Chapter D. Minimal models of surfaces via Mori theory

The final two chapters explain the proof of the classification of surfaces. The treatment of the classification of surfaces in these chapters is nonclassical, with emphasis on a new point of view, motivated in part by recent developments in 3-folds. For classical treatments, see [Beauville], or [3 authors].

Summary

D.0. Introduction and preview

Let X be a projective nonsingular surface defined over k; it's my responsibility to make everything work over any algebraically closed field k, but the student who is so inclined may suppose that $k = \mathbb{C}$ to make things easier. The classification of surfaces breaks up into two parts. (For the definition of K_X and of a nef divisor, see the end of D.1 below, or various places earlier in these notes.)

Case K_X not nef. This corresponds to studying the choice of birational model of X, the theory of minimal models and the characterisation of ruled and rational surfaces. Very roughly, the main result is that every surface is birational either to a surface X with K_X nef or to $\mathbb{P}^2$ or $\mathbb{P}^1 \times C$ with C a curve.

The main point to note is the dichotomy: after choosing a suitable birational model, either K_X is nef, or $-K_X$ is ample, or relatively ample on a nontrivial fibration $X \to C$. In this form, the result remains true over a nonclosed field, and is capable of extension to higher dimensions, although of course everything becomes harder.

Case K_X nef. This part is concerned with dividing surfaces into classes according to the numerical properties of K_X, and then proving that this coincides with the classification by the Kodaira dimension $\kappa(X)$. Very roughly, the main result is that X has "sufficiently many" holomorphic differential forms. More precisely:

(i) if $K_X \overset{\text{num}}{\not\sim} 0$ (numerical equivalence) then $h^0(mK_X) \to \infty$ with m;

(ii) if $K_X \overset{\text{num}}{\sim} 0$ then $mK_X \overset{\text{lin}}{\sim} 0$ for some m. In other words, the topological condition $c_1(X) = 0 \in H^2(X, \mathbb{R})$ implies that $\mathcal{O}_X(mK_X)$ is analytically isomorphic to $\mathcal{O}_X$.

Remark. In both parts, the conditions can be relaxed to allow any field k, and X to have various singularities; it can also make sense to study a *log surface*, that is, a surface X together with a marked divisor D (say reduced, normal crossing), and classify the set-up X with D, replacing K_X by $K_X + D$.

D.1. Preliminaries: 1-cycles versus codimension 1 cycles.

Let X be any projective scheme. Write $\operatorname{Pic} X = H^1(X, \mathcal{O}_X^*)$ for the group of invertible sheaves (or divisor classes) on X, and $\{\text{1-cycles}(X)\}$ for the group of 1-cycles on X, that is, the free group on all curves $\Gamma \subset X$. Then there is a pairing

$$\operatorname{Pic} X \times \{\text{1-cycles}(X)\} \to \mathbb{Z}$$

defined by $(\mathcal{L}, \Gamma) \mapsto \mathcal{L}\Gamma = \deg_\Gamma \mathcal{L}_{|\Gamma}$, and extended by linearity.

D.1.1. Definition of numerical equivalence. Two invertible sheaves $\mathcal{L}_1, \mathcal{L}_2 \in \operatorname{Pic} X$ are *numerically equivalent* if $\mathcal{L}_1\Gamma = \mathcal{L}_2\Gamma$ for every curve $\Gamma \subset X$; we write $\mathcal{L}_1 \overset{\text{num}}{\sim} \mathcal{L}_2$. Similarly, two 1-cycles C_1, C_2 are *numerically equivalent* (written $C_1 \overset{\text{num}}{\sim} C_2$) if $\mathcal{L}C_1 = \mathcal{L}C_2$ for every $\mathcal{L} \in \operatorname{Pic} X$. Define

$$N^1 X = (\operatorname{Pic} X / \overset{\text{num}}{\sim}) \otimes \mathbb{R} \quad \text{and} \quad N_1 X = (\text{1-cycles} \otimes \mathbb{R}) / \overset{\text{num}}{\sim};$$

obviously, by definition N^1X and N_1X are dual $\mathbb{R}$-vector spaces, and $\overset{\text{num}}{\sim}$ is the smallest equivalence relation for which this holds. Clearly, N^1X contains $\mathbb{Z}$- and $\mathbb{Q}$-forms, denoted by $N^1_{\mathbb{Z}}X$, $N^1_{\mathbb{Q}}X$ respectively. The fact that $\rho = \dim_{\mathbb{R}} N^1X$ is finite is the Néron–Severi theorem.

