Chapter E. Classification of surfaces with K nef

E.1. Statement of the main result

The first part of the classification of surfaces, the theory of minimal models, was concerned with the case K_X not nef. Thus I assume throughout this chapter that K_X is nef. It is convenient to summarise the main theorem of the classification of surfaces in the following form:

Table–Main Theorem. *A nonsingular projective surface X with K_X nef belongs to just one of the following cases.*

Name of Case:	$\nu = 2$	$\nu = 1$	$\nu = 0$
Definition:	$K_X^2 > 0$	$K_X^2 = 0, K_X \overset{\text{num}}{\not\sim} 0$	$K_X \overset{\text{num}}{\sim} 0$
P_m and κ :	$P_m \sim m^2, \kappa = 2$	$P_m \sim m, \kappa = 1$	$P_m = 1\ \exists m, \kappa = 0$
Effective results:	$P_m \geq 2$ for all $m \geq 2$	$P_m \geq 1$ for some $m \in \{1,2,3,4,6\}$ $P_m \geq 2$ for some $m \leq 42$	$mK_X \overset{\text{lin}}{\sim} 0$ for some $m \in \{1,2,3,4,6\}$
Structure result:	$\exists$ canonical model $\varphi\colon X \to Y$	$\exists$ elliptic fibre space $\varphi\colon X \to C$ (or more special structure in char. 2 or 3)	either Abelian or $K3$ or etale quotient by $\mathbb{Z}/m$ (same m)

In this table, the first two rows define the invariant $\nu(X) = \nu(K_X) = \kappa_{\text{num}}(X)$, the *numerical Kodaira dimension* of X: the three possible cases for K_X^2 and K_X enumerated in the second row are exclusive and cover all surfaces with K_X nef, and ν is defined to be 2, 1 or 0 accordingly; more generally, for any nef divisor on a projective variety X, the *characteristic dimension* of D is defined by

$$\nu(X,D) = \max\bigl\{k \bigm| D^k \overset{\text{num}}{\not\sim} 0\bigr\}.$$

The third row states the theorem that $\nu(X)$ is equal to the Kodaira dimension $\kappa(X)$, which, as everyone knows, is defined by

$$\begin{aligned}
\kappa = 2 &\iff P_m \sim m^2 \text{ for } m \gg 0,\\
\kappa = 1 &\iff P_m \sim m \text{ for } m \gg 0,\\
\kappa = 0 &\iff P_m \leq 1 \text{ for all } m, \text{ and } = 1 \text{ for some } m.
\end{aligned}$$

The fourth row is a slightly more precise version of the same result, giving effective values of m for which P_m grows. The last row summarises the structure results in each case.

The special structures referred to in the final line are as follows: a surface with $\kappa = 1$ has in any case a fibre space structure $\varphi\colon X \to C$ with fibres of arithmetic genus 1, but in characteristic 2 and 3 there is an extra possible case, a "quasielliptic" fibre space: it can happen that every geometric fibre is a singular curve isomorphic to a cuspidal plane cubic. A surface with $\kappa = 0$ is either as stated in the table, or (in characteristic 2 or 3 only) a quotient of an Abelian surface by a nonreduced finite group scheme, or "quasi-bielliptic" (see E.8.4), or an Enriques surface in characteristic 2. I do not have time in these notes to treat these fascinating cases in the detail they deserve.

Crude corollary. *If K_X is nef then there exists a morphism $\varphi\colon X \to Y$ to a projective variety Y such that*

$$K_X \overset{\text{num}}{\sim} \varphi^*(\text{ample } \mathbb{Q}\text{-divisor on } Y).$$

In higher dimension this statement is called the "abundance conjecture", and remains at the time of writing a hard problem. The 3-fold case has recently been settled in several papers of Miyaoka and Kawamata (see Kollár and others [Utah2], esp. Chapters 13–14 for the latest information). For surfaces, the nontrivial assertion is

$$\nu = 1 \implies \exists \text{ an elliptic fibre space.}$$

This is proved in E.6, and is really the main aim of this chapter. The remainder of the proof of the main theorem can be viewed simply as tidying up after this, using basically similar arguments.

Note that there is essentially nothing to prove in the theorem if $p_g \geq 2$: the case $\nu = 0$ is excluded, and $\nu = 1 \iff K_X^2 = 0$, in which case all the conclusions in the $\nu = 1$ column are satisfied rather trivially.

Corollary (Enriques' criterion)**.** *Let X be a surface, without the condition that K_X nef. Then equivalent conditions*

(1) *X is birationally ruled;*
(2) *$\nexists$ birational model of X with K_X nef;*
(3) *$P_m = 0$ for all $m \in \{1, 2, 3, 4, 6\}$;*
(4) *$\kappa(X) = -\infty$;*
(5) *adjunction terminates;*

etc.

Corollary (Castelnuovo's criterion)**.** *X is rational if and only if $p_g = P_2 = 0$ and $q = 0$.*

In the current view of minimal models of surfaces via Mori theory, ruled surfaces are characterised in the first instance by the numerical property of having no model with K_X nef; hence the proof of Enriques' criterion, in the form

$$\exists \text{ model with } K_X \text{ nef} \implies P_m \neq 0 \text{ for some } m \in \{1, 2, 3, 4, 6\}$$

becomes a problem in the classification of surfaces with K_X nef. Thus minimal model theory is concerned (as a matter of ideology) purely with the numerical properties of K_X, leaving analytic questions such as the dimension of $H^0(mK_X)$ to the second stage of classification theory. The classical arguments of Castelnuovo and Enriques work with $P_2 = 0$ via "adjunction terminates", mixing up analytic constructions such as the Albanese map together with numerical considerations of components of an effective divisor D with $H^0(D + K_X) = 0$. See [Beauville], Chapter 6 for a readable account.

E.2. Traditional numerical game

It turns out to be useful to tabulate the possibilities for the numerical invariants of surfaces with $p_g \leq 1$ (compare Bombieri and Mumford [B–M2], Introduction). For this, use Max Noether's formula

$$(c_1{}^2 + c_2)(X) = 12\chi(\mathcal{O}_X),$$

and interpret the terms on the two sides in terms of other invariants of X.

In characteristic 0 this is very easy (refer to Chapter C for a discussion of the invariants): on the right-hand side $\chi(\mathcal{O}_X) = 1 - q + p_g$, and on the left-hand side $c_1{}^2(X) = K_X{}^2$ and $c_2(X) = E(X) = 2 - 4q + 2p_g + h^{1,1}$.

In characteristic p, (or for non-Kähler complex surfaces) a little care is needed. Write $q' = h^1(\mathcal{O}_X)$, and $q = \dim \operatorname{Pic}^0 X = \dim \operatorname{Alb} X$.

Proposition (Igusa). *(i) $q' \geq q$; (ii) $c_2(X) = 2 - 4q(X) + B_2$ where $B_2 \geq \rho(X)$.*

Discussion of proof. (i) The connected group scheme $\operatorname{Pic}^0 X$ contains an Abelian variety $A = (\operatorname{Pic}^0 X)_{\mathrm{red}}$ such that $(\operatorname{Pic}^0 X)/A$ is a finite group scheme with only one point (possibly nonreduced). Now $q' = h^1(\mathcal{O}_X)$ is the dimension of the Zariski tangent space to $\operatorname{Pic}^0 X$, and so $q' \geq \dim \operatorname{Pic}^0 X = \dim A$; the Albanese variety of X is the dual Abelian variety to A, so that $q = \dim A$.

(ii) The second Chern class $c_2(X) \in H^4_{\mathrm{et}}(X, \mathbb{Q}_\ell)$ equals the Euler characteristic of X in etale cohomology,

$$c_2(X) = B_0 - B_1 + B_2 - B_3 + B_4,$$

where $B_i = B_i(X) = \dim_{\mathbb{Q}_\ell} H^i_{\mathrm{et}}(X, \mathbb{Q}_\ell)$ and ℓ is a prime $\neq p$. Now $B_0 = B_4 = 1$ by Poincaré duality, and

$$B_1 = B_3 = 2q,$$

since H^1_{et} is related to π_1 (more precisely, their ℓ-primary parts), and $\pi_1(X)$ to $\pi_1(\operatorname{Alb} X)$ essentially as in the classical case.

Finally, under the cycle class map $\operatorname{NS}(X) \to H^2_{\mathrm{et}}(X, \mathbb{Q}_\ell)$, the intersection number of divisors goes over to the cup product in etale cohomology, which implies that $\operatorname{NS}(X) \hookrightarrow H^2_{\mathrm{et}}$, and hence $B_2 \geq \rho(X)$. Q.E.D.

Putting everything together gives

$$K_X^2 + 2 - 4q + B_2 = 12 - 12q' + 12p_g;$$

that is,

$$K_X^2 + 12(q' - q) + 8q + B_2 = 10 + 12p_g. \qquad (*)$$

Famous Table. $p_g \leq 1$ *leads to the following table of cases:*

1	$p_g = 1$	$q' = q = 0$	$K_X^2 + B_2 = 22$	(if $K_X \overset{\text{num}}{\sim} 0$ then $K3$)
2	$p_g = 1$	$q' = q = 1$	$K_X^2 + B_2 = 14$	(doesn't exist with $K_X \overset{\text{num}}{\sim} 0$)
3	$p_g = 0$	$q' = q = 0$	$K_X^2 + B_2 = 10$	(if $K_X \overset{\text{num}}{\sim} 0$ then Enriques)
$3'$	$p_g = 1$	$q' = 1, q = 0$	$K_X^2 + B_2 = 10$	(if $K_X \overset{\text{num}}{\sim} 0$ then Enriques, char 2)
4	$p_g = 1$	$q' = q = 2$	$K_X^2 + B_2 = 6$	(if $K_X \overset{\text{num}}{\sim} 0$ then Abelian)
5	$p_g = 0$	$q' = q = 1$	$K_X^2 = 0, B_2 = 2$	(if $K_X \overset{\text{num}}{\sim} 0$ then bielliptic)
$5'$	$p_g = 1$	$q' = 2, q = 1$	$K_X^2 = 0, B_2 = 2$	(if $K_X \overset{\text{num}}{\sim} 0$ then quasi-bielliptic)

Proof. If I restrict to surfaces with $p_g \leq 1$ the right-hand side of $(*)$ is ≤ 22, and the left a sum of positive terms. It's easy to tabulate the possibilities. For the conclusion $K_X^2 = 0$ in (5) and (5′), note that if $q = 1$ then $B_2(X) \geq \rho(X) \geq 2$; indeed, then $\operatorname{Alb} X$ is a curve, and $\alpha\colon X \to \operatorname{Alb} X$ a nonconstant morphism, so that an ample divisor and the fibre of α provide two linearly independent classes in $\mathrm{NS}(X)$. Q.E.D.

Remarks. **(a)** The proof of abundance when $p_g \leq 1$ appeals to Famous Table E.2 several times. However, it's not clear whether the numerical game is really central to the classification of surfaces; if so, then it seems to me that the abundance conjecture in higher dimension is in serious trouble.

(b) The last 3 cases have $\chi(\mathcal{O}_X) = 0$, and otherwise $\chi(\mathcal{O}_X) > 0$. A priori, surfaces with $\chi(\mathcal{O}_X) < 0$ may exist, but not under the assumption $p_g \leq 1$, by what I've just said. Note that in characteristic 0, Bogomolov's inequality ${c_1}^2 \leq 4c_2$ implies that $\chi(\mathcal{O}_X) \geq 0$ and $\chi(\mathcal{O}_X) > 0$ if $K_X^2 > 0$.

