CHAPTER C. GUIDE TO THE CLASSIFICATION OF SURFACES

The classification of surfaces goes back to Castelnuovo and Enriques in the early years of the 20th century. It divides algebraic surfaces into 4 big classes. There will be lots to say about surfaces in each of the 4 classes, and the characteristic methods and results for each class are different. The subject matter thus divides naturally into the 4 separate classes, together with the logical division into cases, or the *proof of classification*.

This chapter introduces the classification of surfaces in overall terms. It is primarily intended as a guide to results that lots of people need to use, and most of the results are given without proof. However, to give a veneer of modernity I cast the classification within the logical structure of Mori theory: the dichotomy K_X nef or otherwise, and the numerical dimension of K_X form the main logical framework, taking precedence over criteria for rationality and ruledness and Kodaira dimension. The underlying material is of course essentially the same as that of Castelnuovo, Enriques and Kodaira. I return to the proof of the classification in Chapters D–E below.

I assume that X is a nonsingular projective surface over $\mathbb{C}$ to make life easier for myself and the reader. It is of course often reasonable to weaken these conditions: a topologist or analyst might want compact complex surfaces without the assumption of algebraicity, a number theorist might want singular projective surfaces over a finite field or a number field, students interested in higher dimensional geometry will need to know how the results for surfaces work in several other contexts. Getting the right level of generality is also important for the internal development of the subject.

SUMMARY

(1) Numerical invariants
(2) Birational versus biregular classification: blowups
(3) Results of minimal model theory
(4) Ruled and rational surfaces, Tsen's theorem
(5) Classification of surfaces with K_X nef by ν
(6) Kodaira dimension, the statement $\kappa = \nu$
(7) More precise description of $\kappa = 0$

C.1. INVARIANTS

C.1.1. Numerical invariants. The main numerical invariants are

(1) the Betti numbers B_i with the Euler number $e(X)$;
(2) the signature (B_2^+, B_2^-) of the quadratic form Q_X on $H^2(X,\mathbb{Z})/\operatorname{Tors}$;
(3) the Hodge numbers $h^{p,q}$ with the Poincaré characteristic $\chi(\mathcal{O}_X)$;
(4) the Chern classes $c_1 = [-K_X] \in H^2(X,\mathbb{Z})$ and $c_2 \in H^4(X,\mathbb{Z}) = \mathbb{Z}$, and the Chern numbers c_1^2 and c_2.

Fortunately there are lots of relations between these numbers: only two or three of them are independent. There are also three other discrete invariants that are not simply numbers, but are closely related to the numerical invariants: the quadratic form Q_X on $H^2(X,\mathbb{Z})/\operatorname{Tors} X$ (up to isomorphism), the torsion subgroup $\operatorname{Tors} X = \operatorname{Tors} H^2(X,\mathbb{Z})$ and the fundamental group $\pi_1(X)$. The intersection form on $H^2(X,\mathbb{Z})$ is completely determined from a knowledge of the numerical invariants (including $c_1 \in H^2(X, \mathbb{Z})$), and in turn, for simply connected surfaces, it is known that this determines the homotopy type and the homeomorphism type of X.

C.1.2. Topological invariants. A complex surface is a compact 4-manifold with a chosen orientation. Accordingly, its Betti numbers $B_i = \operatorname{rank} H^i(X, \mathbb{Z})$ satisfy Poincaré duality

$$B_0 = B_4 = 1, \qquad B_1 = \operatorname{rank} H^1(X, \mathbb{Z}) = B_3.$$

Hence the Euler number is given by

$$e(X) = \sum (-1)^i B_i = 2 - 2B_1 + B_2.$$

(Hirzebruch's convention is to write $e(X)$ for the Euler characteristic, rather than something like $\chi_{\text{top}}(X, \mathbb{Z})$, to distinguish it from Euler–Poincaré characteristics such as $\chi(\mathcal{O}_X)$ in sheaf cohomology; I follow this throughout, and urge you to do likewise.) In addition, Poincaré duality says that the symmetric bilinear form defined by cup product

$$\cup\colon H^2(X, \mathbb{Z}) \times H^2(X, \mathbb{Z}) \to H^4(X, \mathbb{Z}) = \mathbb{Z}$$

is a perfect pairing on $H^2(X, \mathbb{Z})/\operatorname{Tors} X = \operatorname{Hom}(H_2(X, \mathbb{Z}), \mathbb{Z})$. This means, equivalently, that $\cup$ induces an isomorphism $H^2(X, \mathbb{Z})/\operatorname{Tors} X \to \operatorname{Hom}_{\mathbb{Z}}(H^2(X, \mathbb{Z}), \mathbb{Z})$, or that in any $\mathbb{Z}$-basis of $H^2(X, \mathbb{Z})/\operatorname{Tors} X$, the symmetric matrix representing the pairing is unimodular, that is, has determinant ± 1.

