CHAPTER B. Sheaves and coherent cohomology

Sheaf theory is a language to treat geometric data (functions, vector fields, etc.) on a space X in terms of the same kind of data on open sets $U \subset X$. You probably know that in Molière's play *Le Bourgeois Gentilhomme*, M. Jourdain was amazed to learn from his grammar teacher that he had been speaking all of his life *in prose*! In the same way, I hope to persuade you that you have been using some of the ideas of sheaf theory without knowing it ever since your first calculus course.

It is of course out of the question to give a reasonable treatment of sheaves and cohomology in a single lecture. Instead, I give examples of some of the main classes of sheaves occurring in algebraic geometry, and discuss their role in the foundational crises of the subject through the ages. I also try to explain the definition of coherent sheaves, and to highlight the specific features of coherent cohomology which make it different from other cohomology theories used in topology, differential geometry and algebraic geometry.

If all the definitions in this section are intimidating to the younger reader, I assure you that with coherent cohomology, the whole of this type of algebraic geometry becomes a game with fixed rules and just a few standard gambits. This section concludes with a list of "Rules of the Game" for coherent cohomology, which can be taken as axioms, or read up in several references. Remember the Zariski quote[1] we heard a few nights ago: before Serre's [FAC], just a few *maestri* who had spent all their lives contemplating the intricacies of the black arts could say when some restriction map was surjective, and all you could do was to believe them; after [FAC], any idiot could write down exact sequences and deduce any number of such statements.

Summary

(1) You already know lots of sheaves. Continuous functions on $\mathbb{R}$, vector fields on a manifold
(2) Definition of sheaf: sections $\Gamma(U, \mathcal{F})$ over an open and "glueing conditions"
(3) The structure sheaf $\mathcal{O}_X$, intrinsic definition of a variety
(4) Other sheaves on algebraic varieties, $\mathcal{O}_X(D)$, Ω^1_X, $\mathcal{O}_X(K_X) = \Omega^n_X$
(5) Subsheaves, stalks and quotient sheaves. Surjective must be defined on stalks, that's where cohomology comes in
(6) Coherent sheaves. Homomorphisms between locally free sheaves
(7) Rules of coherent cohomology
(8) The Nakai–Moishezon criterion for surfaces

B.1. You already know lots of sheaves

When we talk of functions on a space X, we often actually mean functions defined on some open subset $U \subset X$. For example, in elementary calculus, a function $f(x)$ of a single real variable might be defined on some interval $(a, b) \subset \mathbb{R}$. If f is defined on some big interval (A, B), the property that f is continuous (or differentiable, or real analytic) is defined locally at every point $P \in (A, B)$, and for P, only depends on the behaviour of f on any smaller interval $(a, b) \ni P$. *Voilà, M. Jourdain,* a sheaf!

[1] Compare [Parikh], p. 138; Carol Parikh gave an interesting evening talk on Zariski's life and opinions at the Park City summer school.

In the same way, in a first course on manifolds, we learn that a vector field on a manifold X is a thing that can be written $\sum u_i \frac{\partial}{\partial x_i}$ in terms of local coordinates $x_1, \dots, x_n$ on a coordinate patch $U \subset X$. It is natural in discussing manifolds to work locally over small open sets (for example, coordinate patches), and very awkward to insist that the only vector fields we use are global vector fields on the whole of X. For any open subset $V \subset X$, write

$$\Gamma(V, \mathcal{T}_X) = \bigl\{\text{vector fields on } V\bigr\}$$

for the set of all vector fields on V. The *tangent sheaf* $\mathcal{T}_X$ is this data: all possible vector fields defined on all possible open subsets $V \subset X$. Or in other words, the assignment $V \mapsto \Gamma(V, \mathcal{T}_X)$.

B.2. The structure sheaf $\mathcal{O}_X$ of a variety

Let X be an irreducible quasiprojective variety with its Zariski topology, and $k(X)$ its field of rational functions. For $P \in X$, we know what it means for a rational function $f \in k(X)$ to be *regular* at P (see for example [UAG], §5), and the set of rational functions that are regular at P is the *local ring* $\mathcal{O}_{X,P} \subset k(X)$. For any Zariski open $U \subset X$, define

$$\Gamma(U, \mathcal{O}_X) = \bigl\{f \in k(X) \bigm| f \text{ is regular at all } P \in U\bigr\} = \bigcap_{P \in U} \mathcal{O}_{X,P} \subset k(X).$$

The *structure sheaf* of $\mathcal{O}_X$ is the assignment $U \mapsto \Gamma(U, \mathcal{O}_X)$. In this case everything is very simple, because all the $\Gamma(U, \mathcal{O}_X)$ are subrings of the fixed function field $k(X)$.

For $\emptyset \neq V \subset U$ a smaller Zariski open set, regular functions on U obviously restrict to regular functions on V, defining a inclusion $\Gamma(U, \mathcal{O}_X) \subset \Gamma(V, \mathcal{O}_X)$. Also, since every function regular at P is regular on some open neighbourhood of P, it follows that

$$\mathcal{O}_{X,P} = \bigcup_{U \ni P} \Gamma(U, \mathcal{O}_X) \subset k(X).$$

In sheaf theory, the inclusion $\Gamma(U, \mathcal{O}_X) \subset \Gamma(V, \mathcal{O}_X)$ is called a *restriction map* $\mathrm{Res}_{U,V}$ (or sometimes $\rho_{U,V}$), and $\mathcal{O}_{X,P}$ the *stalk* of $\mathcal{O}_X$ at P.

B.3. The definition of a sheaf

In general, a *presheaf* $\mathcal{F}$ on a topological space X is a way of assigning to every open subset $U \subset X$ a set $\Gamma(U, \mathcal{F})$ (or ring, or group, or object of any category; the symbol $\Gamma(U, \mathcal{F})$ is pronounced "the sections of $\mathcal{F}$ over U"), and to every inclusion $V \subset U$ a restriction map $\mathrm{Res}_{U,V} \colon \Gamma(U, \mathcal{F}) \to \Gamma(V, \mathcal{F})$ (or ring homomorphism, or group homomorphism, or morphism of any category). Restriction is supposed to be transitive, in the obvious sense that $\mathrm{Res}_{V,W} \circ \mathrm{Res}_{U,V} = \mathrm{Res}_{U,W}$ whenever $W \subset V \subset U$.

A *sheaf* is a presheaf with "glueing conditions": a section $s \in \Gamma(U, \mathcal{F})$ is uniquely determined by its restrictions $\mathrm{Res}_{U,U_\alpha}(s)$ to open sets U_α that cover U; and given an open cover, and sections $s_\alpha \in \Gamma(U_\alpha, \mathcal{F})$ having equal restrictions to all overlaps $U_\alpha \cap U_\beta$, the s_α come by restricting a section $s \in \Gamma(U, \mathcal{F})$.