Notice that for a variety defined over $\mathbb{C}$, it is natural to consider N_1X as a subspace of $H_2(X, \mathbb{R})$ and N^1X as the corresponding quotient of $H^2(X, \mathbb{R})$ (in fact both are direct summands). Over $\mathbb{C}$, the finite dimensionality of N^1X and N_1X follows at once from this (see App. to D.1 below); thus the content of the Néron–Severi theorem is a generalisation of this to the purely algebraic set-up. Notice also that N^1X is the same thing as $\mathrm{NS}(X) \otimes \mathbb{R}$, where $\mathrm{NS}(X)$ is the Néron–Severi group of X (that is, Cartier divisors modulo algebraic equivalence).

D.1.2. The case of a surface. I have deliberately avoided the following point: if X is a nonsingular surface then, of course, 1-cycles are the same as codimension 1 cycles, so that $N^1X = N_1X$. This can be seen in two ways:

(1) every curve $\Gamma \subset X$ is a Cartier divisor (that is, locally defined by one equation) and

$$\{\text{1-cycles of } X\} = \operatorname{Div} X \to \operatorname{Pic} X = \operatorname{Div} X / \overset{\text{lin}}{\sim}$$

is defined by

$$\sum n_i C_i \mapsto \mathcal{O}_X(\sum n_i C_i).$$

(2) $\operatorname{Pic} X$ has an intersection pairing defined by

$$\mathcal{L}_1\mathcal{L}_2 = \chi(\mathcal{O}_X) - \chi(\mathcal{L}_1) - \chi(\mathcal{L}_2) + \chi(\mathcal{L}_1 \otimes \mathcal{L}_2),$$

which becomes nondegenerate on N^1X.

Despite the canonical equality $N^1X = N_1X$, it is nevertheless convenient to distinguish between the two spaces. This is useful for dealing with the higher dimensional case, and also for singular surfaces, when it's no longer true that all divisors are Cartier.

Although very simple, this is one of the key ideas of Mori theory, and came as a surprise to anyone who knew the theory of surfaces before 1980: the quadratic intersection form for curves on a nonsingular surface can for most purposes be replaced by the bilinear pairing between N^1 and N_1, and in this form generalises to singular varieties and to higher dimension.

D.1.3. Definition of nef. An element $D \in N^1X$ is *numerically eventually free* or *nef* if $D\Gamma \geq 0$ for every curve $\Gamma \subset X$. This is related to the notion of a free linear system $|D|$: if $|mD|$ is free for some $m > 0$ ("eventually free") then D is nef.

The relation between nef and ample divisors is the content of the Kleiman criterion (see D.2).

D.1.4. The canonical class. If X is a nonsingular n-fold then $\mathcal{O}_X(K_X) = \omega_X = \Omega^n_X$ is a line bundle, locally generated by $\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$. (Compare A.10.) This is the only nontrivial line bundle which can be intrinsically associated with any variety.

Appendix to D.1. Cycle class groups and homology

D.1.5. Proposition. *(i) If X is a variety over $\mathbb{C}$ then $N_1X \subset H_2(X, \mathbb{R})$ and $H^2(X, \mathbb{R}) \twoheadrightarrow N^1X$ is the dual quotient.*

(ii) In terms of Hodge theory,

$$N^1X = H^2(X, \mathbb{R}) \cap H^{1,1} = H^2(X, \mathbb{R}) / \big(H^{2,0} \oplus H^{0,2} \cap H^2(X, \mathbb{R})\big).$$

(iii) Algebraic equivalence of 1*-cycles implies numerical equivalence.*

Sketch Proofs. (i) The inclusion takes a curve C into $[C]$ the 2-cycle obtained by triangulating C. The degree of a line bundle $\mathcal{L}$ on a curve is the same thing as the first Chern class $c_1(\mathcal{L}) \in H^2(X, \mathbb{R}) \twoheadrightarrow N^1X$ evaluated on the 2-cycle $[C] \in H_2(X, \mathbb{R})$.

(ii) Write down the exponential sequence of sheaves.

(iii) This follows easily from the fact that the degree of a line bundle is invariant in a flat family of curves. $\square$

D.2. Easy consequences of RR

Let X be a nonsingular surface. Then RR takes the form

$$\chi(\mathcal{O}_X(D)) = \chi(\mathcal{O}_X) + \tfrac{1}{2}D(D - K_X).$$

The fact that this is a quadratic function of D has an immediate corollary.