C.1.3. Analytic invariants. Most of the analytic invariants are defined in terms of the sheaves of differentials: recall that T_X is a rank 2 complex vector bundle over X, whose sections are vector fields on (open subsets of) X; the dual bundle corresponds to the sheaf of Kähler differentials $\Omega^1_X = \operatorname{Hom}_{\mathcal{O}_X}(T_X, \mathcal{O}_X)$. This is a locally free sheaf of $\mathcal{O}_X$ modules of rank 2, based locally by $\mathrm{d}x, \mathrm{d}y$ where x, y are local coordinates. The sheaf of canonical differentials $\Omega^2_X = \mathcal{O}_X(K_X)$ is the sheaf of holomorphic 2-forms, defined as $\bigwedge^2 \Omega^1_X$, and is the locally free sheaf of $\mathcal{O}_X$ modules of rank 1, based locally by $\mathrm{d}x \wedge \mathrm{d}y$.

Global sections of these bundles give us important invariants: the *geometric genus*

$$p_g(X) = h^0(X, \Omega^2_X) = \dim \Gamma(X, \Omega^2_X),$$

that is, the number of global holomorphic canonical differentials, and the *irregularity* $q = h^0(X, \Omega^1_X)$. *Hodge theory* allows us to express some of the other invariants of X in terms of these: firstly, it defines a decomposition

$$H^1(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} = H^1(X, \mathbb{C}) = H^{0,1} \oplus H^{1,0},$$

where $H^{0,1} = H^1(X, \mathcal{O}_X)$ and $H^{1,0} = H^0(X, \Omega^1_X)$; in this decomposition $H^{0,1}$ is the complex conjugate of $H^{1,0}$, so that in particular

$$h^{0,1} = h^{1,0} = q \quad \text{and} \quad B_1 = \operatorname{rank} H^1(X, \mathbb{Z}) = \dim_{\mathbb{C}} H^1(X, \mathbb{C}) = 2q.$$

Next, it tells us the signature of the pairing on $H^2(X, \mathbb{R}) = H^2(X, \mathbb{Z}) \otimes \mathbb{R}$: diagonalising this quadratic form over $\mathbb{R}$ gives the pairing as a diagonal matrix with B_2^+ entries $+1$ and B_2^- entries -1; the pair (B_2^+, B_2^-) is the Sylvester inertia index of the form. Now Hodge theory tells us that $B_2^+ = 2p_g + 1$. In particular, p_g is an invariant of the (oriented) homotopy type of X.

More precisely, the Hodge decomposition of $H^2(X, \mathbb{C})$ is

$$H^2(X, \mathbb{C}) = H^{0,2} \oplus H^{1,1} \oplus H^{2,0},$$

where $H^{2,0} = H^0(X, \Omega_X^2)$ and $H^{0,2}$ is the complex conjugate of $H^{2,0}$, so that the vector subspace

$$V_{\mathbb{C}}^+ = H^{2,0} \oplus H^{0,2} \subset H^2(X, \mathbb{C})$$

is invariant under complex conjugation, and corresponds to a $2p_g$-dimensional real subspace $V^+ \subset H^2(X, \mathbb{R})$. Then if $L \in H^2(X, \mathbb{Z})$ is the class of a hyperplane section of X (the first Chern class of the line bundle $\mathcal{O}_X(1)$ for some embedding $X \subset \mathbb{P}^N$), by Hodge theory $L \in H^{1,1}$, and the Hodge index theorem says that the intersection form is positive definite on $V^+ \oplus \mathbb{R} \cdot L$, and negative definite on the "primitive part" $H_0^{1,1}$, the orthogonal complement of L in $H^{1,1}$.

Remark. More generally, the set-up of Hodge theory on a Kähler manifold is

$$H^n(X, \mathbb{C}) = \bigoplus_{p+q=n} H^{p,q}(X), \tag{$*$}$$

where $H^{p,q}(X) = H^q(X, \Omega_X^p)$. One proves that $H^{p,q}$ and $H^{q,p}$ are complex conjugates, and that the Hodge decomposition $(*)$ determines the signature of the intersection pairing on the "primitive" cohomology, etc. What I said above is enough for complex projective surfaces.

C.1.4. Chern classes. A crude consequence of the presence of the complex structure is that the tangent bundle T_X has the structure of a complex vector bundle with fibre $\mathbb{C}^2$. This vector bundle has characteristic classes $c_1(T_X) \in H^2(X, \mathbb{Z})$ and $c_2(T_X) \in H^4(X, \mathbb{Z})$. Each of these objects has an alternative interpretation: since Ω_X^1 is the dual of T_X and Ω_X^2 is the determinant bundle $\bigwedge^2 \Omega_X^1$,

$$c_1(T_X) = -c_1(\Omega_X^1) = -c_1(\Omega_X^2) = -[K_X] \in H^2(X, \mathbb{Z}).$$

Also, a general result is that $c_2(T_X) = e(X)$: morally speaking, the top Chern class of T_X should be the number of zeros of a generic section, but the number of zeros of a generic vector field is the Euler number.