If $\mathcal{F}$ is a presheaf whose $\Gamma(U, \mathcal{F})$ are sets of maps defined on U (say, with values in some set Σ), then the first glueing condition holds trivially, and the second holds only if the restrictions placed on maps $U \to \Sigma$ are "local in nature". For example, continuous or differentiable functions on a C^∞ manifold form a sheaf, since continuity and differentiability are local properties of a function. In the same way, sections of a vector bundle $\pi\colon E \to X$, or more generally, the continuous sections of any map $\pi\colon F \to X$ form a sheaf; see Ex. B.4–5 for details.

B.4. The definition of a variety

The first key service that the language of sheaves performs is to give a satisfactory definition of variety: irreducible affine algebraic sets and their sheaf of regular functions were discussed in B.2. A *variety* $X, \mathcal{O}_X$ is a topological space X with a sheaf $\mathcal{O}_X$ of rings of functions $U \to k$ such that $X, \mathcal{O}_X$ is locally isomorphic to an irreducible affine algebraic set with its sheaf of regular functions. (This can be easily generalised to allow reducible varieties, or indeed general k-schemes.)

Thus sheaves solve an important foundational problem of algebraic geometry, the intrinsic definition of variety. You know from a first course in algebraic geometry (for example, [UAG] or [Sh]) that quasiprojective varieties are quite convenient to define and treat using the tricks of homogeneous coordinates. However, it's unsatisfactory to take this as the formal definition of variety, because you only define X together with extrinsic data of an embedding $X \subset \mathbb{P}^N$.

Compare how the topologist, in defining a manifold, gets away without using sheaves: a topological manifold M is a topological space locally homeomorphic to a ball in $\mathbb{R}^n$; continuity of functions is completely determined by the topology (of course), so that there is no need to specify the sheaf of continuous functions. For a differentiable manifold, you require in addition that the local charts $\varphi_i\colon U_i \to$ (ball in $\mathbb{R}^n$) satisfy $\varphi_i \circ \varphi_j^{-1}$ is differentiable (wherever defined). Because of this assumption, the condition that a function on an open subset $V \subset M$ is differentiable is well defined. An equivalent definition would be to specify the sheaf $\mathcal{E}_{M,\mathrm{diff}}$ of differentiable functions on M, and assume that the local charts are isomorphisms of ringed spaces.

In case anyone still hasn't got the idea of what all the fuss is about, I repeat: you can't just define an algebraic variety X to be a point set, or a set with a Zariski topology, because all plane curves would be homeomorphic. If you define it as embedded in a space (affine or projective), you get the notion of rational function $f \in k(X)$ and regular function, and hence isomorphism of varieties, but you also get extrinsic stuff which may have less to do with X than with the embedding. The sheaf $\mathcal{O}_X$ specifies the regular functions on every open set. A variety is a space X together with a notion of regular function on opens of X. By giving $\mathcal{O}_X$, you give every possible isomorphism of opens of X with subvarieties of affine or projective space.

B.5. Other sheaves on algebraic varieties

I continue the theme that you already know lots of sheaves. If X is a normal variety and $D = \sum n_\Gamma \Gamma$ an effective divisor on X, rational functions on X with poles at worst D form a sheaf $\mathcal{O}_X(D)$, the *divisorial sheaf* of D. Recall that zeros and poles of rational functions are interpreted in terms of discrete valuations v_Γ on the local rings $\mathcal{O}_{X,\Gamma}$ at every prime divisor Γ (compare 1.6). The condition that $0 \neq f \in k(X)$ has a pole of order at worst n along Γ reads $v_\Gamma(f) \geq -n$. To define the sheaf $\mathcal{O}_X(D)$, I tell you what its sections are on every open set $U \subset X$:

$$\Gamma(U, \mathcal{O}_X(D)) = \left\{ f \in k(X) \;\middle|\; v_\Gamma(f) \ge -n_\Gamma \text{ for every } \Gamma \text{ such that } \Gamma \cap U \ne \emptyset \right\}.$$

These are local conditions, so that $\mathcal{O}_X(D)$ is a sheaf. Since $\operatorname{div} f = \sum_\Gamma v_\Gamma(f)\Gamma$, the condition on the right can also be written $\operatorname{div} f + D \ge 0$ on U.

Historical discussion. In the case $U = X$ a projective variety (for example a nonsingular projective curve), the space

$$\mathcal{L}(D) = \Gamma(X, \mathcal{O}_X(D))$$

is traditionally called the *RR space* of D. The RR theorem on a curve gives its dimension:

$$\ell(D) = \dim \mathcal{L}(D) \ge 1 - g + \deg D \qquad \text{with equality if } \deg D > 2g - 2, \qquad \text{(RR)}$$

where $g = g(X)$ is the genus of X. This result was the subject of an earlier foundational crisis in algebraic geometry. It follows from the maximum principle that the only global holomorphic functions on a compact Riemann surface are the constants. To find some maps from X to $\mathbb{P}^1 = \mathbb{C} \cup \infty$, Riemann allowed functions which are everywhere meromorphic, with a finite number of poles, which he wrote as a divisor D. He then claimed a proof of (RR) based on a Dirichlet principle from electrostatics: imagine that your Riemann surface is made of beaten copper, and the poles are point electric charges; then physical intuition says that the potential equations for the electric field must have a solution. Riemann used (RR) to prove that a compact Riemann surface can be embedded in a projective space $\mathbb{P}^N_{\mathbb{C}}$, and is hence an algebraic curve (the *Riemann existence theorem*). Unfortunately, Weierstrass pointed out at once that Riemann's Dirichlet principle was false as stated; in fact, 30 or 40 years later Hilbert gave a revised statement and claimed proof of the Dirichlet principle that was also erroneous, although, by all accounts, Hilbert was so famous by then that nobody dared tell him. In any case, Clebsch, Max Noether and Brill proved (RR) for algebraic curves by purely algebraic means shortly after Riemann.

On an irreducible variety, $\mathcal{O}_X(D)$ is defined as a subsheaf of the constant sheaf $k(X)$. If $\operatorname{div} f = D - D'$ then multiplying by f in $k(X)$ clearly takes $\mathcal{O}_X(D)$ into $\mathcal{O}_X(D')$. Thus linearly equivalent divisors D give rise to isomorphic divisorial sheaves $\mathcal{O}_X(D)$. See Ex. B.13 for more details.

In A.10 I introduced the canonical divisor class K_X as the divisor $\operatorname{div} s$ of a rational n-form $s = g\,\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_n$. And K_X is defined up to linear equivalence, because any other rational n-form s' is of the form $s' = hs$ for some $h \in k(X)$, so that $\operatorname{div}(s') = \operatorname{div} h + \operatorname{div} s$. Therefore, there is a well-defined divisorial sheaf $\mathcal{O}_X(K_X)$ on X. In fact it is easier and more intrinsic to introduce the sheaf $\omega_X = \Omega^n_X$ first, prove that it is divisorial (on a nonsingular X it is *locally free of rank* 1, or *invertible*), and then to define K_X as any divisor such that $\mathcal{O}_X(K_X) = \Omega^n_X$. See Ex. B.16. In fact, although the divisor K_X is only a divisor class, the sheaf $\omega_X = \Omega^n_X = \mathcal{O}_X(K_X)$ is canonical.

