CHAPTER 4. Singularities and surfaces

This chapter turns to surface singularities. The main object of study is a normal (isolated) surface singularity $P \in X$ and a resolution $f\colon Y \to X$, with $f^{-1}P = \bigcup \Gamma_i$ a bunch of curves on a nonsingular surface Y. I introduce examples of three classes of singularities, the Du Val singularities, more general rational singularities and elliptic Gorenstein singularities; the latter two classes can be viewed as generalisations of the Du Val singularities. These singularities play an important role all over the classification of surfaces and 3-folds. At the same time, I discuss invariants of singularities, mainly concerned with quantifying the difference between X and Y in the resolution $f\colon Y \to X$. The results here are mostly taken from Artin [A1–2], Laufer [L1–3] and my unpublished manuscript [R], although much of it was known in some form to Du Val in the 1930s. Any area of math can be given a superior treatment using several decades' hindsight (and not least, Mori's notes [Mori]), and in this spirit, this chapter contains new proofs of the foundational results on rational and elliptic Gorenstein surface singularities.

Even for the reader only interested in nonsingular varieties or complex manifolds, there are several reasons for studying singularities. A philosophical point is that the study of varieties and the study of singularities each contains the other many times over. The techniques of this chapter mostly involve a bunch of curves $\{\Gamma_i\}$ on a surface Y, and don't really need the singularity $P \in X$ as such; they apply in other situations, notably to a fibre $f^{-1}P$ of a surface fibred over a curve $f\colon X \to C$. The main technical methods used here are those of the previous chapters, divisors on a nonsingular surface, their cohomology, and so on. As I said for K3s at the start of Chapter 3, this material provides excellent practice for calculations involving intersection numbers, coherent cohomology, and results based on these.

Summary

(1) Examples: the ordinary double point and how it arises, the remaining Du Val singularities A_n, D_n, E_6, E_7, E_8 and their resolutions
(2) Numerical cycle and multiplicity; there is a unique minimal exceptional divisor $0 \neq Z_{\mathrm{num}}$, called the *numerical cycle*, with $Z_{\mathrm{num}}\Gamma \le 0$ for every exceptional curve Γ. In simple cases, properties of $P \in X$ can be expressed in terms of Z_{num}
(3) $R^1f_*\mathcal{O}_Y$, and how you calculate it; there is a unique minimal divisor Z_1 on Y, the *cohomological cycle*, that carries the cohomology $H^1(\mathcal{O}_Y)$
(4) Characterisations of rational singularities
(5) Contractibility of a bunch of curves and application to Castelnuovo's criterion and to Du Val singularities. Minimal resolution, minimal models of surfaces
(6) Gorenstein condition: $P \in X$ is Gorenstein if and only if the canonical class of Y is represented by an exceptional divisor Z_K, the *canonical cycle*
(7) Rational and Du Val singularities; how these relate to canonical models of surfaces, projective models of K3s, elliptic pencils
(8) Elliptic Gorenstein surface singularities, their numerical characterisation
(9) Graded rings on divisors, multiplicity and embedding dimension, proof of the main theorems on rational and elliptic Gorenstein surface singularities

4.1. Example: the ordinary double point

As a first example, consider the ordinary quadratic cone

$$P = (0,0) \in X : (xz = y^2) \subset \mathbb{A}^3.$$

This singularity occurs throughout the theory of algebraic surfaces, and can be used to illustrate a whole catalogue of arguments. Because X is a cone with vertex P, it has the standard "cylinder" resolution $Y \to X$; that is, the cone is a union of generating lines through P, and Y is the disjoint union of the generating lines. In other words, Y is the blowup, or the correspondence between the cone and the base of the cone. This singularity and its resolution have already appeared in Chapter 2, where the surface scroll $\mathbb{F}_2 = \mathbb{F}(0,2)$ has a morphism to $\overline{\mathbb{F}}(0,2) \subset \mathbb{P}^3$ (the ordinary quadratic cone) contracting the negative section.

The exceptional curve $f^{-1}P = \Gamma$ of the resolution is a -2-*curve*, satisfying

$$\Gamma \cong \mathbb{P}^1, \quad \Gamma^2 = -2 \quad (\text{or equivalently, } K_Y\Gamma = 0).$$

A particular case of Contraction Theorem 4.15 is that every -2-curve is obtained in this way by resolving an ordinary double point (that is, a double point with nondegenerate tangent cone). On minimal surfaces (where there are no -1-curves), -2-curves are characterised as the irreducible curves with $K_Y\Gamma \le 0$ and $\Gamma^2 < 0$, and they arise naturally for this reason in many important contexts of the classification of surfaces.

The ordinary double point also appears in Chapter 1 on cubic surfaces, and in the theory of projective embeddings of K3s in Chapter 3. As we saw in Chapter 1, a nonsingular cubic surface is isomorphic to $\mathbb{P}^2$ blown up in 6 "general" points $P_1, \dots, P_6$, embedded by the linear system of cubics through $\{P_1, \dots, P_6\}$. If P_1, P_2, P_3 are collinear (but the $\{P_i\}$ are otherwise "general") then on the blowup $X = \mathrm{Bl}_6(\mathbb{P}^2)$, the line through P_1, P_2, P_3 becomes a -2-curve, and is contracted by $X \to \overline{X} \subset \mathbb{P}^3$ to an ordinary double point of a cubic surface. Similarly, as described in 3.15.1, if X is a K3 and $|D|$ a free nonhyperelliptic linear system on X, the birational morphism $\varphi_D \colon X \to \overline{X} \subset \mathbb{P}^g$ contracts exactly curves Γ with $D\Gamma = 0$; any such curve is a -2-curve (because $K_X\Gamma = 0$ and $\Gamma^2 < 0$), and if there is exactly one then $\varphi_D(\Gamma) = P \in \overline{X}$ is an ordinary double point.

4.2. The Du Val singularities

Example: D_4**.** Consider the singularity

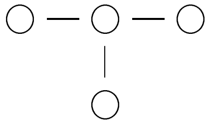
$$P = (0,0,0) \in X : (g = x^2 + y^3 + z^3 = 0) \subset \mathbb{A}^3.$$

The blow-up $X_1 \to X$ is covered by 3 affine pieces, of which I only write down one: consider $\mathbb{A}^3$ with coordinates x_1, y_1, z, and the morphism $\sigma \colon \mathbb{A}^3 \to \mathbb{A}^3$ defined by $x = x_1 z, y = y_1 z, z = z$. The inverse image of X under σ is defined by

$$g(x_1z, y_1z, z) = x_1^2z^2 + y_1^3z^3 + z^3 = z^2 g_1, \qquad \text{where } g_1 = x_1^2 + (y_1^3 + 1)z.$$

Here the factor z^2 vanishes on the exceptional (x_1, y_1)-plane $\mathbb{A}^2 = \sigma^{-1}P \subset \mathbb{A}^3$, and the residual component $X_1 : (g_1 = 0) \subset \mathbb{A}^3$ is the birational transform of X. Now clearly the inverse image of P under $\sigma \colon X_1 \to X$ is the y_1-axis, and X_1 has ordinary double points at the 3 points where $y_1^3+1=0$. (Please check for yourselves that the other affine pieces of the blow-up have no further singular points.) The resolution Y is obtained on blowing up these three points $Y \to X_1 \to X$.

I claim that $f^{-1}P$ consists of -2-curves $\Gamma_0, \Gamma_1, \Gamma_2, \Gamma_3$ meeting as follows:



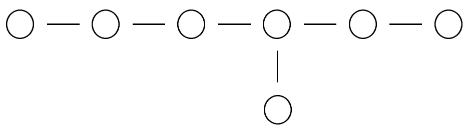
the configuration is the Dynkin diagram D_4. To prove this, it is clear that $\Gamma_1, \Gamma_2, \Gamma_3$ are -2-curves, since they arise from the blowup of ordinary double points. Also the fact that $\Gamma_0 \cong \mathbb{P}^1$, and Γ_0 meets each of Γ_i transversally in 1 point, can be verified directly from the coordinate description of Y. Finally, to see that $\Gamma_0^2 = -2$, note that y is a regular function on Y whose divisor is $\operatorname{div} y = 2\Gamma_0 + \Gamma_1 + \Gamma_2 + \Gamma_3 + C$, where C is the curve $y_1 = 0$ in Y, which also meets Γ_0 transversally in 1 point. Thus

$$0 = (\operatorname{div} y)\Gamma_0 = 2\Gamma_0^2 + \Gamma_0(\Gamma_1 + \Gamma_2 + \Gamma_3 + C) = 2\Gamma_0^2 + 4, \qquad \text{so that } \Gamma_0^2 = -2.$$

The remaining Du Val singularities can be resolved in a similar way (see Ex. 4.3). The equations are

$$\begin{aligned}
A_n &: \quad x^2 + y^2 + z^{n+1} = 0,\\
D_n &: \quad x^2 + y^2 z + z^{n-1} = 0,\\
E_6 &: \quad x^2 + y^3 + z^4 = 0,\\
E_7 &: \quad x^2 + y^3 + yz^3 = 0,\\
E_8 &: \quad x^2 + y^3 + z^5 = 0,
\end{aligned}$$

and the resolution is a bunch of -2-curves whose configuration is given by the corresponding Dynkin diagram; for example, E_7 is the following:



4.3. The general set-up

Let X be an affine surface and $P \in X$ a singular point of X. I want to talk about isolated surface singularities $P \in X$; old-fashioned singularity theorists always write (X, P), but my notation is shorter and self-documenting. It is reasonable to restrict attention to normal singularities, for reasons I will explain. So X is a surface, that is, a 2-dimensional algebraic variety (over an algebraically closed field k, for example, $k = \mathbb{C}$), and $P \in X$ an isolated singular point. I'm usually only interested in a neighbourhood of $P \in X$, so without loss of generality, I can choose X affine and $P \in X$ the only singularity. However, X is contained in a projective variety $\overline{X}$, and for some purposes it may be more convenient to think of X and Y as projective.

A *resolution* of X is a birational projective morphism $f\colon Y \to X$ from a nonsingular surface Y. It is an important and difficult theorem that a resolution always exists. I can assume that f gives $Y \setminus f^{-1}P \xrightarrow{\sim} X \setminus P$. By definition, f is *projective* if it factors as $Y \hookrightarrow X \times \mathbb{P}^N \to X$ with the first map a closed embedding. If X is projective, the condition on f is equivalent to Y also projective. In general, $f^{-1}P$ is a connected bunch of projective curves $\bigcup_{i=1}^n \Gamma_i$; it is traditional to say that f is a *good* resolution (or, more recently, a *log* resolution, meaning resolution in the category of *log varieties*) if $f^{-1}P$ is a union of nonsingular curves meeting transversally, but I do not necessarily assume this.