The final relation between the numerical invariants is Noether's formula:

$$\chi(\mathcal{O}_X) = 1 - q + p_g = \frac{1}{12}(c_1^2 + c_2) = \frac{1}{12}(K_X^2 + e(X));$$

here I interpret c_1^2 and $c_2 \in H^4(X, \mathbb{Z}) = \mathbb{Z}$ as integers. The formula is part of the set-up of Hirzebruch–RR. Since $c_2(X) = e(X) = 2 - 4q + B_2$, and since q and p_g are homotopy invariants, so is c_1^2.

C.1.5. Exercises.

(1) For regular surfaces (that is, surfaces with $q = 0$), show how to express all the numerical invariants in terms of K_X^2 and $\chi = \chi(\mathcal{O}_X)$, and verify the Hirzebruch signature formula

$$B_2^+ - B_2^- = 4\chi - e(X) = \frac{1}{3}(c_1^2 - 2c_2).$$

(2) Assume that $c_1^2 \geq 0$, $c_2 \geq 0$ and $c_1^2 \leq 3c_2$. (Lots of surfaces satisfy these conditions.) Prove that $B_2^- = 0$ is only possible if X satisfies $q = 0$, $B_2 = 1$, $c_1^2 = 9$ and $c_2 = 3$ (that is, X has the numerical invariants of $\mathbb{P}^2$). This proves that the quadratic form on H^2 is either **1** or is indefinite.

Solutions.

(1) $q = 0$, $p_g = \chi - 1$, $e(X) = 12\chi - K_X^2$,
$B_2 = e(X) - 2$, $B_2^+ = 2p_g + 1 = 2\chi - 1$, $B_2^- = e(X) - 2\chi - 1$.
(It is easy to write out similar formulas for any q.)

(2) $B_2^- = B_2 - 2p_g - 1$
$= e(X) - 2 + 4q - 2p_g - 1$
$= c_2 - \frac{1}{6}(c_1^2 + c_2) + 2q - 1$
$= \frac{1}{3}c_2 + \frac{1}{6}(3c_2 - c_1^2) + 2q - 1.$
So $B_2^- = 0$ gives $q = 0$ and $2c_2 + (3c_2 - c_1^2) = 6$, which has only one solution compatible with $c_1^2 + c_2 \equiv 0 \bmod 12$.

C.1.6. Quadratic form on $H^2(X, \mathbb{Z})$. Consider a quadratic form (H, Q) over $\mathbb{Z}$, that is, a free $\mathbb{Z}$-module (lattice) $H \cong \mathbb{Z}^n$, together with a quadratic map $Q\colon H \to \mathbb{Z}$. There is a whole theory of isomorphism classes of quadratic forms over $\mathbb{Z}$; let me start by going through the easy bits of this theory; a nondegenerate quadratic form (H, Q) has a *signature* (n^+, n^-), a *discriminant* $\det Q$, and a *parity*. Parity just means the following: if $Q(x) \in \mathbb{Z}$ is even for all $x \in H$ then the quadratic form is *even*, otherwise it's *odd*.

For $H^2(X, \mathbb{Z})$ of a complex surface, all these invariants are under control: the signature (B_2^+, B_2^-) is discussed in C.1.3 above; the quadratic form Q_X is unimodular (that is, $\det Q_X = 1$) by Poincaré duality; and the parity is determined by $c_1(X)$ or K_X:

$$Q_X(x) \equiv c_1 \cdot x \bmod 2 \quad \text{for all } x \in H^2(X, \mathbb{Z}).$$

Hence Q_X is even if and only if $c_1(X)$ maps to $0 \in H^2(X, \mathbb{Z}/2)$, or equivalently, K_X is divisible by 2 as a divisor class. A surface having this property is *even* or is a *spin surface*.

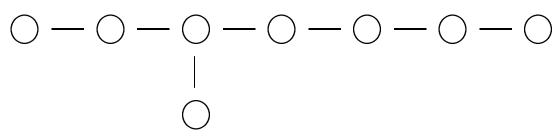
A unimodular quadratic form (H, Q), assumed to be indefinite, is determined up to isomorphism by its signature (n^+, n^-) and its parity. In fact if Q is odd then it is isomorphic to the diagonal form:

$$Q \text{ is odd} \implies (H, Q) \cong \mathbf{1}^{n^+} \oplus (-\mathbf{1})^{n^-},$$

and there is a similar standard form for an even indefinite form

$$Q \text{ is even} \implies (H, Q) \cong \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^a \oplus (\pm E_8)^b,$$

where $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is the standard hyperbolic lattice, E_8 is the lattice based by 8 vectors e_i in bijection with the vertexes of the Dynkin diagram E_8



and quadratic form given by $e_i^2 = 2$, $e_i e_j = 0$ or -1 according as the corresponding vertexes are unjoined or joined, and $-E_8$ the same with the opposite signs $e_i^2 = -2$,

$e_ie_j = 0$ or $+1$. That is, $\pm E_8 = \mathbb{Z}^8$ with the quadratic form

$$\pm\begin{pmatrix} -2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 \end{pmatrix}$$

Definite quadratic forms over $\mathbb{Z}$ are of course much more complicated, but fortunately, we never have to deal with them (compare Exercise 1.5 above).