(c) Notice that one reads directly from (3) and (5) of the table that $p_g = 0$ implies $q = q'$, that is, that $\operatorname{Pic}^0 X$ is reduced. More generally, for any projective scheme X, in the same way that $H^1(\mathcal{O}_X)$ is the space of 1st order infinitesimal deformations of a line bundle, the vector space $H^2(\mathcal{O}_X)$ is the obstruction space for deformations, hence $H^2(\mathcal{O}_X) = 0$ implies that $\operatorname{Pic}^0 X$ is reduced, so that $q = q'$. Mumford [M1], Lecture 27 shows how the tangent space to $(\operatorname{Pic}^0 X)_{\mathrm{red}}$ as a vector subspace of $H^1(\mathcal{O}_X)$ can be determined more generally. However, none of this fancy stuff is required for the current proof.

E.3. General type

For the proof of the main theorem, I start by separating off the case $\nu = 2$.

Theorem. *Suppose $K_X^2 > 0$. Then the curves (if any) with $K_X\Gamma = 0$ are all -2-curves (that is, $\Gamma \cong \mathbb{P}^1$ and $\Gamma^2 = -2$), and there is at most a finite set $\{\Gamma_i\}$ of them; each connected component of $\bigcup \Gamma_i$ supports only rational cycles, and can be contracted by a morphism $\varphi\colon X \to Y$, so that Y has only Du Val singularities, and K_Y is an ample Cartier divisor. Thus Y is the canonical model of X. Hence X is of general type, and its canonical ring $R(X, K_X) = R(Y, K_Y)$ is finitely generated over k.*

Moreover $P_m \geq 2$ for all $m \geq 2$.

Proof. The first part is standard and the proof is omitted; see Chapter 4 for Du Val singularities (especially Contraction Theorem 4.15), or [M2]. If $p_g \geq 2$ then $|K_X|$ is positive dimensional, hence also $|mK_X|$ for each $m \geq 1$. Thus to prove $P_m \geq 2$, I can assume $p_g \leq 1$. This is the first application of Famous Table E.2: I can deal separately with the different cases.

Cases (5) and (5′) are excluded since $K_X^2 > 0$. Cases (1), (2), (3) and (3′) all satisfy $\chi(\mathcal{O}_X) \geq 1$, so RR gives

$$h^0(mK_X) \geq \chi(\mathcal{O}_X(mK_X)) \geq 1 + \binom{m}{2} K_X^2 \geq 2 \qquad \text{for all } m \geq 2.$$

The only remaining case is (4). Then $\chi(\mathcal{O}_X) = 0$ and $c_2 = -K_X^2 < 0$; in characteristic zero, this is impossible, since surfaces of general type have $c_2 > 0$ (for example by Bogomolov's inequality $c_1{}^2 \le 4c_2$). In any case,

$$h^0(mK_X) \ge \chi(\mathcal{O}_X(mK_X)) \ge \binom{m}{2} K_X^2 \ge 1 \qquad \text{for all } m \ge 2,$$

and ≥ 2 except possibly for $m = 2$ and $K_X^2 = 1$. Together with the assumptions of case (4), this gives

$$p_g = 1, \quad q = q' = 2 \quad \text{and} \quad K_X^2 = 1. \tag{$**$}$$

Claim (Shepherd-Barron). $(**)$ *is impossible.*

Proof. Consider the Albanese morphism $X \to \operatorname{Alb} X$ and its Stein factorisation $\alpha\colon X \to A$. By [Sh-Barron], Theorem 7, the image of the Albanese morphism $\alpha\colon X \to A \subset \operatorname{Alb} X$ is a curve, which must be of genus 2 since $q = 2$. (In [Sh-Barron], it then follows that the fibres are geometrically rational curves.)

Let $C \in |K_X|$ be the unique curve. Then since $K_X^2 = 1$, C has a unique irreducible component with $K_X\Gamma = 1$, possibly together with some -2-curves. If C is reducible, or irreducible but singular, then Γ has geometric genus ≤ 1, so has no nonconstant morphism to A. It would follow that α contracts the whole of C to a point of A, which contradicts the index theorem since $C^2 = K_X^2 = 1$. Therefore C is irreducible and nonsingular.

The restriction of α to C is either an isomorphism, or has $\deg \alpha > 1$ (then α is necessarily inseparable, since $g(C) = g(A) = 2$). Either case leads to a contradiction: if $C \to A$ is an isomorphism, C meets any fibre F of α transversally in one point, so $CF = 1$; but then $K_XF = CF = 1$ contradicts $F^2 = 0$.

Now since $X \to A$ is its own Stein factorisation, the function field $k(A)$ is algebraically closed in $k(X)$. It follows that the differential of α is a nonzero map $s : \alpha^*(\Omega^1_A) \to \Omega^1_X$. Let $k \ge 0$ be the maximum number such that s vanishes k times along C, so that $s\colon \alpha^*(\Omega^1_A) \to \Omega^1_X(-kC)$ has nonzero restriction to C. Then the diagram

$$\begin{array}{ccccccccc}
 & & & & \alpha^*(\Omega^1_A) & & & & \\
 & & & & \downarrow & & & & \\
 & & & & \Omega^1_X(-kC) & & & & \\
 & & & & \downarrow & & & & \\
0 & \to & \mathcal{O}_C(-(k+1)C) & \to & \Omega^1_X(-kC)_{|C} & \to & \Omega^1_C(-kC) & \to & 0
\end{array}$$

gives a nonzero map of $\alpha^*(\Omega^1_A)$ to one of

$$\Omega^1_C(-kC) \quad \text{or} \quad \mathcal{O}_C(-(k+1)C).$$

This is a contradiction, since $\alpha^*(\Omega^1_A)$ has degree $2\deg\alpha > 2$, whereas

$$\deg \Omega^1_C(-kC) = 2 - k \quad \text{and} \quad \deg \mathcal{O}_C(-(k+1)C) = -(k+1). \quad \text{Q.E.D.}$$

E.4. The cases with $\chi(\mathcal{O}_X) > 0$

The following result deals with all surfaces with $\chi(\mathcal{O}_X) > 0$ and $p_g \le 1$ (that is, cases (1), (2), (3), (3′) of Famous Table E.2).

Theorem. *Assume that $K_X^2 = 0$ and $\chi(\mathcal{O}_X) = 1 - q' + p_g > 0$. Then*
either $K_X \overset{\text{lin}}{\sim} 0$ and $q' = 0$ (K3 surface);
or $2K_X \overset{\text{lin}}{\sim} 0$ and $p_g = q' = 0$ (classical Enriques surface);
or $\operatorname{char} k = 2$, *$K_X \overset{\text{lin}}{\sim} 0$ and $p_g = q' = 1$, $q = 0$ (nonclassical Enriques surface);*
or $P_6 \ge 2$.

Proof. The idea of the proof is to plug the assumption $\chi(\mathcal{O}_X) = 1$ or 2 into RR, giving $\chi(\mathcal{O}_X(mK_X)) \ge 1$. For $m \ge 2$, note that

$$h^0(mK_X) + h^2(mK_X) \ge \chi(\mathcal{O}_X(mK_X)) \ge 1.$$

But $H^2(mK_X) \overset{\text{d}}{=} H^0(-(m-1)K_X)$ (duality of vector spaces), so that $H^2 \ne 0$ implies that $-(m-1)K_X$ is effective, hence $(m-1)K_X \overset{\text{lin}}{\sim} 0$, since K_X is nef. The theorem follows easily from a small case division based on this. In more detail:

Case (1): $q' = 0$, $p_g = 1$, so that $\chi = 2$. Then $\chi(\mathcal{O}_X(2K_X)) = 2$, and hence either $P_2 = h^0(2K_X) \ge 2$ or $H^2(2K_X) \ne 0$. The second case gives $H^0(-K_X) \ne 0$, so that $K_X \overset{\text{lin}}{\sim} 0$. Note that this leads directly to the definition of K3s as surfaces with $K_X \overset{\text{lin}}{\sim} 0$ and $q' = 0$.

Case (2): $p_g = 1$, $q' = q = 1$; then $\chi(\mathcal{O}_X) = 1$, so RR gives

$$\chi(\mathcal{O}_X(mK_X + \sigma)) = 1 \quad \text{for every } m \text{ and every } \sigma \in \operatorname{Pic}^0 X.$$

Hence there is an effective divisor $D_\sigma \in |K_X + \sigma|$ for every $\sigma \in \operatorname{Pic}^0 X$; divisors of the form $D_\sigma + D_{-\sigma}$ provide infinitely many distinct elements of $|2K_X|$, and therefore $P_2 \ge 2$. In particular $\kappa(X) = 1$.

Case (3): $p_g = q' = 0$; then $\chi(\mathcal{O}_X(2K_X)) = 1$. Now as explained at the start of the proof, $p_g = 0$ implies that $H^2(2K_X) = 0$, so that $H^0(2K_X) \ne 0$. Let $0 \le D_2 \overset{\text{lin}}{\sim} 2K_X$ be an effective divisor. If $D_2 = 0$ then I have the second conclusion of the theorem (a classical Enriques surface). Otherwise $D_2 > 0$ implies that in turn $H^2(3K_X) = 0$, so again by RR $H^0(3K_X) \ne 0$. Choose an effective divisor $0 < D_3 \overset{\text{lin}}{\sim} 3K_X$; now obviously $3D_2 = 2D_3$ would imply that $D_2 = 2D_1$ and $D_3 = 3D_1$, with $0 < D_1 \overset{\text{lin}}{\sim} K_X$, contradicting $p_g = 0$. Hence $|6K_X|$ contains two distinct elements $3D_2$ and $2D_3$, which gives $P_6 \ge 2$.

Case (3′): $p_g = q' = 1$, $q = 0$. Suppose first that $K_X \overset{\text{lin}}{\sim} 0$. The argument of [B–M2], p. 39 shows that in this case $\operatorname{char} k = 2$. (Roughly, if $\operatorname{char} k = p$, a simple subgroup scheme μ_p or α_p of the group scheme $\operatorname{Pic}^0 X$ corresponds to a cover $Y \to X$ which is respectively etale and Galois with group $\mathbb{Z}/p$, or a torsor under α_p; then $\chi(\mathcal{O}_Y) = p{\cdot}\chi(\mathcal{O}_X) = p$, but on the other hand one sees that Y is a reduced irreducible surface with $K_Y \overset{\text{lin}}{\sim} 0$, so that $\chi(\mathcal{O}_Y) \le 2$. Thus $\operatorname{char} k = p \le 2$.) In [B–M2], p. 26, Enriques surfaces are defined by $K_X \overset{\text{num}}{\sim} 0$ and $B_2 = 10$, so that cases

(3) and (3′) of Famous Table E.2 are classical and nonclassical Enriques surfaces by definition.