The surface X is normal if and only if $\mathcal{O}_X = f_*\mathcal{O}_Y$. Here $\mathcal{O}_Y$ is the structure sheaf of Y. The point is just that as a topological space, the normal variety X is the quotient space $Y/\!\sim$ where $\sim$ is the equivalence relation that collapses $f^{-1}P$ to a point; and X has structure sheaf

$$f_*\mathcal{O}_Y = \bigl\{\text{regular functions on } Y \text{ that are constant on } f^{-1}P\bigr\}.$$

Any other variety structure X' on this topological space (having a morphism $Y \to X'$) is obtained by taking a subring $\mathcal{O}_{X'} \subset \mathcal{O}_X$ of finite codimension, that is, $\mathcal{O}_{X'}$ is a subring containing m_P^n for some n, where $m_P \subset \mathcal{O}_{X,P}$ is the maximal ideal. Normal implies nonsingular in codimension 1, and the converse holds for hypersurfaces or complete intersections by the Serre criterion, compare [YPG], App. to §3. If my display of erudition confuses you, just concentrate on the case of isolated hypersurface singularities, which are automatically normal, and lots of fun in their own right.

4.4. Changing to the analytic category

When studying singularities, I do not spend much time worrying about which category of geometry I work in. A natural alternative to the language used here is to work up to local analytic equivalence: any ordinary double point is locally isomorphic to the singularity $y^2 = xz$ of 4.1 by a suitable analytic coordinate change. This does not make sense in the category of algebraic varieties: ordinary double points appear on singular cubic surfaces, or on (contracted) K3 surfaces, etc., and then, of course, no Zariski neighbourhood of $P \in X$ is isomorphic to the cone $y^2 = xz$ in the category of algebraic varieties. My justification for being sloppy here is that everything I need to know about an isolated singularity $P \in X$ of a variety depends not on the local ring $\mathcal{O}_{X,P}$ itself, but only on a quotient $\mathcal{O}_{X,P}/m_P^N$ for some large N.

To be able to use the ideas of singularity theory, especially local normal forms, I could assume that the base field is $\mathbb{C}$, and use local complex geometry (typically, the inverse function theorem). Very roughly, since I am mostly concerned with the invariants of coherent cohomology, affine corresponds to Stein. On the other hand, it is not difficult to work over any field using formal completions or the language of the etale topology. Beware that the normal forms of the singularities used here may not work at all in characteristic p, especially when p is small.

4.5. The numerical cycle of a resolution Z_{num}

Given a resolution $f\colon Y \to X$ of $P \in X$, any nonzero element $g \in m_P \subset \mathcal{O}_{X,P}$ vanishes on the set theoretic fibre $f^{-1}P$, so that $\operatorname{div} g = \sum a_i\Gamma_i + C'$ is as in the proof of Theorem A.7. As we saw there, the exceptional divisor $D = \sum a_i\Gamma_i$ satisfies $D\Gamma_i \le 0$ for every contracted curve Γ_i.

Proposition. *There exists a unique effective exceptional divisor* $Z = Z_{\mathrm{num}} = \sum b_i\Gamma_i > 0$ *such that* $Z > 0$,

$$Z\Gamma_i \le 0 \qquad \textit{for every curve } \Gamma_i, \tag{1}$$

and Z *is minimal with this property.*

Proof. There certainly exists some cycle Z with this property, as I just said, so the only thing to prove is that if Z_1 and Z_2 both satisfy (1) then so does $Z = \gcd(Z_1, Z_2)$. This is more or less obvious: I can write $Z_1 = Z + A_1$ and $Z_2 = Z + A_2$, where $A_1, A_2 \ge 0$ have no common components; then any $\Gamma \not\subset A_1$ has $Z\Gamma = Z_1\Gamma - A_1\Gamma \le Z_1\Gamma \le 0$, and similarly for any $\Gamma \not\subset A_2$. Q.E.D.

Definition. Z_{num} is called the *numerical cycle* (or *numerical fundamental cycle*) of the bunch $\{\Gamma_i\}$. The discussion preceding the proposition shows that every $g \in m_P$ vanishes at least along Z_{num}, that is, $\operatorname{div} g \ge Z_{\mathrm{num}}$.

The numerical cycle is often called the *fundamental cycle* in the literature, but lots of other cycles are equally fundamental. In 4.6, I introduce the *fibre cycle* Z_{f}, which is closely related to the scheme theoretic fibre, and later in this chapter the *cohomological cycle* Z_1, which carries the cohomology $H^1(Y, \mathcal{O}_Y)$ (see 4.8), and the *canonical cycle* Z_K for a Gorenstein singularity $P \in X$, which carries the canonical class of Y in a neighbourhood of $f^{-1}P$ (see 4.19).

Note that Z_{num} is very easy to calculate from the intersection matrix $\Gamma_i\Gamma_j$ (see Ex. 4.4).

4.6. Fibre cycle Z_{f} and multiplicity

I define the *fibre cycle* Z_{f} to be the greatest divisor contained in every $\operatorname{div} g$, that is,

$$Z_{\mathrm{f}} = \gcd\{\operatorname{div} g \mid \text{for all } g \in m_P \subset \mathcal{O}_{X,P}\}.$$

In other words, consider the ideal sheaf $\mathcal{I} = m_P\mathcal{O}_Y$ of $\mathcal{O}_Y$ generated by all $g \in m_P$. The subscheme of Y defined by $\mathcal{I}$ is the scheme theoretic fibre of the morphism f, that is, the subscheme of Y defined by $g = 0$ taken over all $g \in m_P$. By definition the fibre cycle Z_{f} is the maximal effective divisor contained in the scheme theoretic fibre, that is, such that $\mathcal{I} \subset \mathcal{O}_Y(-Z_{\mathrm{f}})$. In more geometric terms, take a finite dimensional vector subspace $V \subset \mathcal{O}_{X,P}$ which generates m_P as a $\mathcal{O}_{X,P}$-module, and for $g \in V$, consider the divisor H_g, a variable hyperplane through P. Then for fixed g, $\operatorname{div} g = f^*(H_g) = L + \sum b_i\Gamma_i$, where L has no components in common with $\{\Gamma_i\}$ and $\sum b_i\Gamma_i$ is an effective divisor. If $g \in V$ is "fairly general" then $\sum b_i\Gamma_i$ takes on a certain minimal value, which is Z_{f}.

It follows from the remarks in 4.5 that $Z_{\mathrm{f}} \ge Z_{\mathrm{num}}$; equality holds in some important cases: see Theorem 4.17 for rational singularities and Theorem 4.23 for elliptic Gorenstein singularities. However, in more complicated cases, for example, when some of the curves Γ_i have large genus, Z_{f} is not determined by simple numerical properties of the configuration $\{\Gamma_i\}$, and it is not hard to find examples with $Z_{\mathrm{f}} > Z_{\mathrm{num}}$ (see Ex. 4.13).

Definition. I take the *multiplicity* of a normal surface singularity $P \in X$ in the pedestrian sense: $\operatorname{mult}_P X$ is the dimension over k of $\mathcal{O}_{X,P}/(h_1, h_2)$ (also called its *length*), where $h_1, h_2 \in m_P$ are sufficiently general elements. In other words, this is the local intersection number at P of two general hyperplane sections through P. If $X \subset \mathbb{P}^N$ is normal and of degree d then any two hyperplane sections H_1, H_2 with no common components meet in d points counted with multiplicity, so that two general hyperplanes through P intersect in P counted $\operatorname{mult}_P X$ times plus $d - \operatorname{mult}_P X$ free intersection points outside P. Thus the multiplicity of P is the number of intersection points of H_1, H_2 absorbed[4] into P.

Lemma. *(i) Suppose that* $m_P\mathcal{O}_Y = \mathcal{O}_Y(-Z_{\mathrm{f}})$ *with* Z_{f} *the fibre cycle. Then* $\operatorname{mult}_P X = -Z_{\mathrm{f}}^2$.

(ii) More generally, if $m_P\mathcal{O}_Y = \mathcal{I}\mathcal{O}_Y(-Z_{\mathrm{f}})$ *with* $\mathcal{I}$ *an ideal defining a zero dimensional subscheme of* Y*, then* $\operatorname{mult}_P X \geq -Z_{\mathrm{f}}^2 + \dim_k \mathcal{O}_Y/\mathcal{I}$.

Remark. $m_P\mathcal{O}_Y = \mathcal{O}_Y(-Z_{\mathrm{f}})$ holds if and only if the resolution dominates the ordinary blowup of $P \in X$ (that is, the blowup of the maximal ideal $m_P \subset \mathcal{O}_X$). Because either condition says that the ratio between generators of m_P is a regular function on Y. Thus, by appropriate choice of resolution, I can always arrange that case (i) holds.

Proof. What's going on is the following: think of X as a projective surface $X \subset \mathbb{P}^N$, and consider the linear system $|m_P\mathcal{O}_X(1)|$ of hyperplane sections $H \ni P$. Taking f^* of these divisors defines a linear system $f^*|m_P\mathcal{O}_X(1)|$ on Y with some base locus, which, following tradition, I break up as the divisorial part Z_{f} and the codimension ≥ 2 part. The intersection of two elements $H_1, H_2 \in |m_P\mathcal{O}_X(1)|$ consists of P with multiplicity $\operatorname{mult}_P X$ plus a free part; I can calculate the free part just as well on Y, and I get the intersection outside $f^{-1}P$ of two elements of $|f^*H - Z_{\mathrm{f}}|$.

In case (i), $|f^*H - Z_{\mathrm{f}}|$ is free. If $h_1, h_2 \in \mathcal{O}_{X,P}$ are two general elements, and $H_1, H_2 \subset X$ the corresponding divisors, then $f^*(H_1) = M_1 + Z_{\mathrm{f}}$ and $f^*(H_2) = M_2 + Z_{\mathrm{f}}$ with M_1 and M_2 disjoint over P, that is, $M_1 \cap M_2 \cap f^{-1}P = \emptyset$. Then M_1M_2 is the free intersection of H_1, H_2 outside P.

Then since $(f^*H)Z_{\mathrm{f}} = 0$, I get

$$\begin{aligned} H^2 = f^*(H_1)f^*(H_2) &= f^*(H_1)(M_2 + Z_{\mathrm{f}}) = f^*(H_1)M_2 \\ &= (M_1 + Z_{\mathrm{f}})M_2 = M_1M_2 + Z_{\mathrm{f}}(f^*(H_2) - Z_{\mathrm{f}}) \\ &= M_1M_2 - Z_{\mathrm{f}}^2. \end{aligned}$$

That is, the base locus Z_{f} of $f^*|m_P\mathcal{O}_X(1)|$ absorbs $-Z_{\mathrm{f}}^2$ points of intersection, as required.

(ii) is similar; you have to prove that two general curves through a zero dimensional base scheme at $P \in Y$ have local intersection number $\geq \dim_k \mathcal{O}_{Y,P}/\mathcal{I}_P$. It is in any case obvious that any points of a zero dimensional base locus make a strictly positive contribution. Q.E.D.