C.1.7. Exercise. All these invariants are usually easy enough to calculate if your surface is given fairly explicitly. For example, suppose that $X = X_d \subset \mathbb{P}^3_{\mathbb{C}}$ is a nonsingular hypersurface of degree d. It is simply connected by Lefschetz theory, and in particular $q = 0$. (See also Ex. 3.3.) The adjunction formula gives $K_X = (-4+d)H = \mathcal{O}_X(-4+d)$, so that

$$p_g = \begin{cases} 0 & \text{for } d \le 3 \\ \binom{d-1}{3} & \text{for } d \ge 4 \end{cases} \qquad \text{and } K_X^2 = d(d-4)^2.$$

Finally, K_X is divisible by 2 in $\operatorname{Pic} X$ if and only if $d-4$ is even, that is, d is even. It's an easy exercise to determine all the invariants of X from this.

C.2. Birational versus biregular classification: blowups

There are several different points of view on what may be meant by the classification of algebraic varieties, among them the following three:

(1) birational, that is, up to birational equivalence;
(2) biregular, that is, up to isomorphism;
(3) projective, that is, up to projective equivalence.

To say that X is a projective surface means that X is capable of being embedded as a closed subvariety of $\mathbb{P}^N$ for some N. The gap between (2) and (3) is the problem of finding a projective embedding of an abstract surface X, or separating off intrinsic properties of $X \subset \mathbb{P}^N$ from the properties of the ambient space and of the embedding. I leave this for the moment. Most of this section and the next is concerned with the gap between (1) and (2).

Notice that for nonsingular projective curves, (1) = (2): for C and C' any two nonsingular projective curves, any birational map from C to C' is an isomorphism. The blowup (see 1.10) shows that this is not the case for surfaces. The main point of this section is that blowups account for the whole of the gap between (1) and (2).

Remark. For surfaces having a minimal model with K_X nef, it turns out that the minimal model is unique, which restores (1) = (2). A birational map between two surfaces with K_X nef is necessarily an isomorphism.

C.2.1. Proposition. (a) Blowups. *Given a surface X and a point $P \in X$, there exists a surface X_1 and a morphism $\sigma\colon X_1 \to X$ such that $\sigma^{-1}P = E$ is a curve, and σ restricts to an isomorphism $X_1 \setminus E \to X \setminus P$.*

Here $E \cong \mathbb{P}^1$ and $E^2 = -1$, so that E is a -1-curve; if C is a curve of X then $\sigma^*C = C' + mE$, where $m = \operatorname{mult}_P C$, C' is the birational transform of C (that is, the closure in X_1 of $\sigma^{-1}(C \setminus P)$), and the points of C' over P correspond one-to-one to the distinct tangent lines to C.

(b) Castelnuovo's contractibility criterion. *Given a surface X and a -1-curve $E \subset X$, there exists a contraction $\sigma\colon X \to Y$ such that E maps to a point P of a (nonsingular projective) surface Y, and σ is the blowup of $P \in Y$.*

For (a), see 1.10. (b) is proved in Chapter 4, Contraction Theorem 4.15.

C.2.2. Theorem. (a) Resolution of indeterminacies. *Let $f\colon X \dashrightarrow \mathbb{P}^N$ be a rational map. Then there exists a commutative diagram*

$$\begin{tikzcd} & Y \arrow[dl, "g"'] \arrow[dr, "h"] & \\ X \arrow[rr, dashed] & & \mathbb{P}^N \end{tikzcd}$$

where g is a composite $Y = X_n \to \cdots \to X_1 \xrightarrow{\sigma_1} X$ of blowups, and h is a morphism.

(b) Factorisation of birational morphisms. *Let $h\colon X \to Y$ be a birational morphism between nonsingular projective surfaces. Then h is the composite of a chain of blowups $X \xrightarrow{\sigma_1} Y_1 \to \cdots \to Y_N = Y$.*

See for example [Beauville], Chapter 2 or [Sh], Chapter IV, §3.

C.2.3. Topological view of a blowup. For surfaces over $\mathbb{C}$, a blowup $\sigma\colon X_1 \to X$ corresponds to taking a connected sum $X\#Q$ of X with the 4-manifold $Q = \overline{\mathbb{P}^2_{\mathbb{C}}}$, that is, $\mathbb{P}^2_{\mathbb{C}}$ with the opposite orientation.

The idea is that it replaces a small disc around P by a tubular neighbourhood of the -1-curve E; it is not hard to see that this is diffeomorphic to $\mathbb{P}^2_{\mathbb{C}} \setminus (\text{disc})$, but with the opposite orientation (both of them are disc bundles over $\mathbb{P}^1_{\mathbb{C}} \cong S^2$, but the "core" copy of $\mathbb{P}^1_{\mathbb{C}}$ has selfintersection ± 1 in the two cases).