D.2.1. Proposition. *$D^2 > 0$ implies either $h^0(nD) \neq 0$ or $h^0(-nD) \neq 0$ for $n \gg 0$. The sign of HD (for any ample H) distinguishes the two cases.*

Proof. If $D^2 > 0$ then

$$h^0(nD) + h^2(nD) \geq \chi(\mathcal{O}_X(nD)) \sim n^2D^2/2,$$

so that either $h^0(nD)$ or $h^2(nD)$ goes to infinity with n. Now by Serre duality,

$$h^2(nD) = h^0(K_X - nD).$$

I can write down the same argument with D replaced by $-D$, and conclude that either $h^0(-nD)$ or $h^0(K_X + nD)$ goes to infinity with n.

Now obviously, $h^0(K_X - nD)$ and $h^0(K_X + nD)$ cannot both go to infinity with n; indeed, if $h^0(K_X - nD) \neq 0$, multiplying by a nonzero section $s \in H^0(K_X - nD)$ gives an inclusion $H^0(K_X + nD) \hookrightarrow H^0(2K_X)$ so that $h^0(K_X + nD) \leq h^0(2K_X)$. Therefore either $h^0(nD)$ or $h^0(-nD)$ grows quadratically with n.

Here is an alternative way of concluding the argument: it is easy to see that for any divisor L on X,

$$|h^0(-nD) - h^0(L - nD)| < \text{const.}n.$$

(Just write $L = A - B$ with A, B curves and consider restriction maps.) Hence if $h^2(nD) = h^0(K_X - nD)$ grows quadratically with n, so does $h^0(-nD)$. Q.E.D.

D.2.2. Corollary (the Index Theorem). *If H is ample on X then $HD = 0$ implies $D^2 \leq 0$. Moreover if $D^2 = 0$ then $D \overset{\text{num}}{\sim} 0$.*

In other words, the intersection pairing on N^1X has signature $(+1, -(\rho - 1))$. The cone $D^2 > 0$ looks like the "light cone": the "past" and "future" components of the cone are separated by the hyperplane $H^\perp$. Another way of stating this, which is very useful in calculations, is that if D_1, D_2 are divisors and $(\lambda D_1 + \mu D_2)^2 > 0$ for some $\lambda, \mu \in \mathbb{R}$ then the determinant

$$\begin{vmatrix} D_1^2 & D_1D_2 \\ D_1D_2 & D_2^2 \end{vmatrix} \leq 0,$$

with equality if and only if D_1, D_2 are $\mathbb{Q}$-linearly dependent in N^1X.

Proof. If $D^2 > 0$ then either nD or $-nD$ is equivalent to a nonzero effective divisor for $n \gg 0$. Since H is ample, either of these conditions implies $HD \neq 0$. This proves the first statement.

The final part is left as an exercise. [Hint: If $DA \neq 0$ for some curve A then replace A by $B = A - \lambda H$ so that $HB = 0$; now if $DA \neq 0$ also $DB \neq 0$, and some linear combination of D and B has $(D + \alpha B)^2 > 0$, contradicting the first part.] Q.E.D.

D.2.3. Corollary (weak form of the Kleiman criterion). *D is nef implies*

(1) *$D^2 \geq 0$ and*
(2) *$D + \varepsilon H$ is ample for any $\varepsilon \in \mathbb{Q}$, $\varepsilon > 0$.*

Proof. Introduce the quadratic polynomial $p(t) = (D + tH)^2$. Then $p(t)$ is a continuous increasing function of $t \in \mathbb{Q}$ for $t > 0$, and $p(t) > 0$ for sufficiently large t. The following assertion then obviously implies (1):

D.2.4. Claim. *Let $t \in \mathbb{Q}$, $t > 0$; then $p(t) > 0$ implies that also $p(t/2) > 0$.*

Proof. For $(D + tH)^2 > 0$, together with $H(D + tH) > 0$ implies that $n(D + tH)$ is equivalent to an effective divisor for suitable $n \gg 0$. By the assumption that D is nef, $D(D + tH) \geq 0$, and hence $(D + (t/2)H)^2 = D(D + tH) + (t/2)^2H^2 > 0$.

For (2) it is enough to notice that $(D + \varepsilon H)\Gamma > 0$ for every curve $\Gamma \subset X$ and $(D + \varepsilon H)^2 > 0$, so that $D + \varepsilon H$ satisfies the conditions of the Nakai–Moishezon criterion (see for example [H2], Chapter I or [H1], Chapter V, 1.10). Q.E.D.