[4]This is proved exactly as the Bezout theorem in A.9. However, note that it is essential for this that $P \in X$ is normal, so that sufficiently general $h_1, h_2 \in m_P$ form a regular sequence, and the usual Koszul sequence $0 \to \mathcal{O}_{X,P} \to \mathcal{O}_{X,P} \oplus \mathcal{O}_{X,P} \to (h_1, h_2) \subset \mathcal{O}_{X,P}$ defined by h_1, h_2 is exact. In general, the right definition of the multiplicity $\operatorname{mult}_P X$ is in terms of the leading term of the Hilbert–Samuel function: $h(k) = \dim m_P^k/m_P^{k+1} = (\operatorname{mult}_P X)k + \text{const}$. For a nonnormal isolated surface singularity, the Koszul complex for any h_1, h_2 is *never* exact in the middle, and $\dim \mathcal{O}_{X,P}/(h_1, h_2)$ is strictly bigger than $\operatorname{mult}_P X$. See [YPG], App. to §3 for more discussion.

4.7. $R^1 f_* \mathcal{O}_Y$, AND HOW TO CALCULATE IT

Here I'm interested in the sheaf $R^1 f_* \mathcal{O}_Y$ for a resolution $f\colon Y \to X$. Assuming that X is affine, we will see that this is the finite dimensional vector space $H^1(Y, \mathcal{O}_Y)$ at P.

Proposition. *Let $P \in X$ be a surface singularity, $f\colon Y \to X$ its resolution and $P \in U \subset X$ an affine neighbourhood of P. Then for any coherent sheaf $\mathcal{F}$ on Y,*

(1) *$H^p(f^{-1}U, \mathcal{F}) = 0$ for $p \geq 2$, and*
(2) *$H^1(f^{-1}U, \mathcal{F})$ is independent of U.*

Definition. $R^1 f_* \mathcal{F}$ is defined to be $H^1(f^{-1}U, \mathcal{F})$, viewed as a sheaf supported at the point $P \in X$. Note that $R^1 f_* \mathcal{F} = \varinjlim_U H^1(f^{-1}U, \mathcal{F})$, because by (2), the injective limit is constant for affine U. Since $H^1(f^{-1}U, \mathcal{F})$ is a $\Gamma(U, \mathcal{O}_X)$-module, it is clear that $R^1 f_* \mathcal{F}$ is a module over $\mathcal{O}_{X,P}$. I will prove shortly that it is a finite dimensional vector space.

Proof. $f^{-1}U$ is covered by 2 affine open sets V_1, V_2, so that the first assertion follows at once by calculating $H^p(f^{-1}U, \mathcal{F})$ by the Čech complex. If $U = \operatorname{Spec} A$ then arbitrarily small neighbourhoods of P are given by $U_g = \operatorname{Spec} A[g^{-1}]$ for $g \in A$; the Čech complex of the corresponding cover $V_i \cap f^{-1}(U_g)$ is obtained by applying the exact functor $\otimes_A A[g^{-1}]$, and therefore

$$H^1(f^{-1}(U_g), \mathcal{F}) = H^1(f^{-1}U, \mathcal{F}) \otimes_A A[g^{-1}] = H^1(f^{-1}U, \mathcal{F}). \quad \text{Q.E.D.}$$

Discussion. My treatment here is self-contained, but I give a brief discussion of some background in sheaf theory for the interested reader. Quite generally, for a morphism $f\colon Y \to X$ and sheaf $\mathcal{F}$ on Y, the sheaf $R^p f_* \mathcal{F}$ is defined as the associated sheaf of the presheaf $U \mapsto H^p(f^{-1}U, \mathcal{F})$. This means that $R^p f_* \mathcal{F}$ is the sheaf on X whose stalk at a point $P \in X$ is $(R^p f_* \mathcal{F})_P = \varinjlim_U H^p(f^{-1}U, \mathcal{F})$, where the $\varinjlim$ runs over opens $U \ni P$.

Note that $R^p f_* \mathcal{F}$ is defined in terms of cohomology of thinner and thinner open neighbourhoods of the fibre $f^{-1}P$. In contrast, the holomorphic functions theorem of Zariski and Grothendieck interprets $(R^p f_* \mathcal{F})_P$ for a coherent sheaf $\mathcal{F}$ in terms of the fibre $f^{-1}P$ and sheaves supported on it:

$$(R^p f_* \mathcal{F})_P = \varprojlim_Z H^p(f^{-1}P, \mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{O}_Z),$$

where the limit is taken over subschemes Z with support in $f^{-1}P$, which in practice means a limit over fatter and fatter infinitesimal neighbourhoods of the fibre. I prove this in the simple case I need in the next section (the general case is done in [EGA III_1], 4.1.5 or [H1], III.11).

The etymology comes from the idea that an element of the left-hand side is something like a Taylor series expansion of a cohomology class around the fibre, and the point is to prove that it converges to a "holomorphic" cohomology class on a neighbourhood of the fibre.

4.8. COHOMOLOGICAL CYCLE

Let $f^{-1}P = \bigcup_{i=1}^n \Gamma_i$; as in 3.10, given a divisor $D = \sum n_i \Gamma_i$ with $n_i \in \mathbb{Z}$, $n_i \geq 0$, write $\mathcal{O}_Y(-D) \subset \mathcal{O}_Y$ for the ideal sheaf of regular functions on Y vanishing along

D, and $\mathcal{O}_D = \mathcal{O}_Y/\mathcal{O}_Y(-D)$. This is a sheaf of rings on $f^{-1}P$, the structure sheaf of a subscheme $D \subset Y$. If $D_1 \le D$ is another divisor supported on $\bigcup \Gamma_i$ then $D = D_1 + D_2$ with $D_1, D_2 > 0$, so by Lemma 3.10, the inclusion $\mathcal{O}_Y(-D) \subset \mathcal{O}_Y(-D_1)$ induces a short exact sequence

$$0 \to \mathcal{O}_{D_2}(-D_1) \to \mathcal{O}_D \to \mathcal{O}_{D_1} \to 0 \tag{2}$$

Theorem. *(a) For every $D_1 \le D$ the induced map $H^1(\mathcal{O}_D) \to H^1(\mathcal{O}_{D_1})$ is surjective; the dimension $h^1(\mathcal{O}_D)$ is bounded above.*

(b) There is a unique smallest Z_1 such that $h^1(\mathcal{O}_{Z_1})$ takes the maximal value. Then for every $D \ge Z_1$, the map of (a) is an isomorphism $H^1(\mathcal{O}_D) = H^1(\mathcal{O}_{Z_1})$, and $h^1(\mathcal{O}_D) < h^1(\mathcal{O}_{Z_1})$ if $D \not\ge Z_1$.

(c) The exact sequence of sheaves $0 \to \mathcal{O}_Y(-D) \to \mathcal{O}_Y \to \mathcal{O}_D \to 0$ induces a surjective homomorphism $R^1 f_* \mathcal{O}_Y \to H^1(\mathcal{O}_D)$, which is an isomorphism for any $D \ge Z_1$.

Definition. The divisor Z_1 is called the *cohomological cycle* of $P \in X$.

The point of the theorem is that $R^1 f_* \mathcal{O}_Y$, defined in 4.7 in terms of open neighbourhoods $f^{-1}U$ of the fibre, is equal to the cohomology group $H^1(\mathcal{O}_{Z_1})$ of an exceptional divisor. This is the same thing as $\varprojlim_D H^1(f^{-1}P, \mathcal{O}_D)$, where the limit runs over fatter and fatter exceptional divisors D, as discussed in 4.7. Moreover, the limit is achieved at Z_1. The theorem is due to M. Artin in modern form, although it seems to have been known in substance to P. Du Val some 30 years earlier.

Proof. (a) The surjectivity of $H^1(\mathcal{O}_D) \to H^1(\mathcal{O}_{D_1})$ follows from (2) since by dimension

$$H^2(\mathcal{O}_{D_2}(-D_1)) = 0.$$

To prove boundedness, note that there exists an effective exceptional divisor $Z = Z_{\text{ample}} = \sum a_i \Gamma_i$ such that $-Z\Gamma_i > 0$ for every Γ_i. This follows from negative definiteness. In other words, $-Z$ is relatively ample for f.

It follows that there exists some exceptional divisor D such that $-D\Gamma_i \ge K_Y \Gamma_i$ for every i (for example, take D to be a large multiple of Z_{ample}). Then for every $D' > 0$,

$$H^1(\mathcal{O}_{D'}(-D)) \overset{\mathrm{d}}{=} H^0(\mathcal{O}_{D'}(K_Y + D' + D))$$

(by Serre duality on D', see 4.10, where the relation $\overset{\mathrm{d}}{=}$ is pronounced "equal to the dual of"). But you can see that $H^0(\mathcal{O}_{D'}(D') \otimes \mathcal{L}) = 0$ if $\deg_{\Gamma_i} \mathcal{L} \le 0$ for all i as an easy exercise (compare Ex. 4.14). Therefore $H^1(\mathcal{O}_{D+D'}) \to H^1(\mathcal{O}_D)$ is also injective for every $D' \ge 0$.

(b) I need to prove that if D' and D'' are divisors for which $h^1(\mathcal{O}_{D'}) = h^1(\mathcal{O}_{D''})$ both take the maximum value, then $A = \gcd(D', D'')$ has the same property, that is, $h^1(\mathcal{O}_A) = h^1(\mathcal{O}_{D'}) = h^1(\mathcal{O}_{D''})$.

So let $D' = A + B'$ and $D'' = A + B''$, where B' and B'' have no common components, and $C = A + B' + B'' = \operatorname{lcm}(D', D'')$. Then I claim that there exists an exact sequence

$$H^1(\mathcal{O}_C) \to H^1(\mathcal{O}_{D'}) \oplus H^1(\mathcal{O}_{D''}) \to H^1(\mathcal{O}_A) \to 0. \tag{3}$$

This proves what I want, since C, D', D'' all have the same value of h^1, so

$$h^1(\mathcal{O}_A) \ge h^1(\mathcal{O}_{D'}) + h^1(\mathcal{O}_{D''}) - h^1(\mathcal{O}_C).$$

(3) comes by chasing the diagram

$$
\begin{array}{ccccccc}
 & & H^1(\mathcal{O}_{B''}(-D')) & \to & H^1(\mathcal{O}_{B''}(-A)) & \to & 0 \\
 & & \downarrow & & \downarrow & & \\
H^1(\mathcal{O}_{B'}(-D'')) & \to & H^1(\mathcal{O}_C) & \to & H^1(\mathcal{O}_{D''}) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
H^1(\mathcal{O}_{B'}(-A)) & \to & H^1(\mathcal{O}_{D'}) & \to & H^1(\mathcal{O}_A) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & &
\end{array}
$$

here B' and B'' are without common components, so that $\mathcal{O}_{B'\cap B''}$ has support the finite set $B' \cap B''$, and $H^1(\mathcal{O}_{B'\cap B''}) = 0$ together with the exact sequence

$$0 \to \mathcal{O}_{B''}(-D') \to \mathcal{O}_{B''}(-A) \to \mathcal{O}_{B'\cap B''} \to 0$$

provides the exactness of the top row and the left-hand column. I leave you the pleasure of the diagram chase.