C.2.4. Fundamental asymmetry. It's important to note the asymmetry here: taking connected sum with Q is a blowup, so is an operation of algebraic geometry. But taking connected sum with $\mathbb{P}^2_{\mathbb{C}}$ itself cannot be an operation of algebraic geometry: it can never happen that X and $X\#\mathbb{P}^2_{\mathbb{C}}$ are both diffeomorphic to algebraic surfaces (with oriented connected sum and diffeomorphism preserving the orientation), if only because B_2^+ passes from odd to even.

At a much deeper level, the contrast can be seen in a very striking way from the following two results:

Theorem (Moishezon–Mandelbaum). *For most naturally occurring large classes of algebraic surfaces, the connected sum $M\#\mathbb{P}^2_{\mathbb{C}}$ has a decomposition*

$$M\#\mathbb{P}^2_{\mathbb{C}} \overset{\text{diffeo}}{\sim} (\mathbb{P}^2_{\mathbb{C}})^{\#n}\#Q^{\#m}.$$

Theorem (Donaldson). *If M is the 4-manifold underlying an algebraic surface, there does not exist any connected sum decomposition*

$$M \overset{\text{diffeo}}{\sim} M_1\#M_2$$

with each of M_1, M_2 having $B_2^+ > 0$.

C.3. Results of minimal model theory

The following is a modern statement of minimal model theory. It is proved in Chapter D. In the classical theory, it would be a corollary of a whole chain of results. Recall that the *canonical divisor* K_X is the divisor corresponding to the invertible sheaf $\Omega_X^2 = \mathcal{O}_X(K_X)$ and that a divisor D on a surface X is *nef* ("numerically eventually free") if $DC \geq 0$ for every curve $C \subset X$.

C.3.1. Theorem. *Let Y be any nonsingular projective surface. Then there is a chain*

$$Y \xrightarrow{\sigma_1} Y_1 \to \cdots \to Y_N = X$$

such that each $\sigma_i\colon Y_{i-1} \to Y_i$ is the contraction of a single -1-curve E_i, and X satisfies

either (i) K_X is nef;

or (ii) $X \cong \mathbb{P}^2$ or a $\mathbb{P}^1$-bundle over a curve.

More crudely, every surface is birational to a surface with K_X nef, or to a $\mathbb{P}^1$-bundle over a curve or $\mathbb{P}^2$. This form of the result is easy to remember, and is very useful. On the other hand, the information it contains can be analysed to give results that are more complicated, but much more general. Notice that as stated, the theorem refers to three different situations:

(1) a contraction $\sigma\colon X \to X_1$ of a -1-curve E, with $E \cong \mathbb{P}^1$, $E^2 = -1$, $K_X E = -1$;
(2) a $\mathbb{P}^1$-bundle $X \to C$ with fibre $F \cong \mathbb{P}^1$, $F^2 = 0$, $K_X F = -2$;
(3) $X \cong \mathbb{P}^2$; here I can write the constant morphism $\mathbb{P}^2 \to$ pt., and consider a line $L \subset \mathbb{P}^2$ with $L \cong \mathbb{P}^1$, $L^2 = 1$, $K_X L = -3$.

All three of these situations can be classed together as a morphism $\varphi\colon X \to Y$ such that $-K_X C > 0$ for every curve C in a fibre of f. Thus Theorem C.3.1 is closely related to the following statement.

C.3.2. Theorem. *Let X be a surface, and suppose that K_X is not nef. Then there exists a morphism $\varphi\colon X \to Y$ (with $\dim Y = 0, 1$ or 2) contracting at least one curve of X to a point, and such that $-K_X C > 0$ for every curve C in a fibre of φ. There is no loss of generality in assuming $\varphi_* \mathcal{O}_X = \mathcal{O}_Y$, which means that Y is normal and the general fibre of φ is connected.*

Theorem C.3.2 is more primitive than Theorem C.3.1, but much more general in scope: it holds for surfaces over any field k, for a large class of singular surfaces, and for nonsingular varieties of any dimension over $\mathbb{C}$. The modern strategy for proving Theorem C.3.1 is first to prove Theorem C.3.2, then to analyse the morphism $\varphi\colon X \to Y$ for which $-K_X$ is ample.

C.3.3. Proposition. *Let $\varphi\colon X \to Y$ be as in the conclusion of Theorem C.3.2. Then one of the following 3 cases holds.*

$\dim Y = 2$. *Then Y is a nonsingular surface, and $\varphi\colon X \to Y$ is the blowdown of a number of disjoint -1-curves;*

$\dim Y = 1$. *Then Y is a nonsingular curve, and $\varphi\colon X \to Y$ is a conic bundle; the general fibre of φ is isomorphic to $\mathbb{P}^1$, whereas a special fibre is isomorphic to a line pair in $\mathbb{P}^2$;*

$Y =$ pt. *In this case, the constant morphism $\varphi\colon X \to Y$ does not give any information, but $-K_X$ is an ample divisor.*

Surfaces in the final case are called *del Pezzo surfaces*, and they are similar in most of their properties to the cubic surfaces $X_3 \subset \mathbb{P}^3$ discussed in Chapter 1. The main step in deducing Theorem C.3.1 from Theorem C.3.2 is to prove that if X is a del Pezzo surface then either $B_2(X) \geq 2$ and X admits another contraction $X \to X'$ of one of the other two types; or $B_2(X) = 1$ and $X \cong \mathbb{P}^2$.

