b,c)\phi(a,d)\phi(f,a)\phi(g,a)\phi(d,c)\phi(c,e)\phi(e,f)$$

over each of these colourings, which gives the products listed in table 7.7.

	A_0	A_1	A_2	A_3
1	1	1	1	1
2	1	1	1	1
3	s	1	1	t^2
4	1	1	s	t^2
5	1	1	1	1
6	1	1	1	1
7	1	t^2	s	1
8	s	t^2	1	1

Table 7.7: Products in $\Phi_\phi(7^2_7)$

These expressions sum to give values of $6+2s$, $6+2t^2$, $6+2s$, and $6+2t^2$ for the four state sum polynomials, thus confirming that the two links are distinct.

Afterword

There now follow brief outlines and discussions of questions raised and/or unanswered by this thesis.

Disparity

The most obvious questions raised by this thesis relate to the disparity between the existing rack and quandle homology theories, and the derived functor theories developed in chapter 3; and the conjectured inequivalence of both to the cotriple theory examined in chapter 4.

The reason for the difference between the first two theories may lie in the method of their construction. The standard theory is strongly related to the (topological) homology of the rack space BX, which is defined in terms of cubical sets. Conversely, the derived functors are altogether more categorical in nature and hence somewhat more simplicial in origin.

It is unclear which of these, if either, the cotriple theory equates to. There seems some evidence to suggest that it is equivalent to neither. It may be that the correct formulation of the standard theory is in terms of some cubical analogue of triples, involving trunks and trunk maps instead of categories and functors.

Nevertheless, all three constructions give valid homology and cohomology theories in the respective categories, some of them more readily computable than others.

Augmented racks

An augmented rack is a set X equipped with a right action by a group G, together with a function $\partial : X \longrightarrow G$ satisfying the **augmentation identity**:

$$\partial(a \cdot g) = g^{-1}(\partial a)g$$

for all $a \in X$ and $g \in G$. That is, an augmented rack is an ordinary rack equipped with a specific operator group.

There is an obvious inclusion functor $\mathsf{Rack} \hookrightarrow \mathsf{AugRack}$ given by the mapping $X \mapsto (X, \operatorname{Op} X)$. It would be interesting to devise algebraic homology theories for augmented racks which generalise those for racks. The first task would be to describe the appropriate category of coefficient modules, which would probably be similar to the rack modules described in chapter 2 but with some additional action by the explicit operator group.

As observed in [15], a **crossed module** is an augmented rack (X, G, ∂) where X is a group, ∂ is a group homomorphism, and the **crossed module identity** holds:

$$a \cdot \partial b = b^{-1}ab$$

for all $a, b \in X$. Crossed modules were first studied, in the context of homotopy theory, by Whitehead [32]. More recently, their cohomology has been studied by Carrasco, Cegarra, and Grandjean [7], and by Paoli [27]. It would be interesting to see if a (co)homology theory for augmented racks agrees with crossed-module (co)homology in the appropriate cases.

Cotriple homology

Another potentially rich topic for further study is that of the nonstandard cotriple group homology theory briefly discussed at the end of chapter 4.

Several interesting questions arise: Is there a concrete and readily computable description of this theory in terms of a standard resolution of some sort? Is this group homology theory equivalent to the usual one? Is there a related cotriple homology theory for racks which is equivalent to any of the three theories studied so far?

Spectral sequences

A powerful theorem of Grothendieck concerns the construction of spectral sequences from composite pairs of functors between abelian categories. In group homology this gives rise to the Lyndon/Hochschild-Serre spectral sequence.

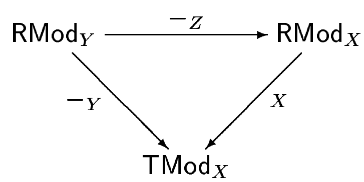
A possible candidate for the derived functor definition of rack homology may be as follows:

Conjecture 7.3

Given three racks X, Y, Z and a Y-module $\mathcal{A}$, where $Z = (Z, \chi, \omega)$ has the structure of a homogeneous X-module, and $Y = Z \rtimes X$, there is a first-quadrant homological spectral sequence

$$E^2_{p,q} = \overline{H}_p(X; \overline{H}_q(Z; \mathcal{A})) \Longrightarrow \overline{H}_{p+q}(Y; \mathcal{A})$$

This relies on there being a commutative diagram



of categories and functors.

There is a cohomological variant which requires the category of rack modules having enough injectives, something which is not proved in this thesis.

Products

In the case where the coefficient module is a wring $\mathcal{W}$, it should prove possible to define products

$$\cup\colon\ H^p(X; \mathcal{W}) \times H^q(X; \mathcal{W}) \longrightarrow H^{p+q}(X; \mathcal{W})$$

and

$$\cap\colon\ H^p(X;\mathcal{W}) \times H_q(X;\mathcal{W}) \to H_{q-p}(X;\mathcal{W}),$$

the first of which should give $H^*(X;\mathcal{W})$ a graded ring structure of some sort. It would be interesting to know whether such operations have any useful applications to knot theory or geometric topology.