(c) The surjectivity of $H^1(\mathcal{O}_Y) \to H^1(\mathcal{O}_D)$ follows from the assertion $H^2 = 0$ of Proposition 4.7. The point is to prove that it is injective for some D, so that in particular $H^1(\mathcal{O}_Y)$ is finite dimensional. If $Z = Z_{\text{ample}}$ is the divisor constructed in (a) then the sheaf $\mathcal{O}_Y(-Z)$ is relatively ample on Y (see Ex. 4.16, essentially by the Nakai–Moishezon ampleness criterion[5]). From this, $H^1(\mathcal{O}_Y(-nZ)) = 0$ for $n \gg 0$ by Serre vanishing, and the result follows. Q.E.D.

4.9. Corollary.

(1) $R^1 f_* \mathcal{O}_Y = 0$ *if and only if* $H^1(\mathcal{O}_D) = 0$ *for every* D.

(2) $R^1 f_* \mathcal{O}_Y$ *is a* 1*-dimensional vector space if and only if* $H^1(\mathcal{O}_D) \le 1$ *for every* D *and* $= 1$ *for some* D.

If $P \in X$ is an affine neighbourhood, then $R^1 f_* \mathcal{O}_Y$ is the sheaf consisting of $H^1(Y, \mathcal{O}_Y)$ supported at P, so that the conditions in (1) and (2) characterise respectively $H^1(Y, \mathcal{O}_Y) = 0$ and $h^1(Y, \mathcal{O}_Y) = 1$.

4.10. Serre duality

Let Y be a projective nonsingular surface and D an effective divisor on Y. I want to use the fact that the line bundle $\mathcal{O}_D(K_Y + D)$ has the properties of a dualising sheaf for D (for example, in the above proof of Theorem 4.8). I don't need the full strength.

Proposition. *If* L *is a divisor on* Y *then there is a natural duality*

$$H^i(\mathcal{O}_D(L)) \overset{\mathrm{d}}{\underline{\quad}} H^{1-i}(\mathcal{O}_D(K_Y + D - L)) \qquad \textit{for } i = 0, 1.$$

[5]This is the central point of the proof. Nakai–Moishezon says that if H is a Cartier divisor on an n-dimensional complete scheme, then $H^r Z > 0$ for every r-dimensional irreducible subvariety Z implies that H is ample. This is completely elementary if you know coherent cohomology: by a dévissage argument, you can see that the numerical condition implies that $H^i(\mathcal{F} \otimes \mathcal{O}(nH)) = 0$ for every $i > 0$, for every coherent sheaf $\mathcal{F}$, and for $n \gg 0$ (see [H1], Chapter V, 1.10). I should have included this as a guided exercise in the section on cohomology.

Compare Ex. 3.26, or see [H1], III.7 and [R1], Theorem 2.12 for a more general treatment of the dualising sheaf ω_D and the adjunction formula

$$\omega_D = \mathcal{O}_D(K_D) = \mathcal{O}_D(K_Y + D) = \omega_Y(D)_{|D}.$$

Proof. This follows formally from duality on Y: the short exact sequence

$$0 \to \mathcal{O}_Y(L-D) \to \mathcal{O}_Y(L) \to \mathcal{O}_D(L) \to 0$$

leads to

$$\begin{aligned} \cdots \to H^i(\mathcal{O}_Y(L-D)) \to H^i(\mathcal{O}_Y(L)) \to H^i(\mathcal{O}_D(L)) \\ \to H^{i+1}(\mathcal{O}_Y(L-D)) \to H^{i+1}(\mathcal{O}_Y(L)) \to \cdots \end{aligned}$$

which by duality on Y is dual term by term to

$$\begin{aligned} \cdots \leftarrow H^{2-i}(\mathcal{O}_Y(K_Y+D-L)) \leftarrow H^{2-i}(\mathcal{O}_Y(K_Y-L)) \leftarrow \cdots \\ \cdots \leftarrow H^{1-i}(\mathcal{O}_Y(K_Y+D-L)) \leftarrow H^{1-i}(\mathcal{O}_Y(K_Y-L)) \leftarrow \cdots \end{aligned}$$

But this can be identified with the cohomology long exact sequence of

$$0 \to \mathcal{O}_Y(K_Y - L) \to \mathcal{O}_Y(K_Y + D - L) \to \mathcal{O}_D(K_Y + D - L) \to 0,$$

which identifies $H^{1-i}(\mathcal{O}_D(K_Y + D - L))$ with the dual of $H^i(\mathcal{O}_D(L))$, as required. Q.E.D.

4.11. RR on a divisor D

In the same notation, define $\deg_D L = DL = \sum_i n_i \deg_{\Gamma_i} L$. Then

$$\chi(\mathcal{O}_D) = -\tfrac{1}{2}(D^2 + DK_Y),$$

and

$$\chi(\mathcal{O}_D(L)) = \chi(\mathcal{O}_D) + \deg_D L.$$

The proof is an exercise in same style as 4.10, using RR on Y. Notice that the formulas discussed in 3.6 involving the arithmetic genus $p_a D$,

$$\chi(\mathcal{O}_D) = 1 - p_a D \quad \text{and} \quad 2p_a D - 2 = (D + K_Y)D$$

are a particular case. If $D = C$ is a nonsingular curve, then $g(C) = p_a C = H^1(\mathcal{O}_C)$, and the formula here reduces to the usual RR formula $\chi(\mathcal{O}_C(L)) = 1 - g + \deg L$.

4.12. Rational singularities

Definition. Let $f\colon Y \to X$ be a resolution of a normal surface singularity $P \in X$. Then $P \in X$ is a *rational singularity* if $R^1 f_* \mathcal{O}_Y = 0$. It is an *elliptic singularity* if $R^1 f_* \mathcal{O}_Y$ is 1-dimensional. Elliptic Gorenstein singularities are treated in 4.21–23 below. Without the Gorenstein condition, elliptic is too weak a condition to be very interesting.

Proposition (Numerical characterisation of rational singularities).

(1) *$P \in X$ is rational if and only if*

$$\chi(\mathcal{O}_D) \geq 1 \quad \textit{for every effective divisor } D \textit{ supported on } f^{-1}P.$$

(2) *For a rational singularity, the numerical cycle Z_{num} is the maximal divisor with $\chi(\mathcal{O}_D) = 1$. In other words,*

$$h^0(\mathcal{O}_D) \geq 2 \quad \textit{for every } D > Z_{\mathrm{num}}.$$

(3) *Z_{num} is numerically 1-connected (see Definition 3.9).*

Remark. By the adjunction formula and RR,

$$(K_Y + D)D = 2p_aD - 2 = -2\chi(\mathcal{O}_D),$$

so that the condition in (1) is equivalent to $p_aD \leq 0$ or $(K_Y + D)D < 0$.

Proof. (1) The implication $\implies$ is clear: $R^1f_*\mathcal{O}_Y = 0$ means that $H^1(\mathcal{O}_D) = 0$ for every D, and $H^0(\mathcal{O}_D) \neq 0$ (because it contains the constant functions $k \subset H^0(\mathcal{O}_Y) \to H^0(\mathcal{O}_D)$), so $\chi(\mathcal{O}_D) \geq 1$.

For $\impliedby$, if $R^1f_*\mathcal{O}_Y \neq 0$, there exists a divisor $D > 0$ such that $H^1(\mathcal{O}_D) \neq 0$, and therefore there exists a minimal one; that is $H^1(\mathcal{O}_D) \neq 0$, but $H^1(\mathcal{O}_{D-\Gamma}) = 0$ for every $\Gamma \subset D$.

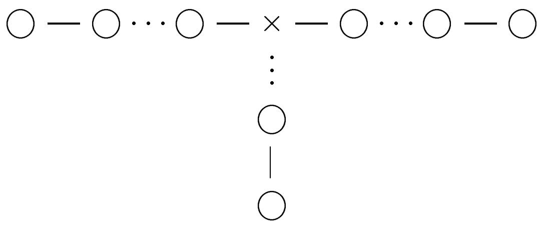
By considering the exact sequence

$$H^1(\mathcal{O}_\Gamma(-D+\Gamma)) \to H^1(\mathcal{O}_D) \to H^1(\mathcal{O}_{D-\Gamma}) = 0,$$

we see that $H^1(\mathcal{O}_\Gamma(-D+\Gamma)) \neq 0$ for every $\Gamma \subset D$. It follows by Serre duality that $H^0(\mathcal{O}_\Gamma(K_Y + \Gamma + D - \Gamma)) \neq 0$, that is, $H^0(\mathcal{O}_\Gamma(K_Y + D)) \neq 0$. Now Γ is an irreducible curve, and $\mathcal{O}_\Gamma(K_Y + D)$ a line bundle over it, so that the existence of a section implies $\deg \geq 0$, that is $\Gamma(K_Y + D) \geq 0$ for every $\Gamma \subset D$. Summing over all components of D gives $D(K_Y + D) \geq 0$, that is $p_aD \geq 1$.

(2) and (3) are easy exercises (see Ex. 4.11–12). Q.E.D.

Example. Consider the bunch of $\mathbb{P}^1$s consisting of a central -3-curve $\times$ meeting 3 simple chains of -2-curves of length $p-1, q-1, r-1$ respectively:



Check that Z_{num} is the reduced sum of all the curves, and that the configuration is rational. It's a fun calculation to obtain this configuration by resolving the singular surface $X \subset \mathbb{C}^4$ defined by

$$\begin{aligned} xy + xt^q + t^{p+q} &= 0 \\ yz + yt^r + t^{q+r} &= 0 \\ xz + zt^p + t^{p+r} &= 0 \end{aligned} \qquad \text{or} \qquad \operatorname{rank} \begin{pmatrix} z + t^r & -x & -t^q \\ -t^r & x + t^p & -y \\ -z & -t^p & y + t^q \end{pmatrix} \leq 1.$$

[Hint: Check that if $p = q = r = 1$, you get the cone over the rational normal curve of degree 3. Then check that if $p = 0$, X has two Du Val singularities of type A_{q-1}, A_{r-1} on the x-axis. Finally, if $p, q, r > 1$, the blowup ($x = x_1t$, etc.) gives 3 coordinate axes as exceptional locus, through a point of the same type with numbers $p-1, q-1, r-1$, and you continue inductively.]

4.13. Structure of Pic D

Let $\{\Gamma_i\}_{i=1}^k$ be a bunch of curves on Y and $D = \sum n_i\Gamma_i$ an effective divisor with $n_i > 0$ for each i. $\mathcal{O}_D^*$ denotes the sheaf of units of $\mathcal{O}_D$, so that $\mathcal{O}_Y^* \to \mathcal{O}_D^*$ is surjective (sheaves for the Zariski topology). Then

$$H^1(\mathcal{O}_D^*) = \{\text{isomorphism classes of invertible sheaves over } \mathcal{O}_D\} = \operatorname{Pic} D.$$

For any curve $\Gamma = \Gamma_i$, there is a natural group homomorphism

$$\rho_\Gamma\colon H^1(\mathcal{O}_D^*) \to H^1(\mathcal{O}_\Gamma^*) = \operatorname{Pic}\Gamma.$$

Geometrically, this takes an invertible sheaf L over D to $L_{|\Gamma} = L \otimes \mathcal{O}_\Gamma$. Now compose with the degree map $\operatorname{Pic}\Gamma \to \mathbb{Z}$ to get

$$\deg_\Gamma\colon H^1(\mathcal{O}_D^*) = \operatorname{Pic} D \to \mathbb{Z}.$$

Putting together this map for each component Γ_i gives a map

$$\deg\colon H^1(\mathcal{O}_D^*) \to \mathbb{Z}^k$$

defined by $L \mapsto \deg_{\Gamma_i} L$. It is easy to see that deg is surjective: choose a general point Q on any component $\Gamma \subset D$, and construct a Cartier divisor with support Q and degree 1 on Γ, whose local equation $s \in \mathcal{O}_{D,Q}$ restricts to a local equation of Q in $\mathcal{O}_{\Gamma,Q}$. (This operation is important in what follows, and is called a *transverse cut* at Q.)