C.4. Ruled and rational surfaces, Tsen's theorem

I define a *ruled surface* to mean a nonsingular projective surface X together with a fixed morphism $\varphi\colon X \to B$ to a base curve B, whose fibre $F_{k(B)}$ over the generic point is a nonsingular curve of genus 0. The main theorem of this section, proved below, is Tsen's theorem: $F_{k(B)}$ is isomorphic over $k(B)$ to $\mathbb{P}^1_{k(B)}$. To discuss the results of minimal model theory, I assume this for the moment. That is, I assume that $\varphi\colon X \to B$ is a given morphism with generic fibre isomorphic to $\mathbb{P}^1_{k(B)}$.

There are many alternatives, going back to the ancients: you could require only that φ exists, without taking the responsibility for specifying it, or you could allow φ to be a rational map $\varphi\colon X \dashrightarrow B$. For the purposes of this introduction, call this "birationally ruled". Thus the plane $\mathbb{P}^2$ or any rational surface is birationally ruled. The $\kappa = -\infty$ part of the classification of surfaces (due to Castelnuovo and Enriques) can be stated in the form

$$P_{12}(X) = 0 \iff \kappa = -\infty \iff X \text{ is birationally ruled.}$$

Compare Corollary E.1.

C.4.1. Models. According to the definition, X has a morphism $X \to B$ with generic fibre $\mathbb{P}^1_{k(B)}$. Choosing an isomorphism of the generic fibre defines a birational map

$$\begin{array}{ccc} X & \dashrightarrow & B \times \mathbb{P}^1 \\ & \searrow \quad \swarrow & \\ & B & \end{array}$$

Since the definition of ruled surface only fixes the generic fibre, the general picture is as follows: X is isomorphic to $B \times \mathbb{P}^1$ over a Zariski open set of B, but over finitely many points $\{P\} \in B$ the fibre can have lots of components, say $\varphi^{-1}P = \bigcup \Gamma_i$. We know that the intersection matrix $(\Gamma_i\Gamma_j)_{ij}$ is negative semidefinite (see Theorem A.7 and Ex. A.17). It's an exercise to see from first principles that if the fibre is not $\mathbb{P}^1$ then at least one of the Γ_i is a -1-curve.

There are other conditions that can be added to a ruled surface that restrict the types of singular fibres. These conditions appear in a natural way in classical work, and more especially in Mori theory:

(1) $-K_X$ ample characterises conic bundles. Every singular fibre of φ is a line pair, a union of two -1-curves $L_1 \cup L_2$ meeting transversally at one point. In fact $-K_X$ is relatively very ample, and embeds X into a $\mathbb{P}^2$-bundle over B.

(2) $-K_X$ ample and $\rho(\varphi) = 1$ characterise $\mathbb{P}^1$-bundles. The condition $\rho(\varphi) = 1$ means that every curve in a fibre is numerically a multiple of the fibre. This implies that every fibre is nonsingular. Using Tsen's theorem, it's not hard to see that then $X = \mathbb{P}(E)$ is the projectivisation of a rank 2 vector bundle over B.

The case of conic bundle is important over an algebraically nonclosed field k: it can happen that $X \to B$ has $-K_X$ ample, so is geometrically a conic bundle, and has $\rho(\varphi) = 1$, so that there are no curves defined over k in the fibres of $X \to B$ that can be contracted out by a birational morphism; but $X \to B$ can still have singular fibres, line pairs $L_1 \cup L_2$ with L_1 and L_2 conjugate over k.

C.4.2. Theorem (Tsen's theorem). *Let k be an algebraically closed field and $k \subset F$ a function field in one variable over k (this means that $F = k(B)$, where B is an irreducible algebraic curve over k). Let X_F be a nonsingular projective curve over F, and assume that X_F is absolutely irreducible, in the sense that it remains irreducible over the algebraic closure of F. Then the following conditions are equivalent:*

(1) *The curve X_F has $g = 0$ in RR;*
(2) *X_F is isomorphic over F to a nonsingular plane conic;*
(3) *X_F becomes isomorphic to $\mathbb{P}^1$ after some field extension $F \subset F'$, that is, $X_F \otimes_F F' \cong \mathbb{P}^1_{F'}$;*
(4) *X_F is isomorphic over F to $\mathbb{P}^1_F$.*

Proof. $2 \implies 3 \implies 1$ and $4 \implies 1$ are obvious. The implication $1 \implies 2$ is standard use of RR: the anticanonical line bundle $-K_{X_F}$ has degree 2, and is very ample by the usual criterion: $\ell(-K_{X_F}) = 3$, and $\ell(-K_{X_F} - D) = 1$ for any divisor D of degree 2. All this works over an arbitrarily field F.