If $K_X \overset{\text{lin}}{\sim} D > 0$, you can prove that P_2, P_3 or $P_4 \geq 2$ as a very instructive exercise in the techniques of E.6, using the assumption $q' = 1$. [Hint: Using the terminology and basic results of E.6.1, write $K_X \overset{\text{lin}}{\sim} D = rE + D_1$, with E a 0-curve, $r \geq 1$ and D_1 disjoint from E. If $D_1 \neq 0$ then D is disconnected, and $h^0(\mathcal{O}_D) \geq 2$; by RR and duality, this implies that $h^0(\omega_D) = h^0(\mathcal{O}_D(K_X + D)) \geq 2$, from which by cohomology $P_2 = h^0(\mathcal{O}_X(K_X + D)) \geq 2$. Otherwise, $D_1 = 0$, so that $K_X \overset{\text{lin}}{\sim} D = rE$, and by adjunction $\mathcal{O}_E((r+1)E) \cong \omega_E \cong \mathcal{O}_E$. Then the exact sequence

$$0 \to \mathcal{O}_E(-(r+1)E) \to \mathcal{O}_{(r+2)E} \to \mathcal{O}_{(r+1)E} \to 0$$

implies that $h^0(\mathcal{O}_{(r+2)E}) \geq 2$. Thus also $h^0(\mathcal{O}_{(r+2)E}(K_X + (r+2)E)) \geq 2$. From this, using the assumption $q' = 1$ and cohomology, $|K_X + (r+2)E| = |(2r+2)E|$ moves, so that $P_m \geq 2$ for all $m \geq 4$ (and for all $m \geq 3$ if $r > 1$).] Q.E.D.

Example. In every case except (3) and (3′) I've proved that $P_2 \geq 2$. An elliptic surface $f\colon X \to \mathbb{P}^1$ with a double fibre $F_2 = 2E_2$ and a triple fibre $F_3 = 3E_3$, and whose Jacobian fibration is a rational elliptic surface has $\chi(\mathcal{O}_X) = 1$ and

$$K_X = f^*\mathcal{O}(-1) + E_2 + 2E_3 = -F + \frac{1}{2}F_2 + \frac{2}{3}F_3;$$

it's easy to see that X has $p_g = 0$, $P_2 = \cdots = P_5 = 1$, $P_6 = 2$. (Compare with Champion E.7.4.)

E.5. Digression: the case $\chi(\mathcal{O}_X) = 0$ in characteristic 0

This section is not needed for the main proof, and is intended just to put the difficulty in its place.

In characteristic 0, $c_2 \geq 0$ (for example by Bogomolov's inequality), so also $\chi(\mathcal{O}_X) = (1/12)c_2 \geq 0$. The arguments given above in the case $\chi(\mathcal{O}_X) > 0$ are rather simple, so treating the remaining cases $K_X^2 = c_2 = 0$ should be considered as the heart of the classification of surfaces. The thing that makes the proof awkward in these cases is that RR gives $\chi(\mathcal{F}) = 0$ for any naturally occurring sheaf on X (compare the discussion in E.9.3). Curiously, although in this case the proof of classification is quite involved, the answer itself is very simple (see E.8.6 for the proof):

E.5.1. Theorem. *If X is an algebraic surface over $\mathbb{C}$ such that $K_X^2 = 0$ and $c_2 = 0$ then X is either an Abelian surface, or a surface of the form $X = (C \times E)/G$, where C is a curve of genus ≥ 1, E a curve of genus 1, and G is a finite Abelian group acting anyhow on C, and on E by translations, in such a way that the diagonal action on $C \times E$ is free.*

E.5.2. Definition. I propose to call the surfaces $(C \times E)/G$ *sesquielliptic surfaces.* When $K_X \overset{\text{num}}{\sim} 0$ then $g(C) = 1$, so that the surfaces here are the *bielliptic surfaces* which occupy the position $p_g = 0$, $q = 1$ in the classification of surfaces with $K_X \overset{\text{num}}{\sim} 0$ (see [Beauville], VI.19–20 and [B–M2], §3). The name is good, since they are characterised by having two elliptic fibrations, to $\mathbb{P}^1$ and $\operatorname{Alb} X$. A sesquielliptic surface also has two projections to C/G and $E/G = \operatorname{Alb} X$, the first of which is an elliptic pencil; (according to the dictionary, the prefix "sesqui-", meaning $1\frac{1}{2}$, is a contraction of Latin *semi* + *que* (and)).

E.5.3. Theorem. *A surface X with $p_g = 0, q = 1$ is sesquielliptic. A sesquielliptic surface $(C \times E)/G$ has $\nu = 0$ if $g(C) = 1$, and $\nu = 1$ if $g(C) \geq 2$.*

Note that the sesquielliptic conclusion can be interpreted as saying that X is a quotient of $\mathbb{C} \times \mathbb{C}$ or $\mathbb{C} \times \mathcal{H}$ (where $\mathcal{H}$ is the complex upper half-plane). Thus Theorem E.5.1 says that $c_1^2(X) = c_2(X) = 0$ implies that X has a locally homogeneous differential geometric structure. It would be reasonable to look for a differential geometric proof of this fact. See E.9.4 for a wild discussion.

E.6. The existence of an elliptic pencil

I now give the proof of abundance: if $\nu(X) = 1$ then X has an elliptic pencil. The proof breaks up into two big steps: (I) X has a 0-curve E (an effective divisor having the numerical properties of a nonmultiple fibre of an elliptic pencil); and (II) if K_X is nef and E a 0-curve then a multiple of E moves in an elliptic pencil.

It's not necessary to assume char $k = 0$, or anything about the values of c_2 or p_g, although if $p_g \geq 2$ or $c_2 > 0$ everything has been proved in the preceding section.

E.6.1. 0-curves. I write down some preliminary material, leaving the easy proofs to the reader. For details, see [Beauville], Chapter VIII, or [3 authors], or [M3], p. 332 (or work them out for yourself).

(a) A divisor *of elliptic fibre type* is a nef divisor $D \overset{\text{num}}{\not\sim} 0$ with $D^2 = DK_X = 0$. It follows by the Index Theorem that $K_X \overset{\text{num}}{\sim} aD$ with $a \in \mathbb{Q}$ and $a \geq 0$. If $D = \sum n_i C_i$ is effective then $DC_i = K_X C_i = 0$ for every component C_i; it is easy to see from this that the intersection matrix $(C_i C_j)$ is negative semidefinite (see Theorem A.7, (2)). In other words, D has the numerical properties of a sum of fibres of an elliptic fibre space.

(b) A 0-*curve* is an effective divisor $E = \sum n_i C_i$ of elliptic fibre type for which $\operatorname{Supp} E$ is connected and $\operatorname{hcf}(n_i) = 1$. Then E is a Gorenstein curve with $\omega_E \cong \mathcal{O}_E$ and $H^0(\mathcal{O}_E) = 1$; E behaves throughout in every respect as if it were an irreducible nonsingular elliptic curve; the crucial property is that for any line bundle $\mathcal{L}$ on E,

$$\deg_{C_i} \mathcal{L} = 0 \text{ for all } i \text{ and } H^0(\mathcal{L}) \neq 0 \quad \Longrightarrow \quad \mathcal{L} \cong \mathcal{O}_E.$$

(c) If $E = \sum n_i C_i$ is a 0-curve and D an effective divisor on X with $DC_i = 0$ for all components of E then

$$D = rE + D',$$

with $r \geq 0$ and D' disjoint from E; in particular the line bundle $\mathcal{O}_X(D')$ is trivial in a neighbourhood of E, so that $\mathcal{O}_{aE}(D') \cong \mathcal{O}_{aE}$ for every $a > 0$.

(d) Any effective divisor D of elliptic fibre type is a sum of 0-curves $D = \sum n_i E_i$.

Mumford's rather regrettable terminology is "canonical type" for "elliptic fibre type" and "indecomposable divisor of canonical type" for "0-curve".

E.6.2. Remark. Miyaoka's inequality for log surfaces [Miyaoka] implies that in characteristic 0, a surface with K_X nef, $K_X^2 = 0$ and $c_2 = 0$ does not contain -2-curves or nodal or cuspidal elliptic curves. So as far as the main case char $k = c_2 = 0$ is concerned, a 0-curve E actually is a nonsingular elliptic curve.

E.6.3. Step I, the existence of a 0-curve. Since $\nu(X) = 1$, it is enough to prove that there exists an effective divisor numerically equivalent to mK_X with $m > 0$, and then apply (d). If $p_g \neq 0$ this is trivial. If $p_g = 0$ then by (3) and (5) of Famous Table E.2, $q' = q \leq 1$; as above, if $q' = 0$ and $K_X \overset{\text{num}}{\not\sim} 0$ then $P_2 > 0$ by RR. This leaves only the case $p_g = 0$, $q' = q = 1$. Then $A = \operatorname{Pic}^0 X$ is reduced, so an elliptic curve.

E.6.4. Proposition. *$H^0(K_X + \sigma) \neq 0$ for some $\sigma \in \operatorname{Pic}^0 X$.*

Proof. The Albanese variety is the dual Abelian variety $\operatorname{Alb} X = A^\vee = A$; write $0 \in A$ for the zero of the group law. On the product $A \times A$, consider the divisor

$$L = \Delta_A - A \times 0 - 0 \times A.$$

L can be thought of as the universal family parametrising line bundles of degree 0 on A, since for each $a \in A$, restricting $\mathcal{O}_{A\times A}(L)$ to $A \times a$ gives $\mathcal{O}_A(a - 0)$.

Consider the Albanese map $\alpha\colon X \to A$ and the pullback $L' = (\alpha \times \mathrm{id})^* L$ via

$$\alpha \times \mathrm{id}\colon X \times A \to A \times A.$$

This is the universal line bundle on $X \times \operatorname{Pic}^0 X$, essentially by definition of the Albanese morphism. Note that $L^2 = -2$, so that $(L')^2$ on $X \times A$ works out as $-2F$, where F is a fibre of α.

E.6.5. Base change. Let p, q denote the projections of $X \times \operatorname{Pic}^0 X$ to its factors. The idea of the proof is to compute $\chi = \chi(X \times A, p^*K_X + L')$ in two different ways: first, by RR on a 3-fold and the following easy calculation

$$\begin{aligned}\chi(X \times A, p^*K_X + L') &= \operatorname{ch}(p^*K_X + L') \cdot \operatorname{Td}_{X\times A}\\ &= \frac{1}{6}(p^*K_X + L')^3 - \frac{1}{4}K_X(p^*K_X + L')^2\\ &= \frac{1}{4}K_X(L')^2 = 1 - f\end{aligned}$$

(the other terms all vanish for simple reasons), one sees that $\chi = 1 - f \leq 0$, where $f = p_aF$ is the genus of a fibre of α. Secondly, by the base change theorem, there is a complex

$$M^\cdot : 0 \to M^0 \xrightarrow{a} M^1 \xrightarrow{b} M^2 \to 0$$

of vector bundles on $A = \operatorname{Pic}^0 X$ such that $R^iq_*(p^*K_X + L')$ is the homology of $M^\cdot$. (To construct $M^\cdot$, take a Čech complex that computes $R^iq_*(p^*K_X + L')$, then use the fact that any bounded complex with coherent cohomology is quasi-isomorphic to a complex of vector bundles; you can truncate the complex so that $M^i = 0$ whenever $H^i = 0$ just by taking ker and coker of maps of constant rank. See [M4], Chapter 2, §5 or [H1], Chapter III.)

Now a and b are maps between locally free sheaves, and a is injective since $H^0(K_X + \sigma) = 0$ for general $\sigma \in A$. The rank of a drops at $\sigma \in A$ if and only if $H^0(K_X + \sigma) \neq 0$, and that of b if and only if $H^2(K_X + \sigma) \neq 0$; either of these only happens at finitely many points $\sigma \in A$, so that $R^iq_*(p^*K_X + L')$ is of finite length; write $r^i(\sigma)$ for the length of the stalk of the sheaf $R^iq_*(p^*K_X + L')$ at σ. Then the Leray spectral sequence gives

$$\chi = \sum_{\sigma\in A} -r^1(\sigma) + r^2(\sigma).$$

Now $r^2(0) \neq 0$ (since $h^2(K_X) = h^0(\mathcal{O}_X) = 1$), and so $\chi \leq 0$ implies $r^1(\sigma) \neq 0$ for some $\sigma \in A$; then the rank of a drops there, as required. Q.E.D.