4.14. Proposition (Artin). *(i) Write $D' = D_{\text{red}}$. Then the two groups*

$$K = \ker\{H^1(\mathcal{O}_D) \to H^1(\mathcal{O}_{D'})\}$$

and

$$K^* = \ker\{H^1(\mathcal{O}_D^*) \to H^1(\mathcal{O}_{D'}^*)\}$$

have filtrations with isomorphic quotients.

(ii) $\deg\colon H^1(\mathcal{O}_D^*) \to \mathbb{Z}^k$ *is an isomorphism if and only if* $H^1(\mathcal{O}_D) = 0$.

Proof. (i) Suppose C is a divisor such that $2C \subset D$; then define the subsheaves $J \subset \mathcal{O}_D$ and $J^* \subset \mathcal{O}_D^*$ to be the kernels in the exact sequences

$$0 \to J \to \mathcal{O}_D \to \mathcal{O}_{D-C} \to 0$$
$$0 \to J^* \to \mathcal{O}_D^* \to \mathcal{O}_{D-C}^* \to 0$$

(here of course $J = \mathcal{O}_C(-D+C)$ by Lemma 3.10); then J and J^* are isomorphic as sheaves of groups, because $J^2 = 0$, so that

$$\varepsilon\colon \Gamma(U, J) \xrightarrow{\sim} \Gamma(U, J^*) \quad \text{defined by } x \mapsto 1 + x$$

gives an isomorphism from the additive to the multiplicative group, a kind of "first order exponential". Also it is clear that $H^0(\mathcal{O}_{D-C}^*) \subset H^0(\mathcal{O}_{D-C})$ (as a set), and contains a Zariski neighbourhood of 1. A trivial cocycle calculation based on this proves that

$$H^0(\mathcal{O}_{D-C}) \to H^1(J) \quad \text{and} \quad H^0(\mathcal{O}^*_{D-C}) \to H^1(J^*) = \varepsilon(H^1(J))$$

have the same images. Hence

$$\ker\{H^1(\mathcal{O}_D) \to H^1(\mathcal{O}_{D-C})\} \cong \ker\{H^1(\mathcal{O}^*_D) \to H^1(\mathcal{O}^*_{D-C})\}.$$

(Of course, the right-hand group is a priori not a k-vector space.) The argument continues by induction from D down to D'; $\ker\{H^1(\mathcal{O}^*_D) \to H^1(\mathcal{O}^*_{D'})\}$ and $\ker\{H^1(\mathcal{O}_D) \to H^1(\mathcal{O}_{D'})\}$ thus have filtrations by subgroups with isomorphic quotients.

(ii) For a reduced curve D', it can be shown easily that

$$\operatorname{Pic} D' \cong \mathbb{Z}^k \iff H^1(\mathcal{O}_{D'}) = 0.$$

$\Longleftarrow$ By RR, a line bundle L on D' with $\deg_\Gamma L = 0$ for every $\Gamma \subset D'$ has

$$h^0(L) \geq 1 - h^1(\mathcal{O}_{D'}),$$

so $H^1(\mathcal{O}_{D'}) = 0 \implies H^0(L) \neq 0$. Then it is not hard to see that the section s has the property that $\mathcal{O}_{D'} \cdot s = L$. (See Lemma 3.11, (iii) and Ex. 3.27.)

$\Longrightarrow$ If L is a line bundle with sufficiently high degree on every component of D' then $H^1(L) = 0$, so by RR, $h^0(L) = 1 - h^1(\mathcal{O}_{D'}) + \deg_{D'} L$. If I pick $\deg_{\Gamma_i} L$ "general" points on each component Γ_i then obviously the resulting set of points $\{P_i\}$ gives a Cartier divisor $\sum P_i$ on D' with the same degrees as L, so $L_0 = L(-\sum P_i)$ is a line bundle of degree 0 on each component. But a simple argument shows that these points can be chosen one at a time to strictly decrease the dimension $h^0(L(-\sum P_i))$ until we get to zero, so that if $H^1(\mathcal{O}_{D'}) \neq 0$, I get a line bundle L_0 of degree zero on each component, but $h^0(L_0) \neq 0$, so that $L_0 \cong \mathcal{O}_{D'}$. Q.E.D.

Remark. For a projective variety X, the group $\operatorname{Pic} X$ is an extension of a discrete group by the group $\operatorname{Pic}^0 X$ of divisors algebraically equivalent to zero modulo divisors linearly equivalent to zero. The elements of $\operatorname{Pic}^0 X$ are naturally parametrised (in the sense of moduli problems) by an algebraic group or algebraic group scheme, the *Picard variety* of X. If X is nonsingular in characteristic zero, then $\operatorname{Pic}^0 X$ is an Abelian variety. The Jacobian of a curve is a well-known example. For singular, or nonreduced X, the algebraic group $\operatorname{Pic}^0 X$ is in general more complicated; however, it has the structure of an extension of an Abelian variety by an affine group (that is, an extension of additive and multiplicative groups $\mathbb{G}_a$s and $\mathbb{G}_m$s), and the Abelian variety comes entirely from the resolution of reduced components of X. Compare the discussion in 3.14.

4.15. Contractibility of a rational bunch of curves

Let Y be a nonsingular quasiprojective surface and $\{\Gamma_i\}$ a bunch of curves on Y. By Propositions 4.12 and 4.14, we know that the following 3 conditions are equivalent:

(1) $H^1(\mathcal{O}_D) = 0$ for every divisor D supported on $\bigcup \Gamma_i$;
(2) $p_a D \leq 0$ for every D;
(3) $\deg\colon H^1(\mathcal{O}^*_D) = \operatorname{Pic} D \xrightarrow{\sim} \mathbb{Z}^k$ for every D, where k is the number of components of D.

If $f\colon Y \to X$ is a resolution of a normal surface singularity $P \in X$ with exceptional locus $\{\Gamma_i\} = f^{-1}P \subset Y$, these conditions are necessary and sufficient for $P \in X$ to be rational.

Contraction Theorem (Artin). *A connected bunch of curves $\{\Gamma_i\}$ on Y is the exceptional locus of a resolution of a rational singularity $P \in X$ if and only if*

(a) the intersection matrix $(\Gamma_i\Gamma_j)$ is negative definite;

(b) $p_a D \le 0$ for every D.

Examples. (i) A -1-*curve* on a surface Y is a curve L satisfying $L \cong \mathbb{P}^1$, $L^2 = -1$ and $K_Y L = -1$. This satisfies the rationality condition, so can be contracted.

Let's check the rationality condition: obviously if $L = \mathbb{P}^1$ then $H^1(\mathcal{O}_L) = 0$, and for $n \ge 2$,

$$0 \to \mathcal{O}_L(-(n-1)L) \to \mathcal{O}_{nL} \to \mathcal{O}_{(n-1)L} \to 0$$

gives

$$0 = H^1(\mathbb{P}^1, \mathcal{O}(n-1)) \to H^1(\mathcal{O}_{nL}) \to H^1(\mathcal{O}_{(n-1)L}) \to 0.$$

So $H^1(\mathcal{O}_{nL}) = 0$ for all n.

This case has the special feature that after contraction, the point $P \in X$ is a nonsingular point, and $Y \to X$ is the usual blowup of P (this is proved in Theorem 4.17). In this case the contraction theorem is Castelnuovo's contractibility criterion.

(ii) Similarly, for any $d > 0$, a curve $L \cong \mathbb{P}^1$, with $L^2 = -d$ satisfies the rationality condition. If $d = 1, 2$ we get -1-curve, -2-curve respectively.

(iii) A bunch of -2-curves $\{\Gamma_i\}$ has negative definite intersection matrix if and only if it forms one of the configurations A_n, D_n, E_6, E_7, E_8 (see Ex. 4.5); if this happens then $p_a D \le 0$ for every effective divisor D supported on Γ_i. This can be seen as follows: $K_Y\Gamma_i = 0$ by the adjunction formula, and so $K_Y D = 0$, so $K_Y D + D^2 < 0$, and $p_a D = \frac{1}{2}(K_Y + D)D + 1 \le 0$. In this case, Theorem 4.17 proves that the image $P \in X$ is a hypersurface double point.

4.16. Proof of Theorem

The conditions are clearly necessary (compare Theorem A.7 and Proposition 4.12). I now assume (a) and (b), and prove that the contraction exists. Suppose that H is ample on Y. Then $H\Gamma_i > 0$ for every curve Γ_i. Since the matrix $(\Gamma_i\Gamma_j)$ is nonsingular, there exists rational numbers q_i such that $(H + \sum q_i\Gamma_i)\Gamma_j = 0$ for every j. By taking n to be any common denominators of the q_i, I get a divisor $D = \sum a_i\Gamma_i$ such that $(nH + D)\Gamma_i = 0$ for every Γ_i. Take n to be sufficiently large, in a sense to be specified below. This means in particular that $D\Gamma_i < 0$ for every i.

I claim that $a_i > 0$: indeed, write $D = A - B$ with A, B effective and with no common components. Then $DB = AB - B^2 \ge 0$. So $B = 0$, or in other words $D > 0$; but then the only way that $D\Gamma_i < 0$ is possible is for $\Gamma_i \subset D$, so $a_i > 0$.

Consider the invertible sheaf $\mathcal{L} = \mathcal{O}_Y(nH + D)$; by construction, $\deg_{\Gamma_i} \mathcal{L} = 0$ for each i. So by Proposition 4.14, (ii), $\mathcal{L}_{D'} \cong \mathcal{O}_{D'}$ for any divisor D' supported on $\{\Gamma_i\}$.

Consider the restriction exact sequence

$$0 \to \mathcal{O}_Y(nH) \to \mathcal{O}_Y(nH + D) \to \mathcal{O}_D \to 0.$$

Now $\mathcal{O}_Y(H)$ is ample, so that by Serre vanishing,

$$H^1(\mathcal{O}_Y(nH)) = 0 \quad \text{for every } n \gg 0,$$

and so $H^0(\mathcal{O}_Y(nH+D)) \to H^0(\mathcal{O}_D) \to 0$ is exact.