The point is thus to prove $2 \implies 4$. I explain how to use the assumptions that $F = k(B)$ is a function field in one variable over an algebraically closed field. The point is that $k(B)$ is the field of fractions of a ring $k[B]$ with "linear growth", namely the coordinate ring of an affine model of B. In more detail, $k[B] = \bigcup k[B]_d$ (polynomials of degree $\le d$) with multiplication satisfying $k[B]_{d_1} \cdot k[B]_{d_2} \subset k[B]_{d_1+d_2}$. Moreover, $k[B]_d$ is finite dimensional over k, with dimension growing as a linear function: for some constants a, b_1, b_2

$$ad - b_1 \le \dim k[B]_d \le ad + b_2 \quad \text{(same } a\text{!)}. \tag{1}$$

There are various ways of proving this, using RR on the curve B, or the commutative algebra definition of $\dim k[B] = 1$ in terms of Hilbert–Samuel functions, but in any case the fact that B is 1-dimensional is crucial.

Now X_F is isomorphic over $k(F)$ to a plane conic

$$X_F : q(x, y, z) = 0 \subset \mathbb{P}^2_F \tag{2}$$

defined by a quadratic form q in x, y, z with coefficients in F. Notice that since X_F is a plane conic, to prove that it is isomorphic to $\mathbb{P}^1_F$, it is enough to prove that is has a point defined over F (the argument is given in [UAG], (1.7) and Ex. 1.5).

The point is thus very simple: under the assumption that F is a function field in 1 variable, I can always solve (2). Multiplying through by denominators, I can assume that the coefficients of q are in $k[B]$, and even in $k[B]_c$ for some choice of c. The plan of the proof is to try to solve (2) with $x, y, z \in k[B]_d$ subject to $q(x, y, z) = 0 \in k[B]_{2d+c}$.

The punch line is that by (1) there are $\ge 3(ad - b_1)$ free variables in the choice of x, y, z, and $\le a(2d + c) + b_2$ polynomial conditions on them in the equation

$q(x,y,z) = 0 \in k[B]_{2d+c}$. To explain this gently, x,y,z are any elements of the vector space $k[B]_d$; if $e_1,\dots,e_N$ is a basis then

$$x = \sum u_i e_i, \quad y = \sum v_i e_i, \quad z = \sum w_i e_i$$

give $3N$ variables u_i, v_i, w_i. In a similar way, the equation $q(x,y,z) = 0$ can be written out as $\dim k[B]_{2d+c}$ quadratic equations in these variables.

Since for $d \gg 0$, there are more free variables than equations, over the algebraically closed k, it follows that u_i, v_i, w_i can be given values in k so that x,y,z satisfy $q(x,y,z) = 0$.

C.4.3. Birational maps. If X and Y are rational surfaces, the set of birational maps $X \dashrightarrow Y$, or the group $\operatorname{Bir} X$ of birational selfmaps is very big. This set (resp. group) is the obstruction to the uniqueness of the model of X as a $\mathbb{P}^1$-bundle over a curve or $\mathbb{P}^2$. For $X = C \times \mathbb{P}^1$, $\operatorname{Bir} X$ contains $\mathrm{PGL}(2, k(C))$. For $X = \mathbb{P}^2$, the group $\operatorname{Bir} X$ is the Cremona group, and is qualitatively something like the free group on a continuum of generators.

C.5. Classification assuming K_X nef

Aim. Assuming X a surface with K_X nef, classify into 3 cases according to the numerical properties of K_X (the 3 possibilities $\nu = 0,1,2$ for the numerical dimension of K_X), and according to the analytic properties of K_X (the 3 possibilities $\kappa = 0,1,2$, for the Kodaira dimension of K_X); statement of $\kappa = \nu$.

C.5.1. General fact. D nef $\implies D^2 \ge 0$. Recall that by definition, D is nef if and only if $DC \ge 0$ for all curves $C \subset X$, so that $D \cdot (\sum n_i C_i) \ge 0$ for all effective divisors $\sum n_i C_i$ with $n_i \ge 0$. See D.2 for a proof.

Morally, the reason C.5.1 holds is that

$$D \text{ nef} \implies D \text{ is “close to being effective”} \implies D^2 \ge 0.$$

C.5.2. Definition of $\overset{\text{num}}{\sim}$. Two divisors D_1 and D_2 on a surface X are numerically equivalent (written $D_1 \overset{\text{num}}{\sim} D_2$) if $D_1 C = D_2 C$ for all curves $C \subset X$. Notice that

$$D_1 \overset{\text{lin}}{\sim} D_2 \iff \mathcal{O}_X(D_1) \cong \mathcal{O}_X(D_2) \implies D_1 \overset{\text{num}}{\sim} D_2$$

because $DC = \deg_C \mathcal{O}_X(D)$.