D.2.5. The Kleiman criterion. Corollary D.2.3 can be thought of as saying that nef divisors are limits of ample divisors. It is not hard to get from this to (a weak form of) Kleiman's ampleness criterion, which asserts that conversely, the cone of ample divisors is the interior of the nef cone in N^1X. For this make the following definitions: let $\operatorname{NE} X \subset N_1 X$ be the cone of effective 1-cycles, that is

$$\operatorname{NE} X = \bigl\{ C \in N_1 X \bigm| C = \sum r_i \Gamma_i \text{ with } r_i \in \mathbb{R}, r_i \geq 0 \bigr\}.$$

Let $\overline{\operatorname{NE}} X$ be the closure of $\operatorname{NE} X$ in the real topology of $N_1 X$. This is called the *Kleiman–Mori cone*.

One effect of taking the closure is the following trivial observation, which has many important uses in applications: if $H \in N^1 X$ is positive on $\overline{\operatorname{NE}} X \setminus 0$, then the section $(Hz = 1) \cap \overline{\operatorname{NE}} X$ is compact. Indeed, the projectivisation of the closed cone $\overline{\operatorname{NE}} X$ is a closed subset of $\mathbb{P}^{\rho-1} = \mathbb{P}(N_1 X)$, and therefore compact, and the section $(Hz = 1)$ projects homeomorphically to it. The same also holds for any face or closed subcone of $\overline{\operatorname{NE}} X$.

Theorem. *For $D \in \operatorname{Pic} X$, view the class of D in $N^1 X$ as a linear form on $N_1 X$. Then*

$$D \textit{ is ample} \iff DC > 0 \textit{ for all } C \in \overline{\operatorname{NE}} X \setminus 0.$$

Note that it is not true that $DC > 0$ for every curve $C \subset X$ implies D ample. The necessary and sufficient condition $DC > 0$ for every $C \in \overline{\operatorname{NE}} X \setminus 0$ is just a tiny bit stronger than that. You can interpret it as saying that DC is "reliably bigger than $\|C\|$".

Sketch proof. The implication $\Longleftarrow$ comes at once from Corollary D.2.3, (2), since it is easy to see by a compactness argument that for H ample and sufficiently small $\varepsilon > 0$, $(D - \varepsilon H)C > 0$ for all $C \in \overline{\operatorname{NE}} X \setminus 0$, so $D = (D - \varepsilon H) + \varepsilon H$ is ample by Corollary D.2.3, (2). For $\Longrightarrow$ you need to use a compactness argument to prove that if a norm $\|\ \|$ is chosen on $N_1 X$ then for an ample divisor D there exists $\varepsilon > 0$ such that

$$DC > \varepsilon ||C|| \quad \text{for all } C \in \operatorname{NE} X$$

(see for example [H2], Chapter I).

Remarks. (1) This is only a weak form of Kleiman's criterion, since X is a priori assumed to be projective. The full strength of Kleiman's criterion gives a necessary and sufficient condition for ampleness in terms of the geometry of $\overline{\operatorname{NE}} X$. In particular, it can be used to prove that a variety is projective (without assuming it!). See [Kleiman].

(2) An analogous statement, the Kleiman criterion (weak form), holds for an arbitrary projective k-scheme: if D is a nef Cartier divisor then $D^r Z \geq 0$ for every irreducible r-dimensional subvariety Z, and D is ample if and only if it is positive on $\overline{\operatorname{NE}} X \setminus 0$. The proof is more or less the same as (D.2.3–5), although you have to use cohomology and carry out an induction on the dimension. It goes something like this.

Step 1. (A version of the Nakai–Moishezon criterion.) If $D^r Z > 0$ for every r and every irreducible r-dimensional subvariety Z then

$$H^j(\mathcal{F} \otimes \mathcal{O}_X(nD)) = 0.$$

for every coherent sheaf $\mathcal{F}$, every $j > 0$, and all $n \gg 0$. (Just use induction on $\dim \operatorname{Supp} \mathcal{F}$.)

Step 2. If D is nef and H an ample Cartier divisor, then using RR, $(D+tH)^r Z > 0$ implies that $H^0(\mathcal{O}_Z(N(D+tH))) \neq 0$, so that $(D+tH)_{|Z}$ is an effective $\mathbb{Q}$-divisor. Then as in Claim D.2.4, using a further induction, one proves that for $t > 0$,

$$(D+tH)^r Z > 0 \implies (D+(t/r)H)^r Z > 0.$$

D.3. The rationality lemma and cone theorem. Another easy consequence of the form of RR is the following:

D.3.1. Rationality lemma. *Suppose that H is ample and K_X is not nef. Define the* nef threshold $t_0 = t_0(H)$ *by*

$$t_0 = \sup\{t \mid H + tK_X \text{ is nef}\},$$

so that $t_0 \in \mathbb{R}$, $t_0 > 0$. Then (i) $t_0 \in \mathbb{Q}$, and (ii) its denominator is ≤ 3.