I now claim that a suitable space of sections of $\mathcal{O}_Y(nH+D)$ defines the required contraction $f\colon Y \to X'$, at least set theoretically. Let n and $s_0, \dots, s_N \in H^0(\mathcal{O}_Y(nH))$ be chosen so that $\varphi\colon Y \to \mathbb{P}^N$ defined by $(s_0 : \cdots : s_N)$ is an embedding; and let $s_{N+1} \in H^0(\mathcal{O}_Y(nH+D))$ be any element which maps to the identity section $1 \in H^0(\mathcal{O}_D)$. Consider the sections $(s_0, \dots, s_{N+1}) \in H^0(Y, \mathcal{L})$, and the map $f'\colon Y \to \mathbb{P}^{N+1}$ by $(s_0 : \cdots : s_{N+1})$.

Then (a) at every point $Q \in Y$, some $s_i \in \mathcal{L}$ is a local generator; (take s_i if $Q \notin D$, and s_{N+1} if $Q \in D$).

(b) $f'(D) = P = (0, \dots, 0, 1)$.

(c) $f'_{|(Y \setminus D)}$ is an embedding (since already the ratios $(s_0 : \cdots : s_N)$ give an embedding). Write $X' = f'(Y)$. Then there is a unique way to factorise the map f' as $Y \xrightarrow{f} X \to X'$ where X is normal, and this is the required contraction. X is projective since X' is, and the normalisation $X \to X'$ is finite. Q.E.D.

4.17. Multiplicity and embedding dimension

To understand what singularities arise on making contractions such as those in Theorem 4.15, I need to be able to determine invariants of a singularity $P \in X$ in terms of its resolution Y and the bunch of curves $\{\Gamma_i\}$. Multiplicity and embedding dimension are two simple invariants, which give an easy characterisation of the nonsingular surface point $P \in X$ in Castelnuovo's criterion, and hypersurface double points arising on contracting a bunch of -2-curves.

Multiplicity was defined in 4.6 above. The *embedding dimension* of $P \in X$ is the dimension of the Zariski tangent space $\operatorname{emb}\dim(P \in X) = \dim m_P/m_P^2$. This is clearly the dimension of the smallest ambient space in which an analytic neighbourhood $P \in X$ can be embedded.

Theorem (Artin). *For a rational singularity $P \in X$, let $Z = Z_{\text{num}}$ be the numerical cycle (see Definition 4.5), and set $d = -Z^2$. Then*

(1) $m_P\mathcal{O}_Y = \mathcal{O}_Y(-Z)$, *and therefore* $\operatorname{mult}(P \in X) = d$;
(2) $m_P/m_P^2 = H^0(\mathcal{O}_Z(-Z))$, *and* $\operatorname{emb}\dim(P \in X) = d + 1$.

In particular, $Z^2 = -1$ implies $P \in X$ is nonsingular, and $Z^2 = -2$ implies $P \in X$ is a hypersurface double point.

Remarks. (a) The proof of (2) gives at once that $m_P^k/m_P^{k+1} = H^0(\mathcal{O}_Z(-kZ))$ and $\dim m_P^k/m_P^{k+1} = kd + 1$ for every $k \geq 1$, so that the knowledgeable reader who so prefers can use the more professional definition of multiplicity in terms of the Hilbert–Samuel function (compare the footnote to 4.6).

(b) The example of the cone over the rational normal curve helps remember the numbers, but don't be misled into thinking that rational surface singularities are all that simple.

Lemma. *Let $f\colon Y \to X$ be a resolution of a rational surface singularity $P \in X$. Suppose that X is affine, and let $\mathcal{L}$ be a line bundle on Y which is nef on $f^{-1}P$ (that is, $\mathcal{L}\Gamma_i \geq 0$ for each i).*

Then $H^1(Y, \mathcal{L}) = 0$ *and* $|\mathcal{L}|$ *is free near* $f^{-1}P$.

Proof. Given any exceptional divisor D, I can construct a divisor A on D as a sum of local transverse cuts through general points of the Γ_i, so that A is an effective Cartier divisor on D with the same degree as $\mathcal{L}$; then by Proposition 4.14, (ii), $\mathcal{L}_{|D} \cong \mathcal{O}_D(A)$. Now the exact sequence of sheaves

$$0 \to \mathcal{O}_D \to \mathcal{O}_D(A) \to \mathcal{O}_A(A) \to 0$$

shows that $H^1(\mathcal{O}_D(A)) = 0$. Now $H^1(Y, \mathcal{L}) = R^1 f_* \mathcal{L}$ can be calculated by the arguments of 4.8 as an inverse limit of the groups $H^1(\mathcal{L}_{|D})$, all of which are zero (compare Ex. 4.15).

The transverse cuts making up A can be chosen freely, so that the linear system $|\mathcal{L}|_D = |A|_D$ is free on any D. Apply this to $D = Z$. Then $\mathcal{L}(-Z)$ is nef, so that by the first part, $H^1(\mathcal{L}(-Z)) = 0$, and therefore

$$H^0(Y, \mathcal{L}) \twoheadrightarrow H^0(Z, \mathcal{L}_{|Z}).$$

Thus $|\mathcal{L}|$ on Y cuts out the complete linear system $|\mathcal{L}|_Z$ on Z, and I just said that this is free. Q.E.D.

Proof of Theorem 4.17, (1). As discussed above, every function $g \in m_P \subset \mathcal{O}_{X,P}$ vanishes on $Z = Z_{\text{num}}$, so that $f_* \mathcal{O}_Y(-Z) = m_P$. By the lemma, $|\mathcal{O}_Y(-Z)| = |-Z|$ is free, which means that $m_P \mathcal{O}_Y = \mathcal{O}_Y(-Z)$ (compare the discussion in 4.6). This means $Z = Z_{\mathfrak{f}}$ and the assumption of Lemma 4.6, (i) holds, so $\operatorname{mult}(P \in X) = -Z^2$ follows.

4.18. Graded rings over divisors

Before proving (2), I make some general remarks on the graded ring

$$R(Z, \mathcal{L}) = \bigoplus_{k \geq 0} H^0(Z, \mathcal{L}^{\otimes k})$$

corresponding to a divisor $Z = \sum n_i \Gamma_i$ on a nonsingular surface and a nef line bundle $\mathcal{L} \in \operatorname{Pic} Z$ on Z with $d = \deg \mathcal{L} > 0$.

I add a brief word of explanation in case you haven't seen this kind of thing before. There is a graded ring $R(V, \mathcal{L})$ corresponding to any line bundle over a variety (or scheme) V: by definition of tensor product of sheaves, there are local multiplication maps $\mathcal{L} \times \mathcal{L} \to \mathcal{L}^{\otimes 2}$ (more generally, $\mathcal{L}^{\otimes a} \times \mathcal{L}^{\otimes b} \to \mathcal{L}^{\otimes a+b}$). The graded ring $R(V, \mathcal{L})$ puts together all global sections of all powers of $\mathcal{L}$, with multiplication between the H^0 induced by the local maps of sheaves.

The case to bear in mind is when $\mathcal{L}$ is very ample, so that $V \subset \mathbb{P}^n$ with $H^0(\mathcal{L}) = \langle x_0, \dots, x_n \rangle$, the vector space of homogeneous coordinates of $\mathbb{P}^n$. Then the multiplication $H^0(\mathcal{L}) \times \cdots \times H^0(\mathcal{L}) \to H^0(\mathcal{L}^{\otimes k})$ just views monomials in the x_i as restrictions to V of homogeneous forms in $H^0(\mathbb{P}^n, \mathcal{O}(k))$. In this case, $R(V, \mathcal{L})$ generated in degree 1 means

$$H^0(\mathbb{P}^n, \mathcal{O}(k)) \twoheadrightarrow H^0(V, \mathcal{L}^{\otimes k}) \quad \text{for all } k.$$

If V itself is a normal variety, this condition is called *projective normality*, and is equivalent to saying that the affine cone over V is normal. See 3.2 for an important example.

In the immediate application, Z is the numerical cycle of a rational singularity or bunch of curves, and $\mathcal{L} = \mathcal{O}_Z(-Z)$. Thus Z satisfies the assumptions of the next theorem.

Theorem. *Let $Z = \sum n_i\Gamma_i$ be a divisor on a nonsingular surface, and assume that Z is numerically 1-connected and $H^1(\mathcal{O}_Z) = 0$. Let $\mathcal{L} \in \operatorname{Pic} Z$ be a nef line bundle on Z with $d = \deg \mathcal{L} > 0$. Then the graded ring $R(Z, \mathcal{L}) = \bigoplus_{k\geq 0} H^0(Z, \mathcal{L}^{\otimes k})$ is generated in degree 1. In particular, the multiplication map*

$$H^0(\mathcal{L}) \times H^0(\mathcal{L}) \to H^0(\mathcal{L}^{\otimes 2})$$

is onto.

Proof. The proof is completely elementary: apart from the language, it is just the simplest case of Castelnuovo's free pencil trick.

Write $\deg_{\Gamma_i} \mathcal{L} = d_i$ (so that $d = \sum n_i d_i$), and choose d_i disjoint transverse cuts A_{ij} through Γ_i (for $j = 1, \dots, d_i$). As a scheme, $A_{ij} \cong \operatorname{Spec} k[x]/x^{n_i}$. Their sum $A = \sum A_{ij}$ is a Cartier divisor A with the same degree as $\mathcal{L}$ on every component of Z, so that Proposition 4.14 gives $\mathcal{L} \cong \mathcal{O}_Z(A)$. In other words, $\mathcal{L}$ has a section s_0 with divisor of zeros equal to A. As in Lemma 4.17, the cuts making up A can be chosen arbitrarily, so that $|\mathcal{L}| = |A|$ is a free linear system on Z.

Consider two general sections $s, s_0 \in H^0(\mathcal{L})$ (the eponymous free pencil), and write $A = \operatorname{div}(s_0)$; I assume that s is a local basis of $\mathcal{L}$ at every point of $\operatorname{Supp} A$. Then s_0 defines the short exact sequence

$$0 \to \mathcal{O}_Z \to \mathcal{L} \to \mathcal{O}_A \to 0.$$

Here I use the second section s to identify $\mathcal{L} = \mathcal{O}_Z(A)$ with $\mathcal{O}_Z$ near A, which simplifies the notation $\mathcal{L}_{|A} = \mathcal{O}_A(A)$ to $\mathcal{O}_A$.

Now the assumption $H^1(\mathcal{O}_Z) = 0$ obviously gives $H^1(\mathcal{L}) = 0$ and $H^0(\mathcal{L}) = d+1$. Let $s_1, \dots, s_d \in H^0(\mathcal{L})$ map to a basis of $H^0(\mathcal{O}_A)$. Then the theorem follows from the more precise claim: for any $k \geq 0$, the vector space $H^0(\mathcal{L}^{\otimes k})$ is based by the $kd + 1$ monomials

$$s_0^k, \quad \text{and} \quad s_0^{k-a} s^{a-1} s_i \quad \text{for } a = 1, \dots, k \text{ and } i = 1, \dots d.$$

Indeed, $H^0(\mathcal{O}_Z)$ is just the constants (because Z is 1-connected), which gives the assertion for $k = 0$; for $k \geq 1$, it follows by induction from the exact sequence

$$H^0(\mathcal{L}^{\otimes k-1}) \hookrightarrow H^0(\mathcal{L}^{\otimes k}) \twoheadrightarrow H^0(\mathcal{O}_A). \quad \text{Q.E.D.}$$

The proof amounts to saying that $\mathcal{L}$ embeds the Artinian scheme A into $\mathbb{P}^{d-1}$ so that $H^0(\mathbb{P}^{d-1}, \mathcal{O}(1)) \to \mathcal{O}_A$ is onto. That is, the image $A \subset \mathbb{P}^{d-1}$ behaves like d points in linearly general position.