E.6.6. Enriques' argument. If the Albanese map $\alpha\colon X \to \operatorname{Alb} X$ has fibre genus $f = 1$ then the existence of an elliptic pencil is established. At this point Enriques (see [M3], p. 331) argues that if $f \geq 2$ and $F = \alpha^{-1}P$ is a general fibre then the linear system $|2K_X + F|$ contains an element vanishing on some other fibre F'; hence $|2K_X + \sigma| \neq \emptyset$ where $\sigma = F - F' \in \operatorname{Pic}^0 X$. Enriques' argument is more geometric and much more picturesque, but it depends on one further dichotomy, and the result it proves directly is slightly weaker than Proposition E.6.4.

Step II, a 0-curve moves.

E.6.7. Lemma. *Let $E = \sum n_i C_i$ be a 0-curve and D a divisor with $DC_i = 0$ for all i; suppose that n, $b > 0$. Then the exact sequence*

$$0 \to \mathcal{O}_{(n+b-1)E}(D + (b-1)E) \to \mathcal{O}_{(n+b)E}(D + bE) \to \mathcal{O}_E(D + bE) \to 0$$

gives rise to an inclusion $H^0(\mathcal{O}_{(n+b-1)E}(D + (b-1)E)) \subset H^0(\mathcal{O}_{(n+b)E}(D + bE))$, and

$$\text{this inclusion is strict} \quad \iff \quad \mathcal{O}_{(n+b)E}(D + bE) \cong \mathcal{O}_{(n+b)E}.$$

Proof. If there exists a section s having nonzero restriction to E, then s is a global basis of $\mathcal{O}_{(n+b)E}(D + bE)$ by E.6.1, (b). Q.E.D.

E.6.8. Corollary. *If the inclusion $H^0(\mathcal{O}_{nE}(D)) \subsetneq H^0(\mathcal{O}_{(n+b)E}(D+bE))$ is strict then there exists b' with $0 < b' \leq b$ such that*

$$\mathcal{O}_{(n+b')E}(D + b'E) \cong \mathcal{O}_{(n+b')E}.$$

Proof. At least one of the inclusions in the chain

$$\begin{aligned} H^0(\mathcal{O}_{nE}(D)) \subset \cdots \subset H^0(\mathcal{O}_{(n'-1)E}(D + (n' - n - 1)E)) \\ \subset H^0(\mathcal{O}_{n'E}(D + (n' - n)E)) \subset \cdots \end{aligned}$$

must be strict, so that this follows from Lemma E.6.7. Q.E.D.

E.6.9. Theorem. *Let X be a surface with K_X nef and E a 0-curve on X.*

(i) For every $\mu \geq 2$,

$$H^0(\mathcal{O}_X(\mu(K_X + E))) \neq 0.$$

(ii) There exist effective divisors D_1, D_2 disjoint from E such that

$$\mu K_X + \nu E \overset{\text{lin}}{\sim} D_1 - D_2.$$

with $\mu, \nu \in \mathbb{Z}$ and $\mu > \nu$, $\mu > 0$. In particular

$$\mathcal{O}_{nE}(\mu K_X + \nu E) \cong \mathcal{O}_{nE} \quad \textit{for all } n \geq 1.$$

(iii) The main point: (ii) implies that

$$H^0(\mathcal{O}_{nE}(K_X + nE)) \to \infty \quad \textit{as} \quad n \to \infty;$$

therefore eventually $h^0(\mathcal{O}_{nE}(K_X+nE)) \geq h^1(\mathcal{O}_X)+2$, *and hence by cohomology*

$$\dim |K_X+nE| \geq 1.$$

E.6.10. *Proof.* (i) Consider the exact sequence

$$0 \to \mathcal{O}_X(\mu K_X + (\mu-1)E) \to \mathcal{O}_X(\mu(K_X+E)) \to \mathcal{O}_E \to 0$$

(where I substitute $\mathcal{O}_E(K_X+E) = \omega_E \cong \mathcal{O}_E$ in the last term). Since $H^1(\mathcal{O}_E) \neq 0$ and $H^2(\mathcal{O}_X(\mu K_X + (\mu-1)E)) = 0$ for $\mu \geq 2$ (by duality, using the fact that K_X is nef), it follows that $h^1(\mathcal{O}_X(\mu(K_X+E))) \geq 1$, and therefore

$$h^0(\mathcal{O}_X(\mu(K_X+E))) = \chi(\mathcal{O}_X) + h^1(\mathcal{O}_X(\mu(K_X+E))) \geq \chi(\mathcal{O}_X)+1.$$

Now if $p_g \neq 0$ then $\mu(K_X+E) = \mu K_X + \mu E$ is obviously effective for $\mu \geq 1$, and there is no problem. If $p_g = 0$ then by (3) and (5) of Famous Table E.2, $\chi(\mathcal{O}_X) \geq 0$. Thus in any case $|\mu(K_X+E)| \neq \emptyset$. Q.E.D.

(ii) Let $D \in |\mu(K_X+E)|$; then by E.6.1, (c),

$$D = rE + D_1,$$

with $r \geq 0$ and D_1 disjoint from E. Now I have a dichotomy:
Case $r > 0$. Then (ii) is satisfied with $\nu = \mu - r$.
Case $r = 0$. Then $D_1 = D \neq 0$, so that it contains a 0-curve E_1 disjoint from E; in this case apply (i) to E_1 to get $D' \in |\mu(K_X+E_1)|$; if

$$D' = sE + D_2$$

with $s \geq 0$ and D_2 disjoint from E then

$$\mu K_X - sE \overset{\text{lin}}{\sim} D_2 - \mu E_1,$$

which proves (ii) in this case. Q.E.D.

E.6.11. *Proof of* (ii) $\implies$ (iii). Let E be a 0-curve. By Lemma E.6.7, in the chain

$$0 \subset H^0(\mathcal{O}_E(K_X+E)) \subset \cdots \subset H^0(\mathcal{O}_{(n-1)E}(K_X+(n-1)E)) \\ \subset H^0(\mathcal{O}_{nE}(K_X+nE)) \subset \cdots,$$

the inclusion from $(n-1)$ up to n is strict if and only if $\mathcal{O}_{nE}(K_X+nE) \cong \mathcal{O}_{nE}$; thus (iii) says exactly that this coincidence occurs infinitely often. It first occurs when $n=1$, since $\omega_E = \mathcal{O}_E(K_X+E) \cong \mathcal{O}_E$. It's clearly enough to prove the following:

E.6.12. Claim. *Suppose that* $\mathcal{O}_{nE}(K_X+nE) \cong \mathcal{O}_{nE}$ *for some* $n \geq 1$. *Then*

$$\mathcal{O}_{n'E}(K_X+n'E) \cong \mathcal{O}_{n'E}$$

for some $n' > n$.

To prove this, note that by (ii), also $\mathcal{O}_{nE}(\mu K_X + \nu E) \cong \mathcal{O}_{nE}$. Thus

$$\mathcal{O}_{nE} \cong \mathcal{O}_{nE}(\mu(K_X+nE) - (\mu K_X + \nu E)) \cong \mathcal{O}_{nE}(bE), \quad \text{where } b = \mu n - \nu > 0.$$

(notice that $\mu > \nu$ is needed for the case $n=1$). The claim therefore follows from the next lemma.

E.6.13. Lemma. *Let E be a 0-curve and n, $b > 0$. Suppose that $\mathcal{O}_{nE}(bE) \cong \mathcal{O}_{nE}$; then there exists n' with $n < n' \le n + b$ such that*

$$\omega_{n'E} = \mathcal{O}_{n'E}(K_X + n'E) \cong \mathcal{O}_{n'E}.$$

Proof. The restriction map

$$H^0(\mathcal{O}_{(n+b)E}) \to H^0(\mathcal{O}_{bE})$$

is nonzero (because of the constant sections). Therefore I have a strict inclusion

$$H^0(\mathcal{O}_{nE}(-bE)) \subsetneq H^0(\mathcal{O}_{(n+b)E}),$$

and in particular $h^0(\mathcal{O}_{nE}(-bE)) < h^0(\mathcal{O}_{(n+b)E})$. By RR and duality,

$$\begin{aligned} h^0(\mathcal{O}_{nE}(-bE)) &= h^1(\mathcal{O}_{nE}(-bE)) = h^0(\omega_{nE}(bE)) \\ h^0(\mathcal{O}_{(n+b)E}) &= \ h^1(\mathcal{O}_{(n+b)E}) \ = h^0(\omega_{(n+b)E}), \end{aligned}$$

so that the preceding inequality gives $h^0(\omega_{nE}(bE)) < h^0(\omega_{(n+b)E})$. Now using the assumption $\mathcal{O}_{nE}(bE) \cong \mathcal{O}_{nE}$, I conclude that the inclusion

$$H^0(\mathcal{O}_{nE}(K_X + nE)) \subset H^0(\mathcal{O}_{(n+b)E}(K_X + (n+b)E)$$

is also strict. The lemma therefore follows by Corollary E.6.8. Q.E.D.

This completes the proof of Theorem E.6.9.

E.6.14. Remarks. (i) Every step of the proof is absolutely trivial, so what's going on? The subtle point is the curious argument in E.6.13

strict inclusion of vector spaces
$\Longrightarrow$ strict numerical inequality
$\Longrightarrow$ (by RR and duality) another strict numerical inequality
$\Longrightarrow$ another strict inclusion of vector spaces;

thus the bald statement that $h^0(\mathcal{O}_{nE}(-bE)) = h^0(\omega_{nE}(bE))$ conceals the fact that both $H^0(\mathcal{O}_{nE}(-bE))$ and $H^0(\omega_{nE}(bE))$ are modules over $H^0(\mathcal{O}_{nE})$, and the module structures contain nontrivial information about the nilpotents of the scheme nE. Hence the essence of the proof is using the numerical fact given by RR and duality to relate the two different nilpotent structures of the sheaves $\mathcal{O}_{(n+b)E}$ and $\omega_{(n+b)E}$.

(ii) The above argument is adapted from [M3], pp. 334–5. Mumford uses only $\mu = 1$ and applies duality 4 or 6 times more than necessary (for example, he starts his argument by saying $H^2(\mathcal{O}_X) \to H^2(\mathcal{O}_X(rE))$ is not injective, which can more simply be read $|K_X| \ni D = rE + D'$ with D' disjoint from E).

E.7. Effective growth of plurigenera

The remaining issues in Main Theorem E.1 are the effective results on the P_m, and the finer structure results, especially those for $\nu = 0$. As remarked in E.4, if $p_g \geq 2$ or if $\chi(\mathcal{O}_X) \geq 1$ then everything has already been proved. The structure results are treated in E.8. The remaining assertions divide up as follows:

E.7.1. Theorem (see especially [B–M2]).