C.5.3. Division into cases according to ν. Assume K_X is nef, so that $K_X^2 \ge 0$; then the following 3 cases are all-inclusive and mutually exclusive:

$\nu = 0$. $K_X \overset{\text{num}}{\sim} 0$, that is, $K_X C = 0$ for all $C \subset X$;

$\nu = 1$. $K_X C > 0$ for some curve $C \subset X$, but $K_X^2 = 0$;

$\nu = 2$. $K_X^2 > 0$.

You should think of ν as being defined by this division into cases. Formally, one can write down the definition of ν as a formula as follows:

$$\nu = \max\{k \mid K_X^k \overset{\text{num}}{\not\sim} 0\},$$

although this has no other meaning than the case division just given.

C.5.4. Theorem (Weak form of main theorem). *There exists a morphism $\varphi\colon X \to Y$ with $\dim Y = \nu(X)$ and $\varphi_*\mathcal{O}_X = \mathcal{O}_Y$ (that is, Y is normal and the generic fibre of φ is connected), such that*

$$K_X C = 0 \iff \varphi(C) = pt.$$

There are 3 separate statements here:

$\nu = 0$. *The statement is vacuous: φ maps X to a point.*

$\nu = 1$. *This is the most substantial case; here $\varphi\colon X \to Y$ is a fibration of X over a curve, with $K_X(\text{fibre}) = 0$. A general fibre E is a nonsingular curve with $E^2 = 0$, so that $K_X E = 0$ implies that $g(E) = 1$, and thus φ is an elliptic fibration.*

$\nu = 2$. *In this case the morphism $\varphi\colon X \to Y$ is birational, and contracts at most a finite number of connected configurations of -2-curves to Du Val singularities; also, $\mathcal{O}_Y(K_Y)$ is an ample line bundle, and $K_X = \varphi^* K_Y$.*

C.5.5. Definition of Kodaira dimension. The vector spaces $\Gamma(X, (\Omega_X^2)^m) = \Gamma(X, \mathcal{O}_X(mK_X))$ provide further invariants of X and K_X. Define the *plurigenera* P_m of X by $P_m(X) = \dim \Gamma(X, \mathcal{O}_X(mK_X))$. It is formal to see that the P_m must satisfy one of the following:

Case $\kappa = -\infty$. $P_m = 0$ for all $m > 0$;

Case $\kappa = 0$. $P_m \le 1$ for all $m > 0$, and $P_m = 1$ for some m;

Case $\kappa = 1$ or 2. There exist constants $a, b > 0$ such that $am^\kappa < P_m < bm^\kappa$ for all sufficiently large m.

It can be shown that if $\kappa \ge 0$ then $\kappa = \dim \varphi_{mK_X}(X)$, where if $P_m \ne 0$ then $\varphi_{mK_X}\colon X \dashrightarrow \mathbb{P}^{P_m - 1}$ is the rational map defined by $\mathcal{O}_X(mK_X)$.

Note that in contrast to ν, the invariants P_m and κ are analytic invariants of X: P_m is the number of global holomorphic m-times canonical differentials on X.

C.5.6. Statement of $\kappa = \nu$. The statement $\kappa = \nu$ is true, and is one of the central structural results of the theory. It means the following:

Case $\nu = 0$. $K_X \overset{\text{num}}{\sim} 0 \implies \exists m$ such that $mK_X \overset{\text{lin}}{\sim} 0$.

Case $\nu = 1$. $K_X \overset{\text{num}}{\not\sim} 0$ but $K_X^2 = 0 \implies \exists m$ such that $P_m \ge 2$.

Case $\nu = 2$. $K_X^2 > 0 \implies P_m$ grows quadratically in m. (This is easy.)

There is a more precise description of the case $\nu = 0$.

C.5.7. Theorem. *If $\nu = 0$ then one of the following 4 cases holds:*

1. $K_X = 0$, $H^1(\mathcal{O}_X) = 0$, *that is, X is a K3 surface (see Chapter 3);*
2. $K_X = 0$, $h^1(\mathcal{O}_X) = 2$, *and X is an Abelian surface.*
3. $K_X \overset{\text{lin}}{\not\sim} 0$ *but* $2K_X \overset{\text{lin}}{\sim} 0$ *and* $H^1(\mathcal{O}_X) = 0$; *these conditions are the definition of an Enriques surface.*
4. $K_X \overset{\text{lin}}{\not\sim} 0$ *but* $mK_X \overset{\text{lin}}{\sim} 0$ *for* $m = 2, 3, 4$ *or* 6, *and* $h^1(\mathcal{O}_X) = 1$. *Then X is a bielliptic surface, that is, $X = E_1 \times E_2/(\mathbb{Z}/m)$ is a quotient of a product of two elliptic curves by the cyclic group $\mathbb{Z}/m$ acting by translations on one factor and a group automorphism in the other.*

The results sketched in this chapter are taken up again with complete proofs in Chapters D–E below.