Proof of Theorem 4.17, (2). Under the identification $m_P = H^0(\mathcal{O}_Y(-Z))$, it is clear that m_P^2 maps to $H^0(\mathcal{O}_Y(-2Z))$, which gives a map

$$m_P/m_P^2 \to H^0(\mathcal{O}_Y(-Z))/H^0(\mathcal{O}_Y(-2Z)) = H^0(\mathcal{O}_Z(-Z)) \tag{4}$$

which is well defined and surjective. The point is to prove that it is also injective, so that $m_P/m_P^2 \cong H^0(\mathcal{O}_Z(-Z))$. This completes the proof, since by RR,

$$h^0(\mathcal{O}_Z(-Z)) = 1 - p_a Z + \deg(\mathcal{O}_Z(-Z)) = 1 - Z^2.$$

To see that (4) is injective, it is enough to prove that

$$H^0(Y, \mathcal{O}_Y(-2Z)) = m_P^2;$$

in words, a function vanishing along $2Z$ is a linear combination of products of two functions vanishing along Z. I prove this as a corollary of Theorem 4.18. Let $g \in H^0(Y, \mathcal{O}_Y(-2Z))$; its restriction to Z is a section $\overline{g} \in H^0(\mathcal{O}_Z(-2Z))$, so by the theorem, it is in the image of $H^0(\mathcal{O}_Z(-Z)) \otimes H^0(\mathcal{O}_Z(-Z))$; choosing an expression $\overline{g} = \sum \overline{x}_i \overline{y}_i$ and lifting to $x_i, y_i \in H^0(\mathcal{O}_Y(-Z))$ gives

$$g - f_2 \in H^0(\mathcal{O}_Y(-3Z)) \quad \text{with } f_2 = \sum x_i y_i \in m_P^2$$

Proceeding in the same way with $H^0(\mathcal{O}_Z(-3Z))$, etc., gives for any n an expression

$$g - f_2 - f_3 - \cdots - f_{n-1} \in H^0(\mathcal{O}_Y(-nZ)) \quad \text{with } f_i \in m_P^i. \tag{5}$$

On the other hand, I claim that $H^0(\mathcal{O}_Y(-nZ)) \subset m_P^2$ for some $n \gg 0$. Since also $m_P^i \subset m_P^2$, (5) implies that $g \in m_P^2$.

Note that the graded algebra $\bigoplus_{k\geq 0} H^0(\mathcal{O}_Y(-kZ))$ is finitely generated as a $k[X]$-algebra: for $\mathcal{O}_Y(-Z)$ is the pullback of the relatively ample line bundle $\mathcal{O}(1)$ under the morphism $\varphi_{|-Z|}\colon Y \to X \times \mathbb{P}^N$ defined by the free linear system $|-Z|$; (the image is the blowup of m_P). The required finite generation follows by projective normalisation. Each $H^0(\mathcal{O}_Y(-kZ))$ is contained in m_P, so that for n greater than the degrees of all the generators, the group $H^0(\mathcal{O}_Y(-nZ))$ is contained in a sum of products of at least two of these, therefore in m_P^2. Q.E.D.

I'm sorry this proof was so involved. Lots of proofs in algebraic geometry depend on finiteness arguments of this type, and projective normalisation is among the more elementary, provided it's in your vocabulary. A posteriori, the graded algebra is generated in degree 1.

An alternative is to go from (5) to the conclusion $g \in m_P^2$ using some kind of completion; [A2] uses Henselisation. If you think of Y as just a thin tubular neighbourhood of $f^{-1}P$ in the complex topology, any general divisor in $|-2Z|$ consists of a number of transverse cuts through general points of components of $f^{-1}P$, with each transverse cut a connected component of the divisor in the complex topology. By considering degrees, it is easy to see that this can be grouped as a sum of two divisors in $|-Z|$, which proves that $H^0(\mathcal{O}_Y(-2Z)) = m_P^2$.

4.19. The Gorenstein condition and the canonical cycle Z_K

The class of Gorenstein singularities $P \in X$ is a mild generalisation of hypersurface singularities; although I sketch a correct treatment of the definition and simple properties, you could for many practical purposes take the easy way out and assume throughout that $P \in X$ is a hypersurface singularity. Let $P \in X$ be a normal surface singularity; at some future point you need to know that $P \in X$ is automatically Cohen–Macaulay, as discussed in [YPG], App. to §3. Let $f\colon Y \to X$ be a resolution of singularities, subsequently usually assumed to be *minimal* (no -1-curves in the fibre).

Proposition–Definition. *Equivalent conditions:*

(1) *The canonical class of Y can be written as an exceptional divisor plus a divisor disjoint from the exceptional locus:*

$$K_Y \overset{\mathrm{lin}}{\sim} -Z_K + M \quad \text{with} \quad M \cap f^{-1}P = \emptyset.$$

(2) *There exists a rational canonical differential $s \in \mathcal{O}_X(K_X)$ on X which is regular and has no zeros on a punctured neighbourhood of $P \in X$.*
(3) *The line bundle $\Omega^2_{X^0}$ on the nonsingular locus $X^0 = X \setminus P$ extends over X as a line bundle ω_X.*
(4) *The canonical Weil divisor class K_X is Cartier (locally principal) at P.*
(5) *The Serre–Grothendieck dualising sheaf ω_X is invertible at P.*
(6) *The local ring $\mathcal{O}_{X,P}$ is Gorenstein in the sense of commutative algebra textbooks (for example* [Matsumura], *Chapter 6, §18).*

If these conditions hold, $P \in X$ is Gorenstein*; the exceptional divisor Z_K is called the* canonical cycle *of the resolution (or the* antidiscrepancy*).*

Discussion of proof. The canonical divisor K_X of a normal n-fold is the Weil divisor class $\operatorname{div} s$ where $s \in \Omega^n_{k(X)}$ is a rational canonical differential. This was discussed in A.10 under the additional assumption that X is nonsingular, so that s could be compared with a volume form $\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$ at every point of X. However, a rational canonical differential $s \in \Omega^n_{k(X)}$ is a birational notion, and to define its divisor $\operatorname{div} s$ as a Weil divisor, we only need to know X at each prime divisor.

It follows from this discussion that, essentially by definition, (1), (3) and (4) are merely restatements of (2). The (pre-) dualising sheaf ω_X is defined, and its fundamental dualising property proved in [H1], III.7. The equivalence of (4) and (5) comes from the fact that the (pre-) dualising sheaf ω_X equals the divisorial sheaf $\mathcal{O}_X(K_X)$, as discussed in [C-3f], App. to §1.

The point of the equivalence between (5) and (6) is to pass between global duality (properties involving ω_X and the category of quasicoherent sheaves on X) and local duality for the local ring $A = \mathcal{O}_{X,P}$. To do this categorically should be an exercise, but one which I can't do convincingly, partly because the literature on local duality leaves me speechless. A practical alternative uses the calculation of ω_X as an $\mathcal{E}xt$ sheaf: assume that $X \subset \mathbb{P}$ is an embedding into a nonsingular variety; then X Gorenstein at P means that $\mathcal{E}xt^{N-n}_{\mathcal{O}_\mathbb{P}}(\mathcal{O}_X, \omega_\mathbb{P}) = \omega_X$ is invertible at P and $\mathcal{E}xt^i = 0$ for $i \neq N-n$. Localising and playing with injective resolutions, it's not hard to see that this is equivalent to $\operatorname{Ext}^n_A(k, A) = k$ and $\operatorname{Ext}^i = 0$ for $i \neq n$, where $k = A/m_P$. This is one of the many equivalent definitions of a Gorenstein local ring, compare [Matsumura], Theorem 18.1.

Example. If $P \in X \subset \mathbb{C}^3$ is an isolated hypersurface singularity defined by $f = 0$, a basis of $\mathcal{O}_X(K_X)$ near P is given by

$$s = \operatorname{Res}_X \frac{\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z}{f} = \frac{\mathrm{d}x \wedge \mathrm{d}y}{\partial f / \partial z} = \text{etc.}$$

Then s is a rational canonical differential, and its divisor on Y is $\operatorname{div}_Y(s) = -Z_K$. That is, s has Z_K as its divisor of poles on Y. Compare the introductory discussion in [YPG], §1.

4.20. The canonical and cohomological cycles, $Z_1 = Z_K$

From now on I assume that the resolution Y is minimal, that is, K_Y is nef on the exceptional curves Γ_i. Then $-Z_K$ is nef, so that either $Z_K = 0$ or $Z_K \geq Z_{\text{num}}$, where Z_{num} is the numerical cycle of 4.2. In particular $Z_K \geq 0$, so I can view it as a subscheme $Z_K \subset Y$. It is easy to see that $Z_K = 0$ if and only if $Y \to X$ is a minimal resolution of a Du Val singularity.

Theorem. *If $P \in X$ is an isolated Gorenstein surface singularity then $Z_1 = Z_K$. Here Z_1 is the cohomological cycle of Theorem 4.8, and Z_K the canonical cycle defined in 4.19.*

Proof. First, if $D \geq Z_K$ then $H^1(\mathcal{O}_D) \to H^1(\mathcal{O}_{Z_K})$ is an isomorphism: indeed, it is clearly surjective, with kernel a quotient of

$$H^1(\mathcal{O}_{D'}(-Z_K)) \overset{\text{d}}{=} H^0(\mathcal{O}_{D'}(D')), \quad \text{where } D' = D - Z_K.$$

But it is an easy exercise to see that $H^0(\mathcal{O}_{D'}(D')) = 0$ (see Ex. 4.14). Therefore Z_K achieves the maximum value of $H^1(\mathcal{O}_D)$, and so $Z_1 \leq Z_K$ by definition of Z_1 (see Theorem 4.8, (b)).

Next, if $Z_1 < Z_K$, then I claim that $h^1(\mathcal{O}_{Z_1}) < h^1(\mathcal{O}_{Z_K})$, which contradicts the definition of Z_1. The duals of these H^1s are the groups

$$\begin{aligned} H^1(\mathcal{O}_{Z_K}) &\overset{\text{d}}{=} H^0(\omega_{Z_K}) = H^0(Z_K, \mathcal{O}_{Z_K}(K_Y + Z_K)) = H^0(Z_K, \mathcal{O}_{Z_K}) \\ H^1(\mathcal{O}_{Z_1}) &\overset{\text{d}}{=} H^0(\omega_{Z_1}) = H^0(Z_1, \mathcal{O}_{Z_1}(K_Y + Z_1)) = H^0(Z_1, \mathcal{O}_{Z_1}(Z_1 - Z_K)) \end{aligned}$$

which fit in the exact sequence

$$0 \to H^0(Z_1, \mathcal{O}_{Z_1}(Z_1 - Z_K)) \to H^0(Z_K, \mathcal{O}_{Z_K}) \to H^0(\mathcal{O}_{D''}), \tag{6}$$

where $D'' = Z_K - Z_1$. But if $D'' \neq 0$, the last map in (6) is certainly nonzero, because it is the identity on the constant functions. This completes the contradiction, and proves the theorem. Q.E.D.