(1) *Case $\nu = 0$. Then $mK \overset{\text{lin}}{\sim} 0$ for some $m \in \{1, 2, 3, 4, 6\}$.*
(2) *Case $\nu = 1$. Then $P_m \geq 1$ for some $m \in \{1, 2, 3, 4, 6\}$.*
(3) *Case $\nu = 1$. $P_m \geq 2$ for some $m \leq 42$.*

Discussion of proof of 1 and 2. There is nothing to prove if $p_g = 1$, so we are in Case (5) of Famous Table E.2, that is, $p_g = 0$, $q = q' = 1$. The proof works in either case by applying the adjunction formula (canonical bundle formula) to a suitably chosen elliptic fibre space $X \to B$. You are almost certainly guessing that I'm going to use the Albanese morphism $\alpha\colon X \to A$, which is a map to an elliptic curve; however, it is a striking observation that *this never works.* Indeed, in the case $K_X \overset{\text{lin}}{\sim} 0$ the adjunction formula gives $K_X = \alpha^* L$ with $\deg L = 0$, but at this stage of the argument there's no reason why L should be torsion, so there is no conclusion to be drawn. In the case $\nu = 1$, we don't know that the Albanese map is an elliptic fibre space (in fact it never is).

In the case $\nu = 1$, I use the elliptic fibre space $X \to B$ provided by abundance (E.6), and in the case $\nu = 0$, the following result, which is the first step in the analysis of bielliptic and quasi-bielliptic surfaces.

E.7.2. Theorem. *If $K_X \overset{\text{num}}{\sim} 0$ and $p_g = 0$, $q = q' = 1$ then X has an elliptic fibration $\varphi\colon X \to \mathbb{P}^1$* different from the Albanese morphism.

Proof. Given $B_2 = 2$ and the structure of the inner product on $\mathrm{NS}(X)$, if F is the fibre of α, it's not hard to find a class E with $EF > 0$ and $E^2 = 0$. Then, because $\chi(\mathcal{O}_X) = EK_X = E^2$, arguing exactly as in E.6.4–5 gives that

$$\chi(X \times A, p^*E + L') = \frac{1}{6}(p^*E + L')^3 = \frac{1}{2}p^*E(L')^2 = -EF,$$

so that the Leray spectral sequence implies $H^0(X, \mathcal{O}_X(E+\sigma)) \neq 0$ for some element $\sigma \in \mathrm{Pic}^0 X$. Thus E is numerically equivalent to an effective divisor. Now some multiple of E moves in a pencil, by Theorem E.6.9. It necessarily maps to $\mathbb{P}^1$, since otherwise $q \geq 2$. Q.E.D.

E.7.3. Vertical divisors on X and fractional divisors on B. The proof of Theorem E.7.1, 1–2 is based on the adjunction formula for the canonical bundle of an elliptic fibration $\varphi\colon X \to B$. For clarity, I start with the *tame* case, which is defined by the condition that $R^1\varphi_*\mathcal{O}_X$ is torsion free. Then $R^1\varphi_*\mathcal{O}_X = L^{-1}$, where L is a line bundle with $\deg L = \chi(\mathcal{O}_X)$, and the adjunction formula for φ is simply Kodaira's canonical bundle formula

$$K_X = \varphi^*(K_B + L) + \sum_i (m_i - 1)E_i, \tag{$*$}$$

where $F_i = m_i E_i$ are all the multiple fibres (see for example [3 authors], Theorem V.12.1). In all our cases, $\chi(\mathcal{O}_X) = 0$, so that $\deg L = 0$ in $(*)$.

All the calculations in what follows take place at the level of fractional divisors on the curve B. To explain this, I write a divisor on X of the form $f^*D+\sum a_iE_i$ as the pullback $f^*\Delta$ of the fractional divisor on B of the form $\Delta = D+\sum(a_i/m_i)Q_i$. If $B = \mathbb{P}^1$ and $\deg D = d$ then $D \overset{\text{lin}}{\sim} dQ$ for a chosen general point $Q \in \mathbb{P}^1$, so that I write $\Delta = dQ+\sum(a_i/m_i)Q_i$. Note that the divisorial sheaf associated with a $\mathbb{Q}$-divisor is defined by $\mathcal{O}_B(m\Delta) = \mathcal{O}_B([m\Delta])$, where $[\ ]$ is the integral part or round-down. On B, this says that rational functions cannot use up a fractional allocation of poles, and on X, it corresponds exactly to the fact that only multiples of m_iE_i can move in a linear system made up of fibres.

You'll be able to follow the argument much better if you work out the details of the following example. (Compare Katsura and Ueno [K–U].)

E.7.4. Champion!. The worst case, when the estimates $P_6 \geq 1$ and $P_{42} \geq 2$ are best possible, arises as follows: let $X \to \mathbb{P}^1$ be an elliptic surface having trivial Jacobian fibration (that is $J \cong E\times\mathbb{P}^1$ with E an elliptic curve), and having multiple fibres $F_2 = 2E_2$, $F_3 = 3E_3$, $F_7 = 7E_7$. Then just as in the example of E.4,

$$K_X = f^*\mathcal{O}_{\mathbb{P}^1}(-2) + E_2 + 2E_3 + 6E_7 = f^*\Delta,$$

where Δ is the $\mathbb{Q}$-divisor

$$\Delta = -2Q + (1/2)Q_2 + (2/3)Q_3 + (6/7)Q_7$$

on $\mathbb{P}^1$. Then $H^0(mK_X) = H^0(\mathcal{O}_{\mathbb{P}^1}(m\Delta))$, and the canonical ring of X is the graded ring $R(\mathbb{P}^1, D)$. An elementary (but fairly long) calculation shows that this ring is the graded ring $k[x, y, z]/(f)$, where x, y, z have weights 6, 14, 21 and $f = x^7 + y^3 + z^2$. In particular $P_i = 0$ for all $i \leq 5$, $P_i \leq 1$ for all $i \leq 41$. (Also $P_{43} = 0$ and $P_{85} = 1$. This is responsible for the slightly obscure statement that $P_m \geq 2$ for all $m \geq 86$ that sometimes appears in the literature.)

Note that this example shows that the apparently ineffective aspect of the proof in E.6.9–13 is not actually very far from the truth.

E.7.5. Proof of 1 and 2, the tame case. For the proof of 1, I apply $(*)$ to the alternative fibration $\varphi\colon X \to \mathbb{P}^1$ obtained in Theorem E.7.2. Then K_X is the pullback of the fractional divisor

$$\Delta = -2Q + \sum \frac{m_i - 1}{m_i} Q_i \quad \text{on } \mathbb{P}^1,$$

where the $Q_i \in \mathbb{P}^1$ are distinct points, and each $m_i \geq 2$. Clearly $K_X \overset{\text{num}}{\sim} 0$ if and only if the fractional divisor has degree 0, that is,

$$-2 + \sum_i \frac{m_i - 1}{m_i} = 0 \quad \text{or} \quad \sum \frac{1}{m_i} = (\text{number of } i) - 2$$

This obviously has exactly 4 solutions

$$(2, 2, 2, 2), \quad (3, 3, 3), \quad (2, 4, 4), \quad (2, 3, 6).$$

Thus $mK_X \overset{\text{lin}}{\sim} 0$ for m respectively $2, 3, 4, 6$. This proves 1.

To prove 2, I apply $(*)$ to the elliptic or quasielliptic fibre space $X \to B$ provided by abundance, that is, the results of E.6. Thus K_X is the pullback of $\Delta = K_B + L + \sum((m_i - 1)/m_i)Q_i$ and from $\nu = 1$ it follows that $\deg \Delta > 0$. The assumptions of Case (5) imply immediately that $\deg L = \chi(\mathcal{O}_X) = 0$ and $g(B) \le 1$.

Because K_X is nef and nonzero, if $g(B) = 1$ there must be at least one multiple fibre; then every term $2((m_i - 1)/m_i)Q_i$ makes a positive contribution to $[2\Delta]$, so that $\deg[2\Delta] > 0$, and RR on B gives $P_2 = H^0(\mathcal{O}_B([2\Delta]) \ge 1$. (Similarly, $P_3 > 0$ and $P_m \ge 2$ for every $m \ge 4$. Here and below, results like this leak out for free, and are used in the proof of Theorem E.7.1, 3.)

The other possibility is that $B \cong \mathbb{P}^1$. Then K_X is the pullback of $\Delta = -2Q + \sum(m_i - 1/m_i)Q_i$, and K_X nef and nonzero gives

$$-2 + \sum_i \frac{m_i - 1}{m_i} > 0, \quad \text{that is,} \quad \sum \frac{1}{m_i} < (\text{number of } i) - 2$$

Now every point Q_i makes a positive integral contribution to 2Δ, so that $P_2 \ge 2$ if there are at least 5 points Q_i. If there are 4 points, at least one must be ≥ 3, and $P_2 \ge 1$, $P_4 \ge 2$. The final part of the proof is a pleasurable exercise. [Hint: If all $m_i \ge 3$, the smallest case is $(3, 3, 4)$, giving

$$3K_X \overset{\text{lin}}{\sim} \varphi^*(-6Q + 2Q_1 + 2Q_2 + (9/4)Q_3) \overset{\text{lin}}{\sim} E_3 = (1/4)F_3,$$

so $P_3 = 1$ and $P_{12} = 2$. Otherwise, $m_1 = 2$, and if both $m_2, m_3 \ge 4$ the smallest case is $(2, 4, 5)$, giving

$$4K_X \overset{\text{lin}}{\sim} \varphi^*(-8Q + 2Q_1 + 3Q_2 + (16/5)Q_3) \overset{\text{lin}}{\sim} E_3 = (1/5)F_3,$$

so that $P_4 = 1$ and $P_{20} = 2$. Otherwise, $m_1 = 2, m_2 = 3$, and $m_3 \ge 7$. This leads to Champion E.7.4.]

E.7.6. Exercise. Prove that $P_{12} \le 1$ happens only in 7 cases:

$$(2,5,5) \quad (2,4,5) \quad \text{and} \quad (2,3,m) \text{ with } m \le 11.$$

E.7.7. Wild!. Once the terminology and basic properties are established, the proofs in the wild case are essentially the same combinatorial arguments as in E.7.3–6.

Let $\varphi\colon X \to B$ be an elliptic or quasielliptic fibration and $F = mE = \varphi^* P$ a multiple fibre. Following [B–M2], I introduce the following three numerical characteristics of F: the *multiplicity* m, the *order* n of $\mathcal{O}_E(E)$ as a torsion element of $\operatorname{Pic}^0 E$, and the *local canonical invariant*, that is, the integer $a < n$ such that $\mathcal{O}_X(K_X) \cong \mathcal{O}_X(aE)$ in a neighbourhood of F.

In characteristic zero, automatically $a = m - 1$ and $n = m$. Indeed, if t_P is a local parameter at P on B, then locally near E, I have $\operatorname{div}(t_P) = mE$ and $\operatorname{div}(\mathrm{d}t_P) = (m - 1)E$. Also taking the mth root of t_P and normalising gives rise to an etale cyclic cover of a neighbourhood of F such that the inverse image of F is connected. On the other hand, the restriction to E is the cyclic cover defined by $\mathcal{O}_E(E)$, hence this has order m.

Remark. The argument here is essentially topological, and can be viewed as a beautiful illustration of Kollár's philosophy (discussed in 3.12) that *vanishing* is when a coherent cohomology group has a topological interpretation: in this case, $H^0(\mathcal{O}_E(iE)) = 0$ for all $0 < i < m$ because the inverse image of E in a cyclic cover is connected.

In characteristic p, the same argument applies only to the part of m coprime to p: if $m = m'p^a$ with m' not divisible by p then the m'th root of t_P still defines an etale cyclic cover in which F remains connected; therefore the torsion order n is given by $n = m'p^b$ for some $b \le a$.