Note that the condition K_X not nef means exactly that $\overline{\mathrm{NE}}\,X$ meets the half-space $(K_X z < 0)$ of N^1X. Please draw a picture of the hyperplane $H + tK_X$ rotating from its initial position at $t = 0$ outside $\overline{\mathrm{NE}}\,X$ (since H is ample) to its asymptotic position for $t \gg 0$, when it cuts into $\overline{\mathrm{NE}}\,X$; t_0 is the threshold value at which it first hits $\overline{\mathrm{NE}}\,X$.

Discussion. The underlying reason behind the cone theorem and the contraction theorem, and all their generalisations to higher dimensional singular and log varieties is *vanishing.* Although the arguments get technically quite complicated, the main mechanism is very simple: when $t = t_0 + \varepsilon$, just after the critical value, $H + tK_X$ is not nef, but vanishing applies to it, because $H + (t-1)K_X$ is ample.

The rationality lemma is proved in much greater generality in [Kawamata] and [Kollár]. For a good introduction to the problem, see [Utah1].

Step 1. If $n(H + t_1K_X) = D_1$ is effective for some $n > 0$, with $t_1 \in \mathbb{Q}$ and $t_1 > t_0$ (so that D_1 is not nef), then t_0 is determined by

$$t_0 = \min_{\Gamma \subset D_1} \left\{ \frac{H\Gamma}{-K_X\Gamma} \right\},$$

where the minimum runs over the irreducible components Γ of D_1 such that $K_X\Gamma < 0$. In fact for $0 < t < t_1$ the divisor $H + tK_X$ is a positive combination of H and D_1, so that it fails to be nef if and only if $(H + tK_X)\Gamma < 0$ for a component Γ of D_1.

Step 2. Now if $t_0 \notin \mathbb{Q}$ then for $n, m \in \mathbb{Z}$ with $n < mt_0 < n+1$, it follows that $mH + nK_X$ is ample but $mH + (n+1)K_X$ is not nef. Since $mH + nK_X$ is ample, by Kodaira vanishing $h^0(mH + (n+1)K_X)$ is given by RR, and will sometimes be positive; this gives either a proof by Step 1 or a contradiction. More precisely, for $m > 0$, set $mt_0 = n + \alpha$ with $0 \leq \alpha < 1$, and write $D_0 = H + t_0K_X \in N^1X$. Then RR gives

$$\begin{aligned} H^0(mH + (n+1)K_X) &\geq \chi(\mathcal{O}_X) + \tfrac{1}{2}(mH + (n+1)K_X)(mH + nK_X) \\ &= \chi(\mathcal{O}_X) + \tfrac{1}{2}(m^2D_0^2 + m(1-2\alpha)D_0K_X - \alpha(1-\alpha)K_X^2). \end{aligned}$$

(Here ≥ 0 comes from the vanishing of H^2, which is trivial in any characteristic; if $\operatorname{char} k = 0$, the vanishing of H^1 gives equality.)

Hence if $D_0^2 > 0$ then $H^0 \neq 0$ for all large m; if $D_0^2 = 0$ and $D_0 \overset{\text{num}}{\not\sim} 0$ then necessarily $D_0 K_X < 0$ (because $D_0(H + t_0 K_X) = 0$ and $D_0 H > 0$), and $H^0 \neq 0$ if m is large and $1 - 2\alpha$ is bounded away from 0. If $D_0^2 = D_0 K_X = 0$ then also $D_0 H = 0$, so that $D_0 \overset{\text{num}}{\sim} 0$, and this implies that $K_X \overset{\text{num}}{\sim} -(1/t_0)H$; in particular $-K_X$ is ample.

This proves (i).

The proof of (ii) is an exercise. [Hint: If $D_0^2 > 0$ then there must exist a curve L with $D_0 L = 0$, since otherwise D_0 would be ample; it is easy to see that $D_0 L = 0$ implies that L is a -1-curve, so that $t_0 \in \mathbb{Z}$. If $D_0^2 = 0$, one sees that $H + tK_X$ cannot be effective for any $t > t_0$. If $D_0 \neq 0$ and $2t_0 \notin \mathbb{Z}$, the above inequality gives a contradiction by taking $m \gg 0$ with $(1-2\alpha)$ negative and bounded away from zero. Finally if $D_0 \overset{\text{num}}{\sim} 0$ then $-K_X$ is ample, and it is easy to see (in any characteristic, see D.4 or especially E.2) that $\chi(\mathcal{O}_X) = 1$. Now since $H + t_0 K_X \overset{\text{num}}{\sim} 0$, it follows that the denominator of t_0 divides K_X in $\operatorname{Pic} X$. It is now not hard to check that if $-K_X = dL$ with L ample, then $d \leq 3$: indeed, a simple argument of projective geometry shows that $h^0(L) \leq 2 + L^2$, whereas RR gives $h^0(L) \geq 1 + (1/2)(d-1)L^2$. (ii) also follows easily by the method of proof of the contraction theorem.]