B.6. Subsheaves, stalks and quotient sheaves

There are two very different kinds of definitions and arguments in sheaf theory, those that take place at the level of the spaces of sections $\Gamma(U, \mathcal{F})$, and those that are local at every point $P \in X$ and involve stalks. I start with a few instances of the first. A *homomorphism* of sheaves $\varphi\colon \mathcal{F} \to \mathcal{G}$ is a collection of morphisms $\varphi_U\colon \Gamma(U, \mathcal{F}) \to \Gamma(U, \mathcal{G})$ for all opens U that commutes with restrictions $\mathrm{Res}_{U,V}$ of $\mathcal{F}$ and $\mathcal{G}$:

$$\begin{array}{ccc} \Gamma(U,\mathcal{F}) & \to & \Gamma(U,\mathcal{G}) \\ \downarrow & \copyright & \downarrow \\ \Gamma(V,\mathcal{F}) & \to & \Gamma(V,\mathcal{G}) \end{array}$$

Similarly, a *subsheaf* $\mathcal{F} \subset \mathcal{G}$ is a collection of subobjects $\Gamma(U, \mathcal{F}) \subset \Gamma(U, \mathcal{G})$ that themselves form a sheaf under the same restriction maps. If $f\colon X \to Y$ is a continuous map of topological spaces, and $\mathcal{F}$ a sheaf on X, then it is immediate to check that $U \mapsto \Gamma(f^{-1}U, \mathcal{F})$ for open sets $U \subset Y$ gives a sheaf on Y, the sheaf-theoretic *pushforward* $f_*\mathcal{F}$. All of these definitions and constructions can be made for presheaves just as well as for sheaves.

The *stalk* of a sheaf $\mathcal{F}$ at a point $P \in X$ is the direct limit $\mathcal{F}_P = \varinjlim_{U \ni P} \Gamma(U, \mathcal{F})$. The limit looks intimidating, but this is just another case of M. Jourdain's prose. Namely, the direct limit is the set of all sections $s \in \Gamma(U, \mathcal{F})$ over all open sets containing U, modulo the equivalence relation $s = s'$ if they coincide on some smaller neighbourhood of P; in other words, $\mathcal{F}_P$ consists of *germs* of sections at P. For example, if $\mathcal{O}_{\mathrm{an}}$ is the sheaf of holomorphic functions on $\mathbb{C}$ then the stalk $\mathcal{O}_{\mathrm{an},0}$ consists of all power series with positive radius of convergence, and a germ is an analytic function on some neighbourhood of 0; different germs are defined on different neighbourhoods. The stalks of the structure sheaf $\mathcal{O}_X$ of a variety X are the local rings $\mathcal{O}_{X,P}$, and in this case the direct limit is simply a union, as mentioned in B.2. Another example: a common definition of a tangent vector to a manifold M at a point P is as a derivation of functions defined near P. The derivation acts on germs of smooth functions: it looks at the function only in an arbitrarily small neighbourhood of P.

Now we say that a homomorphism of sheaves $\varphi\colon \mathcal{F} \to \mathcal{G}$ is *surjective* if it induces surjective maps $\varphi_P\colon \mathcal{F}_P \to \mathcal{G}_P$ on each stalk. Surjectivity onto $\Gamma(U, \mathcal{G})$ for all U is the wrong requirement. Another way of saying the same thing is as follows: if $s \in \Gamma(U, \mathcal{G})$, I don't require that s itself comes from some $t \in \Gamma(U, \mathcal{F})$, but only that this holds in a small neighbourhood of any $P \in U$. I now give a baby example (see also Ex. B.10).

Let $P_1, P_2, P_3 \in \mathbb{P}^n$ be 3 distinct points, not in the hyperplane $(x_0 = 0)$. On $\mathbb{P}^n$, consider the sheaf $\mathcal{O}_{\mathbb{P}^n}(1)$ of linear forms, which is the sheaf defined by

$$\Gamma(U, \mathcal{O}_{\mathbb{P}^n}(1)) = \left\{ \frac{f}{g} \in k(x_0, \dots, x_n) \;\middle|\; \begin{array}{l} f, g \in k[x_0, \dots, x_n] \text{ homog. of degree} \\ d+1 \text{ resp. } d, \text{ and } g(P) \neq 0 \text{ at } P \in U \end{array} \right\}.$$

Now I can find a linear form not vanishing at P_i, so that the evaluation map $\mathcal{O}_{\mathbb{P}^n}(1) \to k_P$ defined by $s \mapsto (s/x_0)(P)$ is surjective. Here k_P is the "skyscraper sheaf" with sections over U either zero if $P \notin U$ or a copy of k if $P \in U$. The kernel is the sheaf of linear forms vanishing at P, that is, $m_P \cdot \mathcal{O}_{\mathbb{P}^n}(1)$. Now consider the evaluation map at all 3 points at once:

$$0 \to \mathcal{I} \cdot \mathcal{O}_{\mathbb{P}^n}(1) \to \mathcal{O}_{\mathbb{P}^n}(1) \to k_{P_1} \oplus k_{P_2} \oplus k_{P_3} \to 0, \qquad (*)$$

where for brevity I write $\mathcal{I} = m_{P_1} \cdot m_{P_2} \cdot m_{P_3}$ for the ideal sheaf of $\{P_1, P_2, P_3\}$, that is, the subsheaf of $\mathcal{O}_{\mathbb{P}^n}$ consisting of regular functions on an open U vanishing at $\{P_1, P_2, P_3\} \cap U$. Now in sheaf theory we say that the evaluation map in $(*)$ is surjective, because it is surjective locally at every point. Is it surjective on global sections? The global map evaluates linear forms in $x_0, \ldots, x_n$ on 3 points, which is surjective if P_1, P_2, P_3 span a plane in $\mathbb{P}^n$, and not surjective if they are collinear. I thus get the exact sequence

$$\begin{aligned} 0 &\to \text{linear forms vanishing at } P_1, P_2, P_3 \to \langle x_0, \ldots, x_n \rangle \to k_{P_1} \oplus k_{P_2} \oplus k_{P_3} \to \\ &\to \text{linear dependences among } P_1, P_2, P_3 \to 0 \end{aligned}$$

In other words, the homomorphism of sheaves is surjective, but it gives a homomorphism on global sections which is not necessarily surjective. In more general language, the last display is written

$$\begin{aligned} 0 &\to H^0(\mathbb{P}^n, \mathcal{I} \cdot \mathcal{O}_{\mathbb{P}^n}(1)) \to H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) \to H^0(\mathbb{P}^n, k_{P_1} \oplus k_{P_2} \oplus k_{P_3}) \to \\ &\to H^1(\mathbb{P}^n, \mathcal{I} \cdot \mathcal{O}_{\mathbb{P}^n}(1)) \to 0, \end{aligned}$$

where $H^1(\mathbb{P}^n, \mathcal{I} \cdot \mathcal{O}_{\mathbb{P}^n}(1))$ is the first cohomology group of the sheaf $\mathcal{I} \cdot \mathcal{O}_{\mathbb{P}^n}(1)$.