Quite generally, a multiple fibre of $X \to B$ can be reduced to a nonmultiple fibre by taking the normalised pullback by a suitable Galois separable ramified cover $B' \to B$ of the base curve, which induces an etale cover of a neighbourhood of $F \subset X$. However, in characteristic p, a ramified cover can have a complicated p-group as local Galois group (ramification group), and in any case, the typical cyclic cover is of the form $x^p + x = a$ (an *Artin–Schreier extension*).

Definition. The fibre F is *tame* if $n = m$ and *wild* if $n < m$. Obviously since both $\mathcal{O}_E(F) = \mathcal{O}_E(mE)$ and $\omega_E = \mathcal{O}_E((a+1)E)$ are trivial, it follows that n divides m and $a + 1$.

If F is a wild fibre, then clearly $h^0(\mathcal{O}_F) \ge 2$, so that also

$$h^1(\mathcal{O}_F) > 1.$$

In particular, the stalk of $R^1\varphi_*\mathcal{O}_X$ at P needs ≥ 2 generators, and it follows that $R^1\varphi_*\mathcal{O}_X = T \oplus L^{-1}$, where T is a torsion sheaf, nonzero at each wild fibre, and L is a line bundle with $\deg L = \chi(\mathcal{O}_X) + \mathrm{length}(T)$. (This follows simply by RR on B.) The adjunction formula in the general case is thus

$$K_X = \varphi^*(K_B + L) + \sum a_i E_i, \tag{$**$}$$

where $F_i = m_i E_i$ are the multiple fibres, and a_i their local canonical invariants.

E.7.8. Lemma. *If $h^1(\mathcal{O}_X) = 1$ then $a = m - 1$ or $a = m - n - 1$.*

Proof. Since $\mathcal{O}_E(nE) = \mathcal{O}_E$, it follows from Lemma E.6.13 that

$$h^0(\mathcal{O}_{n'E}(K + n'E)) \ge 2 \quad \text{for some } n' \text{ with } 1 < n' \le n + 1.$$

Then by cohomology, $H^0(K_X) \subsetneq H^0(K_X + n'E)$, so that some part of the divisor $(a + n')E$ must move off E, that is, $m \le a + n' \le a + n + 1$. Thus $n \ge m - (a + 1)$, but on the other hand, n divides m and $a + 1$. Therefore, $m - (a + 1) = 0$ or n, as asserted. Q.E.D.

E.7.9. Proof of 1 and 2 in the wild case. As before, to prove 1, I work on the alternative fibration $\varphi\colon X \to \mathbb{P}^1$ provided by Theorem E.7.2. In $(**)$ I have $\deg L = \chi(\mathcal{O}_X) + \mathrm{length}(T)$. If $\deg L \ge 2$ then $(**)$ would give $p_g \ge 1$, a contradiction, so it follows that $\deg L = 1$ and there is exactly one wild fibre with $\mathrm{length}(T) = 1$. Set $F_0 = m_0 E_0$ for the wild fibre, n_0 and a_0 for its period and local canonical invariant, and $F_i = m_i E_i$ for $i = 1, \dots, k$ for the tame multiple fibres. Now argue exactly as in E.7.4: K_X is the pullback of the fractional divisor

$$-Q + \frac{a_0}{m_0} Q_0 + \sum_{i=1}^{k} \frac{m_i - 1}{m_i} Q_i.$$

Moreover $a_0 = m_0 - 1$ or $m_0 - n_0 - 1$. In the first case it is an easy exercise to deduce that $K_X \overset{\text{num}}{\sim} 0$ gives $k = 1$ and $(m_0, m_1) = (2, 2)$ so that $P_2 = 1$.

For the second case, it's useful to note that the wild fibre behaves for numerical purposes exactly like two usual multiple fibres of multiplicity m_0 and m_0/n_0: to see this, it is enough to write the above $\mathbb{Q}$-divisor throughout in the form

$$-2Q + \frac{m_0 - 1}{m_0} Q_0 + \frac{m_0/n_0 - 1}{m_0/n_0} Q_0 + \sum_{i=1}^{k} \frac{m_i - 1}{m_i} Q_i.$$

Thus $K_X \overset{\text{num}}{\sim} 0$ gives

$$k = \frac{1}{m_0} + \frac{n_0}{m_0} + \sum_{i=1}^{k} \frac{1}{m_i}.$$

The right-hand side is a sum of reciprocal integers, and an obvious calculation shows that the set of integers $(m_0, m_0/n_0, \{m_i\}_{i=1}^k)$ is one of the usual list:

$$(2,2,2,2) \quad (3,3,3) \quad (2,4,4) \quad (2,3,6).$$

Thus $mK_X \overset{\text{lin}}{\sim} 0$ for $m = 2, 3, 4$ or 6. This proves 1.

For the purposes of this proof, I didn't need to figure out which of the integers is m_0/n_0. Note that since $n_0 = p^a m_0$, where $p = \operatorname{char} k$, the argument shows that for a surface with $\nu = 0$ in Case (5), a wild fibre can only happen in characteristic 2 or 3.

As before, for 2, I work on the pluricanonical elliptic fibration $X \to B$ provided by abundance. The argument is almost identical to previous work. In view of $p_g = 0$, $q = q' = 1$, the adjunction formula $(**)$ gives that $B = \mathbb{P}^1$, $\deg L = 1$ and there is exactly one wild fibre to which Lemma E.7.8 applies. Now K_X is the pullback of the fractional divisor

$$-Q + \frac{a_0}{m_0} Q_0 + \sum_{i=1}^{k} \frac{m_i - 1}{m_i} Q_i.$$

where $a_0 = m_0 - 1$ or $m_0 - n_0 - 1$. In the first case I have

$$-1 + \frac{m_0 - 1}{m_0} + \sum_{i=1}^{k} \frac{m_i - 1}{m_i} > 0 \quad \text{that is,} \quad \sum_{i=0}^{k} \frac{1}{m_i} < k$$

Thus $k \geq 1$. If $k \geq 2$ then $P_2 \geq 2$; otherwise $k = 1$ and one of $m_0, m_1 \geq 3$, so that $P_2 \geq 1$, $P_3 \geq 1$ and $P_6 \geq 2$ (compare Example E.4). In the second case I get

$$-1 + \frac{m_0 - n_0 - 1}{m_0} + \sum_{i=1}^{k} \frac{m_i - 1}{m_i} > 0 \quad \text{that is,} \quad \frac{1}{m_0} + \frac{n_0}{m_0} + \sum_{i=1}^{k} \frac{1}{m_i} < k$$

Here again $k \geq 1$, and the pleasurable exercise at the end of E.7.5 can be repeated verbatim to prove that $P_m \geq 1$ for some $m \leq 6$ and $P_m \geq 2$ for some $m \leq 42$.

E.7.10. Proof of Theorem E.7.1, 3. If $p_g \geq 2$ there is nothing to prove. If $\chi(\mathcal{O}_X) \geq 1$ and $\nu = 1$ then $P_6 \geq 2$ was proved in Theorem E.4. The remaining case with

$p_g = 0$ is Case (5), and everything has been proved in E.7.5 in the tame case, and in E.7.9 in the wild case.

The remaining cases are (4), (5′), which have $p_g = 1$ and $q' = 2$. In this case, the proof consists of using the construction of E.6.9–13 as an *effective* method. Write $K_X = rE + D'$ with E a 0-curve, $r \geq 1$ and $D' > 0$ a divisor disjoint from E. The arguments of E.6.9–13 give $H^0(\mathcal{O}_{(r+2)E}) \geq 2$, and $H^0(\mathcal{O}_{(r+2)E+D'}) \geq 3$ if $D' \neq 0$, so that by RR and duality $H^0(\mathcal{O}_{(r+2)E+D'}(K_X + (r+2)E + D')) \geq 3$, therefore, by the cohomology long exact sequence, $|K_X + (r+2)E + D'|$ moves. In the worst case $r = 1$, this gives $4K_X > K_X + (r+2)E + D'$, so that $P_4 \geq 2$. If $D' = 0$, then I take the argument of Lemma E.6.13 one step further: then $H^0(\mathcal{O}_{bE}(r+b)E) \cong \mathcal{O}_{bE}$ for some $b \leq r+2$, so that $H^0(\mathcal{O}_{(r+2b)E}) \geq 3$ and $|K_X + (r+2b)E|$ moves. Thus in the worst case $D' = 0$, $r = 1$ and $b = 3$, I get that $|8E|$ moves, so that $P_m \geq 2$ for all $m \geq 8$. This completes the proof of Theorem E.7.1. Q.E.D.

E.7.11. Remark. I'm rather disappointed that the arguments of E.6.9–13 don't seem to give the effective bound $P_{42} \geq 2$ also in the cases $p_g = 0$. The interest of the question is that if you could find a better argument here, you might be able to tidy up the main proof of E.6.9–13.

In this case $q = q' \leq 1$, and you could try writing $mK_X = rE + D'$ for some $m = \{1, 2, 3, 4, 6\}$, with D' effective and disjoint from E. Thus

$$\mathcal{O}_E(mK_X - rE) \cong \mathcal{O}_E \cong \mathcal{O}_E(K_X + E) \implies \mathcal{O}_E((m+r)E) \cong \mathcal{O}_E.$$

Now arguing as in Theorem E.6.9, (i), there exists $c \leq m + r + 1$ such that $H^0(\mathcal{O}_{cE}) \geq 2$ and $\mathcal{O}_{cE}(K_X + cE) \cong \mathcal{O}_{cE}$. If $D' \neq 0$, you get as before a curve $cE + D'$ with $H^0(\mathcal{O}_{cE+D'}) \geq 3$, and $|K_X + cE + D'|$ moves by cohomology. In the worst case $m = 6$, $r = 1$, $c = 8$, this gives $P_m \geq 2$ for $m \geq 49$. If $D' = 0$, the next step is to say that $\mathcal{O}_{cE}((mc + r)E) \cong \mathcal{O}_{cE}$, therefore $H^0(\mathcal{O}_{((m+1)c+r)E}) \geq 3$, so that $|K_X + ((m+1)c + r)E|$ moves. In the worst case $m = 6$, $r = 1$, $c = 8$, so that $|K_X + 57E| = |343K_X|$ moves (notice that $343 = 7^3$). Of course, this is far from best possible. The way to improve it would be to show that $\mathcal{O}_{8E}(7E) \cong \mathcal{O}_{8E}$, that is, replace 49 by 7. Probably you can show that the kernel of $\operatorname{Pic}^0(8E) \to \operatorname{Pic}^0 E$ is a vector space by the same kind of arguments as in 4.13, so has no torsion (at least in characteristic $\neq 7$)?

E.8. Abelian, bielliptic and sesquielliptic surfaces

E.8.1. Theorem (characterisation of Abelian surfaces). *Let X be a surface with $K_X \overset{\text{lin}}{\sim} 0$, $p_g = 1$, $q = q' = 2$ (that is, $\nu = 0$ and Case (4) of Famous Table E.2). Then the Albanese map $\alpha\colon X \to \operatorname{Alb} X$ is an isomorphism, so that X is an Abelian surface.*

E.8.2. Proof. Write $\alpha\colon X \to A = \operatorname{Alb} X$ for the Albanese map. Recall that it has the universal mapping property (UMP) for morphisms of X to an Abelian variety. I exclude the possibility that α maps to a curve $C \subset A$ by saying that C has genus 2, and thus has etale covers of large genus; hence, by pullback, X has etale covers with $q > 2$, therefore $p_g > 1$, which contradicts $K_X = 0$.