The rationality lemma implies the Mori cone theorem by an argument that is pure convex body theory, taken from [Utah1], Lecture 11. (This result is not logically necessary for the proof of classification.)

D.3.2. Cone Theorem. *Set*

$$\overline{\mathrm{NE}}_{K_X} X = \bigl\{ C \in \overline{\mathrm{NE}}\, X \bigm| K_X C \geq 0 \bigr\}.$$

Then

$$\overline{\mathrm{NE}}\, X = \overline{\mathrm{NE}}_{K_X} X + \sum R_i, \tag{$*$}$$

where the R_i are extremal rays of $\overline{\mathrm{NE}}\, X$ contained in $(K_X z < 0)$. Moreover, for any ample Cartier divisor $H > 0$ and any given $\varepsilon > 0$, there are only finitely many extremal rays R_i with $(K_X + \varepsilon H)R_i \leq 0$.

In other words, away from the halfspace $K_X \geq 0$, the cone is locally polyhedral. (Draw a picture for God's sake! I'm imprisoned in a word processor.) It follows by the result of D.4 that each ray R_i is spanned by the class of an irreducible curve.

Idea of the proof. Suppose that $F_L = L^\perp \cap \overline{\mathrm{NE}}\, X$ is a face of the cone $\overline{\mathrm{NE}}\, X$ contained in the region $(K_X < 0)$ of the cone; if by bad luck F_L is positive dimensional, I can wiggle L slightly to make it a ray. This shows that each face of the cone contained in $(K_X < 0)$ is spanned by rays. In turn, I then show that each ray is defined by a set of linear equations with bounded denominators, so that the rays are discrete in $(K_X < 0)$.

To start on the formal proof, fix once and for all a basis of $N^1 X$ of the form $K_X, H_1, \ldots, H_{\rho-1}$, where the H_i are ample and $\rho = \operatorname{rank} N^1$. As in the rationality lemma, for any nef element $L \in N^1 X$, set

$$t_0(L) = \max\bigl\{ t \bigm| L + tK_X \text{ is nef} \bigr\}.$$

If $L \in \operatorname{Pic} X$ is nef and the corresponding face $F_L = L^\perp \cap \overline{\mathrm{NE}}\, X$ is contained in $(K_X z < 0)$ (for example, if L is ample) then the rationality lemma gives $rt_0(L) \in \mathbb{Z}$ for the fixed integer $r = 6$.

D.3.3. Main Claim. *Let $L \in \operatorname{Pic} X$ be a nef divisor which supports a face F_L of $\overline{\mathrm{NE}}\, X$ contained in $(K_X z < 0)$. Consider $\nu L + H_i$ for all i and for $\nu \gg 0$.*

(1) *$t_0(\nu L + H_i)$ in an increasing function of ν, is bounded above, and attains its bound.*
(2) *Let ν_0 be any point after $t_0(\nu L + H_i)$ has attained its upper bound, and suppose $\nu > \nu_0$. Set*
$$L' = 6\big(\nu L + H_i + t_0(\nu L + H_i) K_X\big),$$
(multiplying by 6 is simply to ensure that $L' \in \operatorname{Pic} X$ by the rationality lemma). Then L' supports a face $F_{L'} \subset F_L$.
(3) *If $\dim F_L \geq 2$ then there exists i and $\nu \gg 0$ such that*
$$L'_i = 6\big(\nu L + H_i + t_0(\nu L + H_i) K_X\big)$$
supports a strictly smaller face $F_{L'} \subsetneq F_L$.
(4) *In particular, F_L contains an extremal ray R of $\overline{\mathrm{NE}}\, X$.*
(5) *If $F_L = R$ is an extremal ray and $z \in R$ is a nonzero element then $6H_i z / K_X z \in \mathbb{Z}$.*
(6) *The extremal rays of $\overline{\mathrm{NE}}\, X$ are discrete in $(K_X z < 0)$.*

Proof. (1) is almost obvious: $t_0(\nu L + H_i)$ is an increasing function of ν by construction. It is bounded above, because any point $z \in F_L \setminus 0$ obviously gives $Lz = 0$ and $(H_i + tK_X)z < 0$ for $t \gg 0$. It attains its bounds because t_0 varies in the discrete set $(1/6)\mathbb{Z}$.