I mention two other constructions of sheaf theory that use stalks: the *quotient sheaf* $\mathcal{G}/\mathcal{F}$ is defined to have stalk $\mathcal{G}_P/\mathcal{F}_P$ at every $P \in X$; and the *sheaf-theoretic pullback* $f^{-1}\mathcal{F}$ of a sheaf on Y by a morphism $f\colon X \to Y$ is defined to have stalk $\mathcal{F}_{f(P)}$ at $P \in X$. I omit the definition, which involves the notion of the associated sheaf of a presheaf. (This material is completed in Ex. B.27–29.)

B.7. Coherent sheaves

The sheaves of algebraic geometry $\mathcal{O}_X$, $\mathcal{O}_X(D)$, $\mathcal{O}_{\mathbb{P}^n}(r)$ we have met so far are all coherent sheaves; so is the ideal sheaf $\mathcal{I}_Y \subset \mathcal{O}_X$ of a subvariety $Y \subset X$. The adjective coherent means that they are sheaves of modules over $\mathcal{O}_X$, with a finiteness condition, and closely related to the structure sheaf $\mathcal{O}_X$.

The general progression is presheaf, sheaf, sheaf of $\mathcal{O}_X$-modules, (quasi-) coherent sheaf, locally free sheaf. I have not been through the general definitions particularly carefully; it should be clear what the definition of sheaf of $\mathcal{O}_X$-modules is. If you have trouble see, for example, [FAC] or [H1], Chapter II. The definition of (quasi-)coherent involves tension between the requirements of generality and explicitness: namely, the definition is that $\mathcal{F}$ should be a sheaf of $\mathcal{O}_X$-modules, and $\mathcal{F}$ should be locally isomorphic to the cokernel of a homomorphism between free sheaves. In other words, on local pieces U, there should exist a resolution

$$\mathcal{O}_U^{\oplus N} \to \mathcal{O}_U^{\oplus M} \to \mathcal{F}_{|U} \to 0 \tag{1}$$

(that is, an exact sequence of sheaves of $\mathcal{O}_X$-modules). The tension comes because to have an intrinsic definition you want the condition for all sufficiently small neighbourhoods U, but to have an explicit construction you want only that there exists a cover of X by opens U with the property.

The tension is solved in the best possible way: thanks to cohomology, we can have it both ways! In other words, if I have an open cover $X = \bigcup U_i$ of X by affine sets U_i such that (1) holds for each U_i, then the same condition (1) holds for every affine open set $U \subset X$. This is the content of Rule ii below. It is proved in [FAC], [H1], Chapter II and [Sh], Chapter VII.

Now *quasicoherent* is condition (1) for all U of some (any) open cover with arbitrary cardinals N and M. *Coherent* is the same with finite N and M. Notice that for a fixed affine U, (1) and the vanishing of H^1 (see Rule ii) gives a sequence

$$k[U]^N \to k[U]^M \to \Gamma(U, \mathcal{F}) \to 0, \tag{2}$$

where $k[U] = \Gamma(U, \mathcal{O}_X)$ is the affine coordinate ring of U. In other words, $\Gamma(U, \mathcal{F}) = F$ is just an arbitrary $k[U]$-module; moreover, (1) implies that $\mathcal{F}$ is determined on U by F and localisation. This is the construction of the sheaf $\widetilde{F}$ on U from a $k[U]$-module F.

It is instructive to compare the condition in (1) with the topologist's notion of a map of vector bundles. (1) is a homomorphism of free sheaves $\mathcal{O}_U^{\oplus N} \to \mathcal{O}_U^{\oplus M}$, and so for a choice of bases is determined by a $N \times M$ matrix A with coefficients in $\Gamma(U, \mathcal{O}_X)$. In algebraic geometry, we must allow the rank of the matrix A to vary from point to point. It is upper semicontinuous in any case, since $\operatorname{rank} A \le r$ is a closed condition.

B.8. Examples

Example 1. If $Y \subset X$ is a subvariety (subscheme) of an affine variety defined by $f_1 = \cdots = f_n = 0$ then the structure sheaf $\mathcal{O}_Y$ is determined by the exact sequence

$$\mathcal{O}_X^n \xrightarrow{F} \mathcal{O}_X \to \mathcal{O}_Y \to 0, \qquad \text{where } F = (f_1, \dots, f_n).$$

Obvious F has rank 1 outside Y and rank 0 along Y.

Example 2. Now let $X = \mathbb{A}^n$ and $Y = \mathbb{A}^{n-1} : (x_n = 0) \subset X$, and consider the surjective homomorphism $p\colon \mathcal{O}_X^r \to \mathcal{O}_Y$ determined by $(f_1, \dots, f_r) \mapsto f_{r|Y}$. Obviously the kernel of p is the subsheaf $\mathcal{O}_X^{r-1} \oplus \mathcal{I}_Y \subset \mathcal{O}_X^r$. Since $\mathcal{I}_Y = x_n \cdot \mathcal{O}_X$ is itself a free sheaf, $\ker p \cong \mathcal{O}_X^r$. Using this isomorphism I get a short exact sequence

$$0 \to \mathcal{O}_X^r \xrightarrow{A} \mathcal{O}_X^r \to \mathcal{O}_Y \to 0, \qquad \text{where } A = \operatorname{diag}(1, \dots, 1, x_n). \tag{3}$$

The construction $\ker p$ passes from a locally free sheaf $\mathcal{E}$ to $\mathcal{E}' = \ker p \subset \mathcal{E}$, where p is the composite of restricting to a divisor $Y \subset X$ and a projection of $\mathcal{E}_{|Y}$ to a quotient bundle. This construction is well known as a standard elementary transformation of vector bundles.

Notice that the homomorphism A of sheaves is injective even at points of Y where A drops rank. The point is that the map of sheaves only looks at sections over opens, or stalks, and does not look at the *fibre* of the vector bundle. The stalk looks like $\mathcal{O}_{X,P}^r$, a free module over the local ring, whereas the fibre looks like the quotient $\mathcal{E}_P / m_P \mathcal{E}_P$, which is a k-vector space.