Consider the differential $d\alpha\colon T_X \to \alpha^* T_A$. In characteristic zero, it is generically an isomorphism, so gives an injective map

$$T_X \hookrightarrow \alpha^* T_A \cong \mathcal{O}_X^{\oplus 2},$$

necessarily an isomorphism because $K_X \overset{\text{lin}}{\sim} 0$; thus α is etale. But an etale cover of an Abelian variety is itself Abelian, so that $X = A$ follows from the UMP of $\alpha\colon X \to A$.

In characteristic p, the same proof goes through unchanged if $d\alpha$ is generically injective. On the other hand, $d\alpha$ is not generically zero. Indeed, $d\alpha = 0$ means that $\alpha^* k(A) \subset k(X)^p$ (the subfield of pth powers of elements of $k(X)$), or equivalently, $k(A)^{1/p} \subset k(X)$. This means that α factors through the geometric Frobenius map $A^{(-1)} \to A$, which again contradicts its UMP.[6]

Therefore I can assume that $d\alpha$ has generic rank 1, and argue on foliations as in Rudakov and Shafarevich [R–S] and [Ekedahl]. The foliation $\ker d\alpha \subset T_X$ is a rank 1 subsheaf, and is necessarily saturated; write $\ker d\alpha = \mathcal{O}_X(D)$. Then there is an exact sequence

$$0 \to \mathcal{O}_X(D) \to T_X \to \mathcal{I}_Z \mathcal{O}_X(-D) \to 0,$$

where $\mathcal{I}_Z$ is the ideal sheaf of a zero dimensional subscheme, and $D \geq 0$, since by assumption there is a nonzero homomorphism $\mathcal{I}_Z \mathcal{O}_X(-D) \to \alpha^* T_A \cong \mathcal{O}_X^{\oplus 2}$.

Consider the subfield of the function field $k(X)$ generated by $k(X)^p$ and $k(A)$, that is, the composite field $K = k(A)k(X)^p$. Then since $[k(X) : k(X)^p] = p^2$ and $k(A) \not\subset k(X)^p$, I deduce that $K \subset k(X)$ is inseparable of degree p. It follows that α factorises as $X \to Y \to A$, where Y is the surface obtained as the normalisation of A in K, the morphism $\pi\colon X \to Y$ is inseparable of degree p, and $\ker d\pi = \mathcal{O}_X(D) \subset T_X$. Therefore $\mathcal{O}_X(D)$ is a p-closed foliation and $Y = X/\mathcal{O}_X(D)$ is the corresponding quotient. (See [R–Sh] for the terminology.)

Case $D > 0$. By the adjunction formula for $X \to Y$ of [R–Sh], §2, Proposition 2 (p. 1211 of translation), $\pi^* K_Y = -(p-1)D$, so that $K_{Y'}$ cannot be nef on any birational model Y' of Y, and Y is rational or ruled by the main result of minimal model theory (Corollary D.4.4). This contradicts $Y \to A$ generically finite.

Case $D = 0$. Then $Z = 0$, since $c_2(X) = 0$. By [R–Sh], §1, Corollary of Theorem 1 (p. 1208 of translation), the surface Y is nonsingular, and $K_Y \overset{\text{num}}{\sim} 0$ by the same adjunction formula. Thus $p_g(Y) \leq 1$; but since $q(Y) \geq 2$, it follows that Y is also in Case (4) of Famous Table E.2, that is, Y satisfies all the same hypotheses as X. I can therefore assume by induction on $\deg(Y/A)$ that $Y = A$, that is, $\alpha\colon X \to A$ is inseparable of degree p.

Moreover, by the exact sequence $0 \to \mathcal{O}_X(D) \to T_X \to \mathcal{I}_Z \mathcal{O}_X(-D) \to 0$, the differential $d\alpha\colon T_X \to \alpha^* T_A \cong \mathcal{O}_X^{\oplus 2}$ has image isomorphic to $\mathcal{O}_X$, which is thus a direct summand. Thus $T_A = L_1 \oplus L_2$ is a direct sum of two trivial line bundles $L_i \cong \mathcal{O}_A$, such that $d\alpha$ maps surjectively to $\alpha^* L_1$.

Now $X \to A$ is inseparable of degree p, so that the inclusion $k(X)^p \subset k(A)$ defines a factorisation $F\colon X \to A \to X^{(1)}$ of the Frobenius. On the other hand, since $d\alpha$ has image $\alpha^* L_1$, and $dF = 0$, it follows that $\beta\colon A \to X^{(1)}$ has $\ker d\beta = L_1$, and β is the quotient by L_1. Therefore L_1 is p-closed.

[6] Here $A^{(-1)}$ is the normalisation of A in the field $k(A)^{1/p}$; it is isomorphic as a scheme to A via the absolute Frobenius, so is an Abelian variety in its own right. The point of the funny notation is mainly to avoid offending experts: $A^{(-1)}$ and the morphism $A^{(-1)} \to A$ are defined over k, and $A^{(-1)}$ is conjugate to A by the Frobenius automorphism of k, but is not isomorphic to A over k.

Now also $L_1 \cong \mathcal{O}_A$, and is a direct summand of T_A, so that L_1 is a p-Lie subalgebra of the algebra of tangent fields on A. The Frobenius map $F\colon A \to A^{(1)}$ is a homomorphism of algebraic group schemes, and the vector space $H^0(T_A)$ is in a natural way the p-Lie algebra corresponding to the finite subgroup scheme $\ker F$. From the fact that L_1 is p-closed, it follows that $H^0(L_1)$ is a p-Lie subalgebra, so that by the Lie correspondence ([M4], Chapter III, §14, Theorem), it is the Lie algebra of a subgroup scheme $G \subset \ker F$. Therefore the quotient by L_1 is the same thing as the quotient by G, and it follows that $X^{(1)}$ is itself an Abelian variety. Therefore X is also an Abelian variety, and the factorisation $X \to A$ contradicts the UMP of α as before. Q.E.D.

E.8.3. Remarks. (1) The proof in [B–M2], pp. 40–41 is an extremely weird reduction to a finite field. My proof (based largely on suggestions of Nick Shepherd-Barron) is an expanded and simplified version of the argument of [Ekedahl], Proposition 4.3.

(2) It's an exercise to generalise the treatment of the final case to prove the following theorem of Igusa and Serre ([Serre], §2.6, Theorem 4): if $f\colon X \to A$ is a generically finite surjective morphism of an n-fold to an Abelian variety, the two conditions (a) and (b) are equivalent:

(a) There exists a 1-form $\omega \in H^0(\Omega^1_A)$ such that $f^*\omega = 0 \in \Omega^1_X$.

(b) f factors via $B \to A$, an inseparable morphism of height 1 between Abelian varieties. (This means that $B \to A$ has degree > 1 and factors the Frobenius morphism $B \to A \to B^{(1)}$; or, more simply, $k(B)^p \subset k(A) \subsetneq k(B)$.)

It follows that if the Albanese map $\alpha\colon X \to \operatorname{Alb} X$ is surjective for any variety X, then its differential is an injective map $\alpha^*\colon H^0(\Omega^1_A) \hookrightarrow H^0(\Omega^1_X)$.

E.8.4. Theorem.

(1) *Case* $K_X \overset{\text{num}}{\sim} 0$, $p_g = 0$, $q = q' = 1$ *(that is,* $\nu = 0$ *and (5)). Then* X *is bielliptic (see Definition E.5.2).*

(2) *Case* $K_X \overset{\text{lin}}{\sim} 0$, $p_g = 1$, $q = 1$, $q' = 2$ *(that is,* $\nu = 0$ *and (5′)). Then* $\operatorname{char} k = 2$ *or* 3, *and* X *is quasi-bielliptic.*

Here quasi-bielliptic means that X is a quasielliptic fibre space $X \to A$ over an elliptic curve, which becomes a $\mathbb{P}^1$-bundle on making an inseparable cover of the base curve.

E.8.5. Sketch proof. In either case (5–5′), consider the Albanese map $\alpha\colon X \to A$, and the alternative fibration $\varphi\colon X \to \mathbb{P}^1$ of Theorem E.7.2. Any fibre E of φ is a curve with $p_aE = 1$ having a finite morphism $\pi\colon E \to A$. Thus π is necessarily an isogeny of elliptic curves (after choosing base points), so that E is a nonsingular elliptic curve, and $E \to A = E/G$ is the quotient by the finite subgroup scheme $G = \ker \pi \subset E$.

Suppose first that α is an elliptic fibration, that is, that the general geometric fibre is nonsingular. Then the adjunction formula (∗∗) applied to α implies that $\deg L = 0$, and all $a_i = 0$, and therefore, every fibre F is the same nonsingular elliptic curve. Now pulling back $X \to A$ by the finite cover $E \to A$ gives an elliptic fibre space $X_E = X \times_A E \to E$ having a section E (the diagonal $\Delta_E \subset E \times_A E$), and with all geometric fibres the same curve F. Therefore X_E is isomorphic to the product $E \times F$ and $X_E \to X$ is a quotient by the subgroup scheme $G \subset E$, acting on E by translation, and acting somehow on F. However, since X has

$mK_X \overset{\text{lin}}{\sim} 0$ for some $m \in \{1,2,3,4,6\}$ it follows that the action of G on $H^0(K_F)$ factors via the group scheme μ_m of mth roots of unity, where m is the order of K_X in $\operatorname{Pic} X$. The kernel of this action is a translation subgroup scheme of $E \times F$, and the corresponding quotient is an Abelian variety. Therefore X is isomorphic to the quotient of an Abelian variety by μ_m.

If $\operatorname{char} k \neq 2$ or 3, it follows that X is an etale quotient of an Abelian variety by a cyclic group of order m. In any case, $m = 1$ would give that X is an Abelian surface, and contradict the case assumption (5–5′). But all surfaces with $\nu = 0$ in Case (5′) have $p_g = 1$, therefore $K_X \overset{\text{lin}}{\sim} 0$, so that α cannot be an elliptic fibration for these surfaces.

Next suppose that $\alpha\colon X \to A$ is a quasielliptic fibre space. As before, $\deg L = 0$ and all $a_i = 0$ in the adjunction formula $(**)$, so that every fibre F of α is a reduced irreducible curve with a cusp. Pulling back by the isogeny $E \to A$ gives a ruled surface with a section, so that the normalised pullback X_E is isomorphic to a $\mathbb{P}^1$-bundle over E, and $X_E \to X$ is again a quotient by a subgroup scheme $G \subset E$. Q.E.D.

E.8.6. Sketch proof of Theorem E.5.1 and E.5.3. Consider a surface X over $\mathbb{C}$ with K_X nef, $\nu = 1$ and $c_2 = 0$. We know that X is an elliptic surface $f\colon X \to B$ with $c_2 = 0$. Hence by the Euler number calculation, the only degenerate fibres are multiple nonsingular fibres, say $E_i = m_i e_i$. In particular the modular invariant of the fibre is bounded, and therefore constant. Hence the Jacobian fibration is trivial, $J = E \times B$. Now the canonical bundle formula gives $K_X = f^*K_B + \sum(m_i - 1)e_i$, and it's easy to see that $p_g(X) = g(B)$, $q(X) = g(B) + 1$. By the complete reducibility property of Abelian varieties, $\operatorname{Alb} X$ has a projection to an elliptic curve A which restricts to a nonconstant morphism from the general fibre E of $X \to B$. In particular, the Albanese morphism $\alpha\colon X \to \operatorname{Alb} X$ never equals the canonical elliptic fibre space $f\colon X \to B$. The normalised pullback $X \times_A E$ is then a product. But $E \to A$ must be Galois with Abelian group, so that X is obtained as stated in the theorem. Q.E.D.