(2) Suppose that $t_0 = t_0(\nu L + H_i)$ does not change with $\nu \geq \nu_0$, and let F_0 be the face orthogonal to the nef element $\nu_0 L + H_i + t_0 K_X$. Then for $\nu > \nu_0$, any $z \in F_{L'_i}$ satisfies
$$0 = (\nu L + H_i + t_0 K_X)z = (\nu - \nu_0)Lz + (\nu L + H_i + t_0 K_X)z,$$
and therefore $Lz = 0$.

For (3), consider $\nu \gg 0$, and
$$L'_i = 6\big(\nu L + H_i + t_0(\nu L + H_i) K_X\big)$$
for each i. Since $\nu \gg 0$, these are "small wiggles" of L in $\rho - 1$ linearly independent directions. Each $F_{L'_i} \subset F_L$ is a face of F_L. The intersection of all the $F_{L'_i}$ is contained in the set defined by
$$(\nu L + H_i + t_0(\nu L + H_i) K_X)z = 0,$$
which are $\rho - 1$ linearly independent conditions on z. Therefore at least one of the $F_{L'_i}$ is strictly smaller than F_L. (4) follows obviously from (3).

(5) follows from (2) and the rationality lemma. Indeed, since $F_L = R$ is a ray, (2) implies that $F_{L'_i} = F_L = R$. That is, R is orthogonal to $H_i + t_0 K_X$, and $6t_0 \in \mathbb{Z}$.

Finally, (6) follows from (5), since in $(K_X z < 0)$, every ray contains a unique element z with $K_X z = -1$ and $H_i z \in (1/6)\mathbb{Z}$. Q.E.D.

Proof of Theorem D.3.2. Write $B = \overline{\mathrm{NE}}_{K_X} X + \sum R_i$ for the right-hand side of $(*)$. Note that $B \subset \overline{\mathrm{NE}}\, X$ is a closed convex cone: indeed, by (6) the R_i can only have accumulation points in $(K_X z \geq 0)$, that is, in $\overline{\mathrm{NE}}_{K_X} X$.

Suppose that $B \subsetneq \overline{\mathrm{NE}}\, X$. Then there exists an element $M \in N^1 X$ which is nef, and supports a nonzero face F_M of $\overline{\mathrm{NE}}\, X$ disjoint from B, necessarily contained in $(K_X z < 0)$. By the usual compactness argument, for sufficiently small $\varepsilon > 0$, $M - \varepsilon K_X$ is ample, and $M + \varepsilon K_X$ is not nef, but is positive on B. These are all open conditions on M, and any open neighbourhood of M in $N^1 X$ contains a rational element M, so that by passing to this and taking a multiple, I can assume that $M \in \operatorname{Pic} X$. But now the rationality lemma and Claim D.3.3, (4) imply that there is a ray R of $\overline{\mathrm{NE}}\, X$ with $MR < 0$, so that $R \not\subset B$. This contradicts the choice of B, and proves $(*)$. Q.E.D.

D.4. Contraction theorem and minimal models of surfaces

A consequence of the rationality lemma is that if K_X is not nef then there exists a divisor D on X which is nef but not ample, and such that $D - \varepsilon K_X$ is ample for $\varepsilon > 0$ (namely the divisor $D = n(H + t_0 K_X)$ appearing in the rationality lemma). Similarly, it is clear from the statement of the cone theorem that for each extremal ray $R = R_i$ of $\overline{\mathrm{NE}}\, X$ there is a divisor class D on X such that $D \geq 0$ on $\overline{\mathrm{NE}}\, X$ and $\overline{\mathrm{NE}}\, X \cap D^{\perp} = R$, and this divisor satisfies the same condition.

D.4.1. Contraction theorem. *Let D be a divisor which is nef but not ample, and such that $D - \varepsilon K_X$ is ample for some $\varepsilon > 0$. Then $|mD|$ is free for all $m \gg 0$.*

Equivalently, there exists a morphism $\varphi\colon X \to Y$ to a projective variety Y such that $D = \varphi^*(\text{ample})$. Of course, this means that for a curve $C \subset X$,

$$\varphi(C) = \text{pt.} \iff DC = 0.$$

By taking normalisation, I can assume that $\mathcal{O}_Y = \varphi_* \mathcal{O}_X$. In the 3 possible cases, this morphism is

Case $D^2 > 0$: φ is the contraction of a finite number of disjoint -1-curves L_i to nonsingular points of a surface Y.

Case $D^2 = 0$ but $D \overset{\text{num}}{\not\sim} 0$: φ is a conic bundle structure $X \to$ (curve).