Example 3. A similar example. Suppose that $X = \mathbb{A}^2$ and that $Y \subset X$ is the subscheme defined by $f = g = 0$; suppose for simplicity that Y only lives at one point, that is $V(f, g) = \{P\}$. The subscheme Y is the point P with structure sheaf the finite dimensional ring $\mathcal{O}_Y = \mathcal{O}_{X,P}/(f, g)$. Then $\mathcal{O}_{X,P}$ is a UFD, so it's easy to check that the following sequence

$$0 \to \mathcal{O}_{X,P} \xrightarrow{-g,f} \mathcal{O}_{X,P} \oplus \mathcal{O}_{X,P} \xrightarrow{\substack{f \\ g}} \mathcal{O}_{X,P} \to \mathcal{O}_Y \to 0$$

is exact. It's called the *Koszul complex* of f, g; its construction only depends on the fact that f, g forms a regular sequence in $\mathcal{O}_{X,P}$.

Now the point of this example is that every section of a locally free sheaf of rank 2 with only zeros in codimension 2 looks like this.

As a rule, traditional topologists have only allowed maps of constant rank between vector bundles, which is equivalent to saying that the kernel, image and cokernel are locally direct summands. As we have seen in Examples 1–3, the more general notion of sheaf homomorphism between locally free sheaves is very useful in algebraic geometry.

B.9. Rules of coherent cohomology

This table of rules states the main useful results of coherent cohomology at a fairly simple level of generality. I will take them as axioms throughout. For the proofs, see [FAC]. Anyone complaining that the paper is in French will receive a blast of unpleasant sarcasm. Together with the axioms, it is also very useful to know some of their immediate consequences, such as the cohomology of line bundles on $\mathbb{P}^n$ worked out in Ex. 3.1–2.

Actually, the hard thing is not to get used to these rules, but to understand what a coherent sheaf is.

Data 1. For any variety X over k and any (quasi-) coherent sheaf $\mathcal{F}$ on X there is a k-vector space $H^i(X, \mathcal{F})$, that is functorial in $\mathcal{F}$. In other words a homomorphism of sheaves of $\mathcal{O}_X$-modules $a\colon \mathcal{F} \to \mathcal{G}$ gives rise to a linear map $a_*\colon H^i(X, \mathcal{F}) \to H^i(X, \mathcal{G})$, with obvious compatibilities. (To answer the topologist's immediate question: there is no functoriality for morphisms of varieties X as yet. Sheaf cohomology is a property of the category of sheaves over a fixed X.)

Data 2. If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is a short exact sequence of quasicoherent sheaves on X then there is a coboundary map

$$d_i\colon H^i(X, \mathcal{F}'') \to H^{i+1}(X, \mathcal{F}'),$$

again functorial in exact sequences.

So far, $H^*(X, \text{blank})$ is a *cohomological δ-functor*, if you like that kind of thing. This data satisfies the following conditions:

i. **Sections H^0.**

$$H^0(X, \mathcal{F}) = \Gamma(X, \mathcal{F}),$$

the space of sections of a sheaf, as in the definition of a sheaf.

ii. **Affine varieties X.** If X is affine then

$$H^i(X, \mathcal{F}) = 0 \qquad \text{for all } i > 0.$$

Moreover, $H^0(X, \mathcal{F})$ is a sufficiently big module over the affine coordinate ring $k[X] = H^0(X, \mathcal{O}_X)$, so that the following localisation works:

$$H^0(U, \mathcal{F}) = H^0(X, \mathcal{F}) \otimes_{k[X]} H^0(U, \mathcal{O}_X) \qquad \text{for every open } U \subset X;$$
$$\mathcal{F}_P = H^0(X, \mathcal{F}) \otimes_{k[X]} \mathcal{O}_{X,P} \qquad \text{for every point } P \in X.$$

Actually, you have to prove all this before the notion of coherent sheaf is reasonably intrinsic (compare B.7).

iii. **Dimension.**

$$H^i(X, \mathcal{F}) = 0 \qquad \text{for all } i > \dim X$$

The topologist who at last finds some mild satisfaction should beware that I mean the dimension of X as an algebraic variety, e.g., an algebraic curve has dimension 1 (although over the complex numbers it's a Riemann surface).

iv. **Long exact sequence.** If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is a short exact sequence of quasicoherent sheaves on X then the functoriality homomorphisms of Data 1 and the coboundary homomorphisms of Data 2 give a cohomology long exact sequence

$$\begin{aligned} \cdots \to H^i(X, \mathcal{F}') \to H^i(X, \mathcal{F}) \to H^i(X, \mathcal{F}'') \to \\ \to H^{i+1}(X, \mathcal{F}') \to \cdots \end{aligned}$$

v. **Finite dimensionality.** If $\mathcal{F}$ is coherent and X is proper (for example, projective) then

$$H^i(X, \mathcal{F}) \qquad \text{is finite dimensional over } k \text{ for any } i.$$

One traditionally writes $h^i(X, \mathcal{F}) = \dim_k H^i(X, \mathcal{F})$.

vi. **Ample line bundle, Serre vanishing.** Suppose that $X \subset \mathbb{P}^n$ is a closed subvariety. Let $\mathcal{O}_X(1) = \mathcal{O}_X \otimes \mathcal{O}_{\mathbb{P}^n}(1)$ be the invertible sheaf obtained by restricting $\mathcal{O}_{\mathbb{P}^n}(1)$ to X; this is the sheaf having the homogeneous coordinates $x_0, \dots, x_n$ as sections (compare B.6 for a definition of $\mathcal{O}_{\mathbb{P}^n}(1)$). Write $\mathcal{O}_X(r)$ for the r times tensor product of $\mathcal{O}_X(1)$, and for any quasicoherent sheaf $\mathcal{F}$ on X, write $\mathcal{F}(r) = \mathcal{F} \otimes \mathcal{O}_X(r)$.

Then given a coherent sheaf $\mathcal{F}$, there exists N such that all the following hold for all $r \geq N$: the space of global sections $H^0(X, \mathcal{F}(r))$ is big enough so that

$$\operatorname{Im}\{H^0(X, \mathcal{F}(r)) \to H^0(U, \mathcal{F}(r))\} \qquad \text{generates } H^0(U, \mathcal{F}(r))$$

as a module over $H^0(U, \mathcal{O}_X)$ for every open $U \subset X$, and

$$\operatorname{Im}\{H^0(X, \mathcal{F}(r)) \to \mathcal{F}(r)_P\} \qquad \text{generates } \mathcal{F}(r)_P$$

as a module over $\mathcal{O}_{X,P}$ for every point $P \in X$. In other words, $\mathcal{F}(r)$ is *generated by its H^0*. Moreover,

$$H^i(X, \mathcal{F}(r)) = 0 \qquad \text{for all } i > 0.$$

This is called *Serre vanishing.*

Actually everything in (vi), apart from the language, is a trivial consequence of (ii) applied to the affine cone C_X over X. A coherent sheaf $\mathcal{F}$ on X corresponds to a finitely generated graded module $\bigoplus_{r \geq 0} H^0(X, \mathcal{F}(r))$ over the usual homogeneous coordinate ring of $X \subset \mathbb{P}^n$, the affine coordinate ring of C_X. Thus the language is just a formal way of saying the usual correspondence between homogeneous polynomials and functions on $X \subset \mathbb{P}^n$.

vii. **Serre duality.** Let X be a nonsingular projective n-fold and K_X its canonical divisor class, so that $\mathcal{O}_X(K_X) = \Omega_X^n = \bigwedge^n \Omega_X^1$. Then

$$H^n(X, \mathcal{O}_X(K_X)) \qquad \text{is a 1-dimensional vector space} \cong k.$$

Assuming nobody objects, I pick a generator and write $= k$.