E.9. Any questions or comments?

This chapter has given a detailed, complete, self-contained proof of the main results of the classification of surfaces, following Enriques' argument as rewritten by Mumford. My reason for writing this in this form is my impression that while hundreds of mathematicians (and nowadays also a growing number of theoretical physicists and specialists in science fiction) need to know and use the results of the classification, very few have ever been through a proof. Although the coincidence $\nu = \kappa$ may seem at first sight such an obvious and coarse result, its proof is logically quite intricate and beautiful. Moreover, the role of the bielliptic and sesquielliptic surfaces with $c_2 = 0$ as a logical bottleneck of the proof seems to me to deserve some emphasis.

E.9.1. Abundance as a logical bottleneck. As far as I know, there is only one complete proof of abundance in the literature that does not pass via Enriques' argument at the central point, namely that given in [3 authors]. They first give Ueno's proof of Iitaka's additivity conjecture $C_{2,1}$, which uses the theory of moduli of curves. Recall that if X has $p_g = 0$, $q = 1$ and $K_X^2 = 0$ then X has an Albanese morphism $\alpha\colon X \to \operatorname{Alb} X = A$ to an elliptic curve A, with fibres of genus 1 if $\nu = 0$, or genus ≥ 1 if $\nu = 1$. Then $C_{2,1}$ proves at once that $P_m \geq 1$ for some m, and $P_m > 1$ for some m if the fibres of α have genus ≥ 2, and with a bit more work also if $\nu = 1$.

E.9.2. Finding cohomology when $\chi(\mathcal{F}) = 0$. The central problem in proving abundance is to find nonzero cohomology on X. The point is that

$$\chi(\mathcal{O}_X(nK_X)) = \chi(\mathcal{O}_X) \quad \text{for all } n,$$

so that RR on its own does not imply that $H^0(nK_X) \to \infty$ as $n \to \infty$. But if $P_m \geq 2$ for some m, then $D \in |mK_X|$ will consist of many fibres of the elliptic fibration φ_{mK_X}, so $H^0(\mathcal{O}_D)$ is large. It's interesting to analyse the argument of E.6–7 to note the key points in Proposition E.6.4, Theorem E.6.9 and Proposition E.7 at which one has to work to squeeze out nonzero cohomology one drop at a time.

E.9.3. Finding curves C with $KC = 0$. Abundance can be stated as saying that a suitably defined "moduli space" of maps $\varphi\colon C \to X$ with $K_X\varphi(C) = 0$ has the right dimension. Our experience with Mori theory teaches us that there is essentially only one way of proving that a nonsingular projective variety X with K_X not nef contains rational curves, namely Mori's bending-and-breaking argument: very roughly, a curve $C \subset X$ with $K_XC < 0$ moves in a positive dimensional family (for easy reasons), and must break off a rational component at some point (for delicate easy reasons). Put this crudely, this suggests that a similar argument might be capable of locating the curves with $K_XC = 0$ on a n-fold with K_X nef. Unfortunately, it seems almost certain that this kind of approach cannot work.

The point is that Mori's bending-and-breaking argument actually works with the deformation theory of morphisms $\varphi\colon C \to X$ with C a fixed curve; if you allow the moduli of C to vary, you might gain some parameters, but you completely lose the ability to predict that the family breaks up. Although, if you believe abundance, the moduli space of curves in X with $K_XC = 0$ has the right dimension, it seems impossible to approximate them by curves with K_XC "small".

In the rest of this chapter I discuss three other approaches to the proof of abundance, none of which has been completely worked out (to my knowledge). Assume $k = \mathbb{C}$ for a partial sense of security.

E.9.4. Kähler–Einstein. This is the question left at the end of E.5. Consider minimal surfaces with $K_X^2 = c_2 = 0$. By a theorem of Uhlenbeck–Yau and Donaldson, if we knew that T_X were stable (or a direct sum of stable bundles), it could be given a Hermitian–Einstein metric. Of course, a posteriori this happens only for Abelian surfaces. Nevertheless, it seems to me (as an outsider in the analytic side of differential geometry) quite likely that one can prove a priori that there exists a metric that makes the Kähler structure locally symmetric.

H. Tsuji seems to have some general results on this kind of question in higher dimensions, which I paraphrase, probably rather wildly. Try to construct a "Kähler pseudometric" on the line bundle K_X whose curvature form vanishes along the fibres of the canonical map $\varphi\colon X \to Y$, and is positive in directions transverse to the fibres. Of course, a priori we don't know φ or any of its fibres, but if $K_X^n = 0$ then the curvature of a Kähler pseudometric must vanish on a subspace of the tangent bundle. Then there is a kind of heat equation argument saying that a Kähler pseudometric should exist as a distribution. If you could prove that this exists as an analytic pseudometric then its kernel should give the fibres of φ, at least as a subbundle of T_X or as a complex foliation. Then to prove that the canonical fibration exists is equivalent to proving the regularity of the Kähler pseudometric distribution. (??)

E.9.5. Miyaoka's argument. Let X be a 3-fold with K_X nef and $\nu = 1$ (that is $K_X \overset{\text{num}}{\not\sim} 0$, $K_X^2 \overset{\text{num}}{\sim} 0$), and assume $\kappa \geq 0$ to avoid appealing to Miyaoka's previous hard theorems. Let E be a suitable divisor chosen from the components of a divisor $D \in |mK_X|$, with $E^2 \overset{\text{num}}{\sim} 0$ and $K_{X|E} \overset{\text{num}}{\sim} 0$. Miyaoka proves that some multiple of E moves in a pencil on X. The idea of the proof is to restrict X to a tubular neighbourhood of E; then some easy homotopy theory shows that it is possible to make cyclic covers of X branched along E, and then to pass to a minimal model $F \subset Y$ via Kulikov's results. This model has the following good properties: F is a degenerate Abelian or $K3$ surface (that is, global normal crossings, $K_F \overset{\text{lin}}{\sim} 0$, and all components of F are algebraic surfaces), Y is nonsingular, and $K_{Y|F} \overset{\text{lin}}{\sim} F_{|F} \overset{\text{lin}}{\sim} 0$.

Now F has a local deformation space as an abstract complex space; next, Miyaoka's idea is to prove that there are sufficiently many abstract infinitesimal deformations of F that can be mapped to Y. (Specialists know that Miyaoka's appeal to Friedman's results in deformation theory in the case of degenerate Abelian surfaces is not valid as written, but I understand that his proof has been fixed up by slightly different arguments. Miyaoka's proof contains in any case a whole string of important new ideas. See [Utah2] for the current status of this.)

It seems that in characteristic 0, this argument can be translated back to the surface case without much trouble, and can be used to replace Step II, (E.6.7–13). (??)

E.9.6. There must be a better way. It seems conceivable to me that we are just overlooking some much coarser and much more fundamental reason why abundance must be true for surfaces, an argument that doesn't resort to miserable cookery with the constant term in RR. Here is one attempt at such an argument.

Let X be a surface with $\nu = 1$. Consider the set of curves $E \subset X$ with $K_X E = 0$. By the Hodge index theorem, the intersection form on these is negative semidefinite. Thus either there are only finitely many, or at most a single 1-dimensional family plus finitely many; if there is a positive dimensional family, it must be an elliptic pencil, and everything's lovely. I say that there must be a way of proving directly that there are infinitely many curves E with $K_X E = 0$, or better, that there is a curve E with $K_X E = 0$ through every sufficiently general point $P \in X$.

Let H be a divisor on X with $d = K_X H > 0$. Note that replacing H by $H + nK_X$ does not change $K_X H$, so that there is no loss of generality in assuming that also $H^2 > 0$.

Now resolutely ignoring the constant term in RR gives

$$h^0(\mathcal{O}_X(H + nK_X)) \sim (n/2)d.$$

E.9.7. Conjecture. *There exist integers $n \gg 0$ and $m < \sqrt{(H + nK_X)^2} - 1$, and a decomposition*

$$H + nK_X = D + E \qquad \text{with } D, E > 0 \text{ and } DE \leq m.$$

Moreover, E can be chosen to pass through a sufficiently general point P of X.

The statement itself follows from abundance: just take E to be the fibre of φ_{mK} through P. The conjectural part is that this statement can be proved directly without appealing to abundance.

E.9.8. Lemma. *The conjecture implies abundance.*

Proof. The form $D^2E^2 \le (DE)^2$ of the Hodge index theorem implies at once that D^2 or $E^2 \le 0$, since otherwise

$$(H + nK_X)^2 = D^2 + 2DE + E^2 \le \max_{1\le x\le m^2} \{x + 2m + m^2/x\} = (1 + m)^2$$

contradicts the assumption on m. Exchanging D and E allows me to assume $E^2 \le 0$. Then $D^2 > 0$.

Set $d_1 = K_XD$ and $d_2 = K_XE$, so that $d = d_1 + d_2$ and d_1, $d_2 \ge 0$ (since K_X is nef). Since $D^2 > 0$, the Hodge index theorem gives $d_1 > 0$.

If $d_2 \ne 0$ then $d_1 < d$, and I just replace $H \mapsto D$ and repeat the same argument with a smaller value of d. Since d can only decrease a finite number of times, eventually every decomposition $D+E$ as in the statement of the conjecture satisfies $K_XE = 0$. Q.E.D.

E.9.9. Special clusters. The suggestion for proving that $H+nK_X$ has a decomposition of the required type is in terms of Reider's method. I say that $Z \subset X$ is a *special cluster* of degree m for $|H+(n+1)K_X|$ if Z is a zero dimensional subscheme (for example $Z = P_1 + \cdots + P_m$ with distinct $P_i \in X$) which is special in the sense that it imposes dependent conditions on $|H + (n + 1)K_X|$. In other words, the evaluation homomorphism at Z

$$H^0(H + (n + 1)K_X) \to \mathcal{O}_Z \qquad (*)$$

is not surjective. Z behaves like an analog of a special divisor on a general curve $C \in |H + nK_X|$.

The cokernel of $(*)$ is the group $H^1(I_Z \cdot \mathcal{O}_X(H + (n + 1)K_X))$, which has Serre dual corresponding to extension classes

$$0 \to \mathcal{O}_X \to \mathcal{E} \to I_Z \cdot \mathcal{O}_X(H + nK_X) \to 0;$$

in most cases one can arrange that $\mathcal{E}$ is locally free, so a rank 2 vector bundle.

Now ${c_1}^2(\mathcal{E}) = (H + nK_X)^2$ and $c_2(\mathcal{E}) = m$, so if I take m to be fairly small, $\mathcal{E}$ has the Bogomolov numerical instability property $({c_1}^2 - 4c_2)(\mathcal{E}) > 0$. It's not hard to get from this to a decomposition as in the conjecture.

The hard problem is the existence of special clusters for suitable values of n and m. One can set up a formalism of vector bundles over the Hilbert scheme of clusters, and define locuses W^r_m of special clusters in analogy with the theory of special linear systems on curves. It's easy to do a dimension count in the style of Brill–Noether to prove that suitable locuses W^r_m have dimension > 0, but I don't know how to do the intersection theory to prove that they are nonempty. (??)

E.9.10. Although this proof runs into technical problems, it seems to me to provide at least a possible reason of principle why abundance may be true, not depending on frail numerical considerations: since K_X is not ample, there must be lots of arbitrarily large cohomology groups around; if you can find one large group, e.g. $H^1(I_Z \cdot \mathcal{O}_X(H+nK_X))$, then it should be possible to chase this back to $H^0(nK_X)$.