Case $D \overset{\text{num}}{\sim} 0$: X is a surface with $-K_X$ ample and $p_g = q = 0$ (del Pezzo surface).

Proof. If $D^2 > 0$ then by the Nakai–Moishezon ampleness criterion there must be curves L with $DL = 0$. Now any such curve is a -1-curve, that is, $L \cong \mathbb{P}^1$ and $L^2 = -1$; for by the index theorem $L^2 < 0$, and since $D - \varepsilon K_X$ is ample $K_X L < 0$, and therefore by the adjunction formula $K_X L = L^2 = -1$ and $p_a L = 0$. The index theorem implies that any two of these are disjoint. The standard classical proof of Castelnuovo's contractibility criterion then gives the result.

If $D^2 = 0$ but $D \overset{\text{num}}{\not\sim} 0$ then since $D - \varepsilon K_X$ is ample, $K_X D < 0$, and RR gives $h^0(mD) \sim (-K_X D/2)m$, so in particular $h^0(mD) \to \infty$. The mobile part of $|mD|$ then gives a conic bundle structure $X \to$ (curve), and using this it is not hard to see that $|mD|$ itself is free.

Finally, if $D \overset{\text{num}}{\sim} 0$ then the condition on $D - \varepsilon K_X$ implies that $-K_X$ is ample, and the assertion is just that $D \overset{\text{lin}}{\sim} 0$. The key to this is to prove that $\chi(\mathcal{O}_X) = 1$, therefore $\chi(\mathcal{O}_X(D)) = \chi(\mathcal{O}_X) = 1$, and $H^0(\mathcal{O}_X(D)) \neq 0$, giving $D \overset{\text{lin}}{\sim} 0$ as required. If $\operatorname{char} k = 0$ then by Kodaira vanishing $H^i(\mathcal{O}_X) = 0$ for $i = 1, 2$, and so $\chi(\mathcal{O}_X) = 1$. If $\operatorname{char} k = p$ then $p_g = 0$ and $K_X^2 > 0$, so that the following alternative argument implies that $q = 0$ also in the case $\operatorname{char} k = p$. (See E.2 for how to get rid of the reference to Mumford.)

D.4.2. Traditional argument. Because $p_g = 0$, the Picard scheme is reduced (see E.2, Remark (c), or compare [M1], Lecture 27), and $B_1 = B_3 = 2q$. Hence (in etale cohomology), $c_2 = 2 + B_2 - 4q$, so as $B_2 \geq 1$ and $K_X^2 > 0$ I get

$$12\chi(\mathcal{O}_X) = 12(1-q) = c_1^2 + c_2 \geq 4 - 4q;$$

thus $q \leq 1$, and $q = 1$ is only possible if $B_2 = 1$. But if $q = 1$ then the fibre and section of the Albanese morphism imply $B_2 \geq 2$, so a contradiction. Therefore $\chi(\mathcal{O}_X) = 1$. Q.E.D.

D.4.3. Corollary. *Let R be an extremal ray of $\overline{\mathrm{NE}}\,X$; then there exists a morphism $\varphi\colon X \to Y$ to a projective variety Y such that $-K_X$ is relatively ample for φ, $\mathcal{O}_Y = \varphi_*\mathcal{O}_X$, and for all curves $C \subset X$,*

$$\varphi(C) = \text{pt.} \iff DC = 0.$$

The corresponding classification of extremal rays gives

(1) contraction of a -1-curve;
(2) $\mathbb{P}^1$-bundle $\varphi\colon X \to C$ over a curve C;
(3) $\mathbb{P}^2$.

Note that in (2), the assertion that $X = \mathbb{P}(\mathcal{E})$ for a rank 2 vector bundle is nontrivial: you need to know that $X \to C$ has a rational section, either by Tsen's theorem (see Theorem C.4.2 or [Sh], I.6), or an argument in topology using Poincaré duality, and the fact that because $p_g = 0$ the cohomology $H^2(X, \mathbb{Z})$ is spanned by curves on X (see for example [Beauville], Theorem III.4). Similarly, to know that $X \cong \mathbb{P}^2$ in (3), the main point is to prove that $-K_X$ is divisible by 3 in $\operatorname{Pic} X$, which again follows from $K_X^2 = 9$ (Noether's formula) by Poincaré duality.

D.4.4. Corollary (Main Theorem of minimal model theory). *Let X be any surface. Then there exists a chain of contractions of -1-curves $X \to X_1 \cdots \to X_N = Y$ such that either K_Y is nef, or Y is $\mathbb{P}^2$ or a $\mathbb{P}^1$-bundle over a curve.*