For any invertible sheaf $\mathcal{L} = \mathcal{O}_X(D)$, write $\mathcal{L}^{-1} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X) = \mathcal{O}_X(-D)$. Then there is a canonical pairing

$$H^i(X, \mathcal{L}) \times H^{n-i}(X, \mathcal{L}^{-1} \otimes_{\mathcal{O}_X} \mathcal{O}_X(K_X)) \to k,$$

or

$$H^i(X, \mathcal{O}_X(D)) \times H^{n-i}(X, \mathcal{O}_X(K_X - D)) \to k,$$

which establishes a duality between the two groups.

Remark. You can ignore this remark *en première lecture*. Of course Serre duality can be generalised to singular X and arbitrary coherent sheaves $\mathcal{F}$. However, in the same way that Poincaré duality for singular cohomology requires a manifold, or at least a space satisfying a suitable local duality, the general form is a bit complicated. If X is Cohen–Macaulay and $\dim X = n$ then there exists a sheaf ω_X, the *Grothendieck dualising sheaf*, such that $H^n(X, \omega_X) = k$, and for any coherent sheaf $\mathcal{F}$ there is a canonical pairing

$$H^i(X, \mathcal{F}) \times \operatorname{Ext}^{n-i}_{\mathcal{O}_X}(\mathcal{F}, \omega_X) \to H^n(X, \omega_X) = k$$

which establishes a duality between the two groups.

If X is not Cohen–Macaulay, for example, if it has components of different dimension, then you can't expect a duality that works in a single dimension (i against $n - i$), and ω_X is replaced by a complex.[2]

viii. **Euler–Poincaré characteristic $\chi(X, \mathcal{F})$ and Hilbert polynomial.** Whenever the dimensions are finite, I write $h^i(X, \mathcal{F}) = \dim_k H^i(X, \mathcal{F})$. Define the Euler–Poincaré characteristic of $\mathcal{F}$ by $\chi(X, \mathcal{F}) = \sum_{i=0}^{\infty} (-1)^i h^i(X, \mathcal{F})$. Although its definition involves all the cohomology groups $H^i(X, \mathcal{F})$, this alternating sum is in fact a much more elementary quantity. For example, from the cohomology long exact sequence (iv), it follows at once that

$$\chi(X, \mathcal{F}) = \chi(X, \mathcal{F}') + \chi(X, \mathcal{F}'').$$

For $X \subset \mathbb{P}^n$, the numerical function given by $r \mapsto \chi(X, \mathcal{F}(r))$ is a polynomial, called the *Hilbert polynomial* of $\mathcal{F}$.

ix. **Riemann–Roch.** For a divisor D on an algebraic curve C

$$\chi(\mathcal{O}_C(D)) = h^0(\mathcal{O}_C(D)) - h^1(\mathcal{O}_C(D)) = \chi(\mathcal{O}_C) + \deg D,$$
$$\text{and} \quad \chi(\mathcal{O}_C) = 1 - g, \qquad \text{where } g = g(C) \text{ is the genus.}$$

[2] I have colloquial lecture notes on this topic which I may include in a later edition. See also, for example, [R1], App. to §2.

For a divisor D on an algebraic surface X

$$\chi(\mathcal{O}_X(D)) = h^0(\mathcal{O}_X(D)) - h^1(\mathcal{O}_X(D)) + h^2(\mathcal{O}_X(D)) = \chi(\mathcal{O}_X) + \tfrac{1}{2}D(D-K_X),$$
$$\text{and}\quad \chi(\mathcal{O}_X) = \frac{1}{12}(c_1^2+c_2) = \frac{1}{12}(K_X^2+e(X)),$$

where K_X is the canonical class of X and $e(X) =$ the topological Euler number of X, the alternating sum of Betti numbers. These formulas are discussed further in Ex. B.22–24 and in 3.2 and 3.4.

For an arbitrary sheaf on a projective variety X,

$$\chi(X,\mathcal{F}) = \int \operatorname{ch}\mathcal{F}\cdot \operatorname{Td}_X,$$

where $\operatorname{ch}\mathcal{F}$ and Td_X are certain characteristic classes of the sheaf $\mathcal{F}$ and the tangent sheaf of X, and the integral sign (also pronounced "evaluate on the fundamental class of X") means in practice that you take the sum of homogeneous terms of degree $\dim X$, interpret them as a zero dimensional cycle on X, and then as an integer.

You can't be a grown-up algebraic geometer until you have memorised these formulas. Eventually you have to learn what they mean, and how to calculate with them as well.

Exercises to Chapter B

1. Prove that the constant sheaf $\mathbb{Z}$ cannot be made into a sheaf of $\mathcal{O}_X$-modules.

2. Recall that the stalk $\mathcal{F}_P$ of a sheaf $\mathcal{F}$ at a point P is defined as the direct limit of the sets of sections $\Gamma(U,\mathcal{F})$ taken over all $U \ni P$. If $\varphi\colon \mathcal{F}\to\mathcal{G}$ is a morphism of sheaves, show how to define $\varphi_P\colon \mathcal{F}_P\to\mathcal{G}_P$, and check that it is well defined.

3. If X is a variety with structure sheaf $\mathcal{O}_X$ and $P\in X$ a point, prove that the stalk $\mathcal{O}_{X,P}$ is a local ring.

4. Let $\pi\colon F\to X$ be a continuous map of topological spaces. A section of π over an open set $U\subset X$ is a map $s\colon U\to F$ such that $\pi\circ s = \mathrm{id}_U$. Prove that sections of F form a sheaf $\mathcal{F}$.

5. Let $\pi\colon F\to X$ be a vector bundle of rank r (in the continuous, differentiable, complex analytic or algebraic categories). Prove that $\mathcal{F}$ constructed in the preceding exercise is a locally free sheaf of rank r over the appropriate structure sheaf of X.

6 (harder). Prove that there is an equivalence of categories between vector bundles and locally free sheaves. You'll need to choose one of the continuous, differentiable, complex analytic or algebraic categories, and be careful to ensure that the two sides of your equivalence have the same morphisms; one (boring) possibility is to allow only isomorphisms as morphisms.