Bibliography

[1] Nicolás Andruskiewitsch and Matías Graña.
From racks to pointed Hopf algebras.
Math.QA/0202084, February 2002, http://www.arxiv.org/.

[2] Michael Barr.
Cartan-Eilenberg cohomology and triples.
Journal of Pure and Applied Algebra, 112:219–238, 1996.

[3] Michael Barr and Jon Beck.
Homology and standard constructions.
In *Seminar on Triples and Categorical Homology Theory*, volume 80 of *Lecture Notes in Mathematics*, pages 245–335. Springer-Verlag, 1969.

[4] Michael Barr and Charles Wells.
Toposes, Triples and Theories.
Number 278 in Grundlehren der mathematischen Wissenschaften. Springer-Verlag, 1984.

[5] Jonathan Beck.
Triples, algebras and cohomology.
PhD thesis, Columbia University, 1967.

[6] Tom Blyth.
Categories.
Longman, 1986.

[7] P. Carrasco, A.N. Cegarra, and A.R. Grandjean.
Cohomology of crossed modules.
preprint, 1999.

[8] Henri Cartan and Samuel Eilenberg.
Homological Algebra.
Princeton Landmarks in Mathematics and Physics. Princeton University Press, 1999.

[9] J. Scott Carter, Mohamed Elhamdadi, Matías Graña, and Masahico Saito.
Cocycle knot invariants from quandle modules and generalized quandle cohomology.
Math.GT/0306068, June 2003, `http://www.arxiv.org/`.

[10] J. Scott Carter, Mohamed Elhamdadi, and Masahico Saito.
Twisted quandle homology theory and cocycle knot invariants.
Algebraic and Geometric Topology, 2:95–135, February 2002.

[11] J. Scott Carter, Daniel Jelsovsky, Seiichi Kamada, Laurel Langford, and Masahico Saito.
Quandle cohomology and state-sum invariants of knotted curves and surfaces.
Math.GT/9903135, March 1999, `http://www.arxiv.org/`.

[12] J. Scott Carter, Daniel Jelsovsky, Seiichi Kamada, and Masahico Saito.
Quandle homology groups, their Betti numbers, and virtual knots.
Math.GT/9909161, September 1999, `http://www.arxiv.org/`.

[13] J. Scott Carter, Seiichi Kamada, and Masahico Saito.
Diagrammatic computations for quandles and cocycle knot invariants.
Math.GT/0102092, February 2001, `http://www.arxiv.org/`.

[14] John Conway and Gavin Wraith.
Correspondence, 1959.

[15] Roger Fenn and Colin Rourke.
Racks and links in codimension 2.
Journal of Knot Theory and its Ramifications, 1:343–406, 1992.

[16] Roger Fenn, Colin Rourke, and Brian Sanderson.
Trunks and classifying spaces.
Applied Categorical Structures, 3:321–356, 1995.

[17] Roger Fenn, Colin Rourke, and Brian Sanderson.
James bundles and applications.

`http://www.maths.warwick.ac.uk/~bjs/`.

[18] Jean Flower.
Cyclic Bordism and Rack Spaces.
PhD thesis, Mathematics Institute, University of Warwick, Coventry, CV4 7AL, September 1994.

[19] Peter Freyd.
Abelian Categories.
Harper's Series in Modern Mathematics. Harper and Row, 1964.

[20] Michael Greene.
Some Results in Geometric Topology and Geometry.
PhD thesis, Mathematics Institute, University of Warwick, Coventry, CV4 7AL, September 1997.

[21] Pierre Grillet.
Semigroups: an introduction to the structure theory.
Dekker, 1995.

[22] Peter Hilton and Urs Stammbach.
A Course in Homological Algebra.
Number 4 in Graduate Texts in Mathematics. Springer-Verlag, second edition, 1996.

[23] David Joyce.
A classifying invariant of knots: the knot quandle.
Journal of Pure and Applied Algebra, 23:37–65, 1982.

[24] Saunders Mac Lane.
Homology.
Number 114 in Grundlehren der mathematischen Wissenschaften. Springer-Verlag, 1963.

[25] Saunders Mac Lane.
Categories for the Working Mathematician.
Number 5 in Graduate Texts in Mathematics. Springer-Verlag, 1971.

[26] Barry Mitchell.
Rings with several objects.
Advances in Mathematics, 8:1–161, 1972.

[27] Simona Paoli.
(Co)homology of crossed modules with coefficients in a π_1 module.
Technical Report 2002/17, Mathematics Institute, University of Warwick, April 2002.

[28] Daniel Quillen.
On the (co-)homology of commutative rings.
In *AMS Proceedings of Symposia in Pure Mathematics*, volume XVII, pages 65–87, 1970.

[29] Dale Rolfsen.
Knots and Links.
Publish or Perish, 1976.

[30] Hayley Ryder.
The Structure of Racks.
PhD thesis, Mathematics Institute, University of Warwick, Coventry, CV4 7AL, September 1993.

[31] Charles Weibel.
An Introduction to Homological Algebra.
Number 28 in Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1994.

[32] J.H.C. Whitehead.
On adding relations to homotopy groups.
Annals of Mathematics, 42:409–428, 1941.