7. An *affine structure* on an n-dimensional manifold M is an atlas consisting of charts $\varphi_i\colon U_i \xrightarrow{\sim}$ ball in $\mathbb{R}^n$ such that the glueing maps $\varphi_i\circ\varphi_j^{-1}$ are affine linear transformations $x\mapsto Ax+B$. Show how to introduce a sheaf of affine linear functions on M, and to give an alternative definition of manifold with affine structure based on an affine linear structure sheaf.

8. If $\varphi\colon \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves, define $\operatorname{Im}\varphi \subset \mathcal{G}$ and prove that $\varphi\colon \mathcal{F} \to \operatorname{Im}\varphi$ is surjective. [Hint: As in B.6, use stalks $\mathcal{F}_P$.]

9. Let X be a projective variety, D a divisor on X, and $\mathcal{O}_X(D)$ the divisorial sheaf of D (see B.5). Choose a basis $f_0, \dots, f_n$ of the RR space $\mathcal{L}(D)$, that is, rational functions $f_i \in k(X)$ with $\operatorname{div} f_i + D \geq 0$; write $\varphi_{|D|}\colon X \to \mathbb{P}^n$ for the map defined by the ratio $f_0 : \cdots : f_n$. On the other hand, there is a map $\varphi_{\mathcal{L}}\colon X \to \mathbb{P}^n$ defined by the global sections of $\mathcal{L}$. Think through the definitions, and show that these two maps are identical.

10. On the complex plane $\mathbb{C}$ (with the complex topology), let $\mathcal{O}_{\text{an}}$ be the sheaf of holomorphic functions, and $\mathcal{O}^*_{\text{an}}$ the sheaf of invertible holomorphic functions; check you have mastered the language by writing down displayed formulas with $\{\,|\,\}$ for the sections of $\mathcal{O}_{\text{an}}$ and $\mathcal{O}^*_{\text{an}}$ over an appropriate domain.

Show that the exponential map $f \mapsto \exp(f)$ defines a morphism of sheaves $\exp\colon \mathcal{O}_{\text{an}} \to \mathcal{O}^*_{\text{an}}$, and that it is surjective. Prove that the kernel is the constant sheaf $2\pi i\mathbb{Z}$. Consider the exact sequence

$$0 \to 2\pi i\mathbb{Z} \to \mathcal{O}_{\text{an}} \xrightarrow{\exp} \mathcal{O}^*_{\text{an}} \to 0.$$

We agreed earlier that exp is surjective as a morphism of sheaves. Show that if $U \subset \mathbb{C}$ is the annular region $0 < a < |z| < b$ then $\Gamma(U, \mathcal{O}_{\text{an}}) \to \Gamma(U, \mathcal{O}^*_{\text{an}})$ is not surjective.

Find a necessary and sufficient condition on an open set $U \subset \mathbb{C}$ such that $\Gamma(U, \mathcal{O}_{\text{an}}) \to \Gamma(U, \mathcal{O}^*_{\text{an}})$ is surjective.

(In this question, the sheaves are not coherent algebraic sheaves. Only $\mathcal{O}_{\text{an}}$ is a coherent analytic sheaf. The two sheaves $2\pi i\mathbb{Z}$ and $\mathcal{O}^*_{\text{an}}$ are sheaves of Abelian groups, but obviously cannot be made into $\mathcal{O}_{\text{an}}$-modules.)

11. Show that

$$0 \to \mathcal{I}_Y \to \mathcal{O}_X \to \mathcal{O}_Y \to 0$$

is an exact sequence of sheaves on X whenever Y is a subvariety.

12. (a) Let C be a projective curve. Prove that

$$C \cong \mathbb{P}^1 \iff g(C) = 0.$$

[Hint: Use RR.]

(b) Let $C \subset Y$ be an irreducible curve in a nonsingular surface. Show that if $K_Y C = 0$ and $C^2 = -2$ then $C \cong \mathbb{P}^1$ so that C is a -2-curve.

13. Prove that any locally free sheaf $\mathcal{L}$ of $\mathcal{O}_X$-modules of rank 1 (invertible sheaf) on a nonsingular variety X is of the form $\mathcal{L} \cong \mathcal{O}_X(D)$. Prove that $\mathcal{O}_X(D) \cong \mathcal{O}_X(D')$ as sheaves of $\mathcal{O}_X$-modules if and only if $D \overset{\text{lin}}{\sim} D'$ (linear equivalence was defined in 1.6).

14. Let X be a nonsingular n-fold. Define a *rational* 1-*form* to be an expression $\sum f_i \mathrm{d}g_i$ with $f_i, g_i \in k(X)$ modulo the Leibnitz rules $\mathrm{d}a = 0$ for $a \in k$ and $\mathrm{d}(fg) = f\mathrm{d}g + g\mathrm{d}f$. Write $\Omega^1_{k(X)/k}$ for the set of rational 1-forms. Prove that it is an n-dimensional vector space over $k(X)$ with basis $\mathrm{d}g_1, \dots, \mathrm{d}g_n$, where $g_1, \dots, g_n$ is any (separable) transcendence basis of $k(X)/k$.

15. The sheaf Ω^1_X of regular 1-forms is defined by imposing regularity conditions on rational 1-forms; in other words, if $s \in \Omega^1_{k(X)/k}$, then s is regular at a point $P \in X$

if and only if it can be written $\sum f_i \mathrm{d}g_i$ with $f_i, g_i \in \mathcal{O}_{X,P}$. Prove that if $z_1, \dots, z_n$ are local coordinates at a point $P \in X$ then $\mathrm{d}z_1, \dots, \mathrm{d}z_n$ are local generators of Ω^1_X in a neighbourhood of P.

If you're happy with the tangent sheaf $\mathcal{T}_X$ or tangent bundle T_X of X, show that Ω^1_X can be identified with the sheaf of linear forms on $\mathcal{T}_X$ or T_X. That is, $\Omega^1_X = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{T}_X, \mathcal{O}_X)$ is the sheaf Hom, defined either by setting the stalk at P equal to $\Omega^1_{X,P} = \mathrm{Hom}_{\mathcal{O}_{X,P}}(\mathcal{T}_{X,P}, \mathcal{O}_{X,P})$, or by setting the sections $\Gamma(U, \Omega^1_X)$ over U equal to the set of morphisms $T_{X|U} \to k \times U$ that commute with the projection to U and are linear in each fibre.

16. Set $\Omega^n_X = \bigwedge^n \Omega^1_X$, the sheaf of regular n-forms. Prove that $\Omega^n_X = \mathcal{O}_X(K_X)$.

17. Show that $\mathcal{O}_{\mathbb{P}^n}(H) \cong \mathcal{O}_{\mathbb{P}^n}(1)$ where H is any hyperplane. Extend to $\mathcal{O}_{\mathbb{P}^n}(kH)$ for any $k \in \mathbb{Z}$.

18. Give a definition of $\mathcal{O}_{\mathbb{P}^n}(r)$ in terms of ratios f/g of homogeneous polynomials in $x_1, \dots, x_n$ of degree $d+r$ and d respectively (compare B.6). Do the same for $\mathcal{O}_{\mathbb{F}}(eL + dM)$ on the scrolls in terms of bihomogeneous polynomials. Notice that here the space is constructed in terms of a group action, its structure sheaf in terms of invariant rational functions, and the other eigenspaces (character spaces) of rational functions correspond to divisorial sheaves (locally free sheaves of rank 1).

19. In the notation of Chapter 2, let $\mathbb{F} = \mathbb{F}(a_1, \dots, a_n)$ be the scroll, and M the divisor class linearly equivalent to $D_i + a_i L$. Prove that the pushforward of $\mathcal{O}_X(M)$ (defined in B.6) is a sheaf of $\mathcal{O}_{\mathbb{P}^1}$-modules isomorphic to $\mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_n)$.

20. Construct an example of an invertible sheaf $\mathcal{L}$ on a variety X generated by its H^0, but whose sections do not separate points. Construct an example so that the sections separate points but do not separate tangent vectors.

21. State and prove Bezout's theorem on the surface scroll $\mathbb{F}_a = \mathbb{F}(a, 0)$; in other words, if C and D are curves of bidegree d, e and d', e' with no common components, state a formula for the number of points of $C \cap D$ counted with multiplicities, and prove it by the argument sketched in A.9 of the notes. You'll need to figure out the dimension of the space of forms of bidegree d, e.

22. From now on C is a nonsingular projective curve. I assume known that $\deg \operatorname{div} f = 0$ for any rational function $f \in k(C)$, that is, a rational function has the same number of zeros and poles (counted with multiplicities).

(a) Prove that if $\deg D < 0$ then $H^0(C, \mathcal{O}_C(D)) = 0$.

(b) Prove that $h^0(\mathcal{O}_C(D - P)) = h^0(\mathcal{O}_C(D))$ or $h^0(\mathcal{O}_C(D)) - 1$.

(c) Prove that $h^0(\mathcal{O}_C(D)) \le \deg D + 1$, and that equality holds, with $\deg D \ge 1$, if and only if $C \cong \mathbb{P}^1$.

23. Given that $\chi(\mathcal{O}_C) = 1 - g$, use induction and short exact sequences of the form

$$0 \to \mathcal{O}_C(D - P) \to \mathcal{O}_C(D) \to k_P \to 0$$

to prove RR. [Hint: If $0 \to V_0 \to \cdots \to V_n \to 0$ is an exact sequence of finite dimensional vector spaces then $\sum_{i=0}^n (-1)^i \dim V_i = 0$. This exercise is carried out in [H1], Chapter IV.]

24. Use RR and Serre duality to prove that $\deg K_C = 2g - 2$ and $h^0(K_C) = g$.

To prove that the number g appearing in RR is the same as the number g in the famous picture of the surface with g holes, you have to use that T_C is the dual of

K_C, so has degree $2-2g$, and some form of Gauss–Bonnet: the number of zeros of a regular vector field, counted with their indexes, equals the Euler characteristic. Any argument involving coherent cohomology (polynomials) on one side and topology on the other is automatically deeper than anything purely in algebraic geometry or purely in topology.

25. For points $P_1, \cdots, P_k \in \mathbb{P}^2$, write $h^0(\mathbb{P}^2, \mathcal{I}_{P_1+\cdots+P_k} \cdot \mathcal{O}(2))$ for the vector space of conics through $P_1, \cdots, P_k$, and $h^1(\mathbb{P}^2, \mathcal{I}_{P_1+\cdots+P_k} \cdot \mathcal{O}(2))$ for the space of linear dependence relations between the conditions $P_1, \ldots, P_k$ impose on conics (compare the example in B.6). State and prove the results of [UAG], §1 on the dimension of the space conics through points $P_1, \cdots, P_k$ in terms of coherent cohomology groups

$$h^1(\mathbb{P}^2, \mathcal{I}_{P_1+\cdots+P_k} \cdot \mathcal{O}(2)).$$

26. Let $P_1, \ldots, P_9 \in C \subset \mathbb{P}^2$ be 9 distinct points contained in a nonsingular cubic curve. Suppose that $h^1(\mathbb{P}^2, \mathcal{I}_{P_1+\cdots+P_9} \cdot \mathcal{O}(3)) \neq 0$. Prove that the surface $S = \mathrm{Bl}_{P_1,\ldots,P_9}\mathbb{P}^2$ obtained by blowing up $P_1, \ldots, P_9$ has an elliptic fibration $S \to \mathbb{P}^1$.

27. The sheafication $\mathrm{sh}(\mathcal{F})$ of a presheaf. If $\mathcal{F}$ is a presheaf, there is an *associated sheaf* or *sheafication* $\mathrm{sh}(\mathcal{F})$ which satisfies the universal mapping property for homomorphisms from $\mathcal{F}$ to a sheaf. Construct $\mathrm{sh}(\mathcal{F})$ and prove the universal mapping property. The idea is to consider the stalks $\mathcal{F}_P$, and set

$$\Gamma(U, \mathcal{F}) = \text{good maps } P \mapsto s_P \in \mathcal{F}_P \text{ for all } P \in U,$$

where "good" means that all the s_Q for Q in some small neighbourhood V_P of P are the restrictions of some $s \in \Gamma(V_P, \mathcal{F})$. If you have trouble with this question, refer to [H1], Chap. II or one of the books on sheaf theory.

28. If $\mathcal{F} \subset \mathcal{G}$ is a subsheaf, construct the quotient sheaf $\mathcal{G}/\mathcal{F}$ as the associated sheaf of the presheaf $U \mapsto \Gamma(U, \mathcal{G})/\Gamma(U, \mathcal{F})$, and prove that it has the universal mapping property for maps from $\mathcal{G}$ to a sheaf killing $\mathcal{F}$. Prove also that its stalks are $\mathcal{G}_P/\mathcal{F}_P$, so that the sequence $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{G}/\mathcal{F} \to 0$ is exact.

29. If $f\colon X \to Y$ is a continuous map of topological spaces and $\mathcal{F}$ is a sheaf on Y, construct the sheaf theoretic pullback $f^{-1}\mathcal{F}$, whose stalk at $P \in X$ is $\mathcal{F}_{f(P)}$. Prove that it has the universal mapping property for sheaves $\mathcal{G}$ on X such that there exists a sheaf homomorphism $\mathcal{F} \to f_*\mathcal{G}$.

Incidentally, you mustn't write f^* for f^{-1}, because f^* is usually reserved for the pull back of sheaves of $\mathcal{O}_X$-module, given by $f^{-1}\mathcal{F} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$.