4.21. Elliptic Gorenstein surface singularities

Definition. A normal surface singularity $P \in X$ is an *elliptic Gorenstein singularity* if it is Gorenstein (Proposition–Definition 4.19) and elliptic (Definition 4.12).

The treatment of elliptic Gorenstein singularities in what follows parallels closely that of rational singularities in 4.12–18.

Theorem (Numerical characterisation). *Let $P \in X$ be an elliptic Gorenstein surface singularity, $f\colon Y \to X$ its minimal resolution, and $Z = Z_K$ the canonical cycle. Then Z is the unique effective cycle with $p_a Z = 1$, and $p_a D \leq 0$ for $D \neq Z$. Moreover, Z is numerically 2-connected and $Z = Z_{\text{num}} = Z_1 = Z_K$.*

Conversely, suppose that the minimal resolution $Y \to X \ni P$ of a normal surface singularity has an effective cycle Z satisfying $p_a Z = 1$, and $p_a D \leq 0$ for $D \neq Z$. Then $P \in X$ is elliptic Gorenstein.

Proof. Note that by the adjunction formula 4.10, the canonical cycle Z_K of any Gorenstein singularity automatically satisfies $\omega_Z = \mathcal{O}_Z(K_Y + Z_K) \cong \mathcal{O}_Z$, so that $\chi(\mathcal{O}_{Z_K}) = 0$ and $p_a(Z_K) = 1$.

Suppose first that $P \in X$ is elliptic Gorenstein, and set $Z = Z_1 = Z_K$. Then by Theorem 4.8, (b), any effective exceptional divisor D satisfies $H^1(\mathcal{O}_D) = 0$ if $D \not\geq Z = Z_1$, and so $p_a D \leq 0$. If $D > Z = Z_K$ then

$$2p_a D - 2 = D(D-Z) = (D-Z)^2 + Z(D-Z) \leq (D-Z)^2 < 0, \tag{7}$$

($K_Y = -Z$ is nef on exceptional curves, because Y is minimal), so again $p_a D \leq 0$.

Claim. *If an effective divisor Z on a nonsingular surface satisfies $p_a Z = 1$ and $p_a D \leq 0$ for all $D < Z$ then $D(Z-D) \geq 2$, that is, Z is numerically 2-connected.*

Proof. Try it and see. □

This proves the "moreover" part: $-Z$ is nef (because $-Z \overset{\text{num}}{\sim} K_Y$), and for any divisor D with $0 < D < Z$, the claim implies that $D(Z-D) \geq 2$, so $-D$ is not nef. Therefore $Z = Z_{\text{num}}$.

The point of the converse is to prove that $K_Y + Z \overset{\text{lin}}{\sim} 0$ near $f^{-1}P$, so that $P \in X$ is Gorenstein by Proposition 4.19, (1). I divide this up into easy steps.

Step 1. *$Z = Z_{\text{num}}$, and in particular $f^{-1}P = \operatorname{Supp} Z$.*

Proof. As in (7), applying the assumption $p_a \leq 0$ to $Z + \Gamma_i$ implies that $-Z\Gamma_i \geq 0$ for any exceptional Γ_i:

$$\begin{aligned} -2 \geq 2p_a(Z+\Gamma) - 2 &= (K_Y + Z + \Gamma)(Z+\Gamma) \\ &= (K_Y + Z)Z + (K_Y + \Gamma)\Gamma + 2Z\Gamma \\ &\geq -2 + 2Z\Gamma. \end{aligned} \tag{8}$$

As in the claim, if $0 < D < Z$ then $-D$ is not nef, so that $Z = Z_{\text{num}}$.

Step 2. *$K_Y + Z \overset{\text{num}}{\sim} 0$ on exceptional curves.*

Proof. Since $(K_Y + Z)Z = 0$, there is nothing more to prove if Z is irreducible. If $Z = \sum n_i \Gamma_i$ then $\sum n_i (K_Y + Z)\Gamma_i = 0$, and the assumption $p_a(\Gamma_i) = 0$ together with the recent claim gives

$$(K_Y + Z)\Gamma_i = (K_Y + \Gamma_i)\Gamma_i + (Z - \Gamma_i)\Gamma_i = -2 + (Z - \Gamma_i)\Gamma_i \geq 0.$$

Thus $(K_Y + Z)\Gamma_i = 0$ for each i, as required.

Step 3. *$\mathcal{O}_Z(K_Y + Z) \cong \mathcal{O}_Z$.*

Proof. We know that $\mathcal{O}_Z(K_Y + Z)$ is a numerically trivial line bundle on Z, and that Z is numerically connected, so that by Lemma 3.11, (iii), it is enough to observe that it has a section:

$$H^0(\mathcal{O}_Z(K_Y + Z)) \overset{\text{d}}{=} H^1(\mathcal{O}_Z) \neq 0.$$

Step 4. *$H^0(\mathcal{O}_Y(K_Y + Z)) \to H^0(\mathcal{O}_Z(K_Y + Z))$ is surjective.*

Proof. As in the proof of Theorem 4.8, the cokernel $H^1(\mathcal{O}_Y(-Z))$ can be calculated as an inverse limit $\varprojlim H^1(\mathcal{O}_D(-Z))$, and each of these groups is zero (see Ex. 4.15).

It follows from Steps 3–4 that $\mathcal{O}_Y(K_Y + Z)$ has a section which is a basis at every point of $f^{-1}P$. Therefore $K_Y + Z \overset{\text{lin}}{\sim} 0$ in a neighbourhood of $f^{-1}P$. This completes the proof that $P \in X$ is Gorenstein with canonical cycle $Z_K = Z$. Clearly by Theorems 4.8 and 4.20, $R^1 f_* \mathcal{O}_Y = H^1(\mathcal{O}_Z)$ is 1-dimensional, so that $P \in X$ is elliptic Gorenstein. Q.E.D.

4.22. Examples

Just as the affine cone over a rational normal curve of degree d serves as a model example of a rational singularity of multiplicity d (at least for mnemonic purposes), the affine cone over a normally embedded elliptic curve $E_d \subset \mathbb{P}^{d-1}$ of degree $d \geq 3$ is the model example of an elliptic Gorenstein singularity; in singularity theory these cones are called *simple elliptic singularities*. Note that the fact that the elliptic curve E_d has a modulus (the j-invariant) means that the analytic type of the singularity also depends on a modulus. The typical example is the ordinary hypersurface triple point

$$0 \in X : (x^3 + y^3 + z^3 + \lambda xyz = 0) \subset \mathbb{C}^3$$

These admit many degenerations (for example, $xyz + x^p + y^q + z^r$ for $p, q, r \geq 4$, known to singularity theorists as $T_{p,q,r}$). This class of singularities includes cases that are very complicated from the point of view of classification of singularities.

Elliptic Gorenstein singularities with $d = 1$ or 2 are hypersurface double points, again typified by the simple elliptic singularities

$$x^2 + y^3 + z^6 + \lambda xyz \quad \text{and} \quad x^2 + y^4 + z^4 + \lambda xyz,$$

which this time are cones in a weighted sense: they are the respective affine cones on the curves in weighted projective planes $C_6 \subset \mathbb{P}(3, 2, 1)$ and $C_4 \subset \mathbb{P}(2, 1, 1)$. The statements of the results are slightly different, but the theory is broadly similar to that for $d \geq 3$. They are alternative explicit methods to treat them. Compare Ex. 4.18.

Both [L2] and [R] contain systematic lists of examples of elliptic Gorenstein singularities with $d = 1, 2, 3$.

4.23. The main theorem

Set $d = -Z^2$, defined to be the *degree* of $P \in X$.

Main Theorem ([R], [L2], [Mori]).
(I) If $d \geq 2$ then $m_P \mathcal{O}_Y = \mathcal{O}_Y(-Z)$; in particular $Z = Z_{\mathrm{f}}$ and $|-Z|$ is a free linear system, so that $\operatorname{mult}_P X = d$.
(II) If $d \geq 3$, then $f_* \mathcal{O}_Y(-2Z) = m_P^2$, and $\operatorname{emb\,dim}_P X = d$.
Moreover, $f_* \mathcal{O}_Y(-kZ) = m_P^k$ for all k; that is, the $\mathcal{O}_X$-algebra

$$\bigoplus_{k \geq 0} f_* \mathcal{O}_Y(-kZ) = \bigoplus_{k \geq 0} m_P^k \tag{9}$$

is generated by its component m_P in degree 1.

The results for $d = 1$ or 2 are slightly longer to state, and are deferred to 4.25 for the sake of clarity; however, the proofs are closely related, and are given together when appropriate.

Discussion. (II) says that the ordinary blow $X_1 \to X$ of X at P coincides with the relative canonical model of $Y \to X$. Indeed, the blowup is defined to be Proj of the algebra $\bigoplus m_P^k$ on the right-hand side of (9). On the other hand, by definition of the canonical cycle, $\mathcal{O}_Y(-Z) \cong \mathcal{O}_Y(K_Y)$ in a neighbourhood of $f^{-1}P$, so that the left-hand side of (9) is the relative canonical algebra $\mathcal{R}(Y, K_Y)$ of $Y \to X$. It is easy to see that $\operatorname{Proj} \mathcal{R}(Y, K_Y)$ is the normal surface $\overline{Y}$ with only Du Val singularities and ample $K_{\overline{Y}}$ obtained from Y by contracting exactly the -2-curves of $f^{-1}P$.

Proof of (I). Most of the proof takes place on Z. First note that the restriction $H^0(Y, \mathcal{O}_Y(-Z)) \to H^0(Z, \mathcal{O}_Z(-Z))$ is surjective, by the usual argument: the cokernel is $H^1(Y, \mathcal{O}_Y(-2Z))$, which is zero by Ex. 4.15. For $d = -Z^2 \geq 2$, the following result applied to $\mathcal{L} = \mathcal{O}_Z(-Z)$ gives at once that $|-Z|$ is free on Y, which proves (I).

Lemma. *Let $\mathcal{L}$ be a line bundle on Z which is nef and of degree $d > 0$. Assume if possible that $Q \in Z$ is a closed point at which $|\mathcal{L}|$ is not free. Then $d = 1$, $Q \in Z$ is a nonsingular point (therefore a Cartier divisor), and $\mathcal{L} \cong \mathcal{O}_Z(Q)$. Then $H^0(\mathcal{L})$ consists of the identity inclusion $s\colon \mathcal{O}_Z \hookrightarrow \mathcal{O}_Z(Q)$ and its scalar multiples.*

If $\deg \mathcal{L} = 1$ then $h^0(\mathcal{L}) = 1$, so that $|\mathcal{L}|$ is not free, and all these assertions hold for some point $Q \in Z$.

Proof. The assumption is that the evaluation map $H^0(Z, \mathcal{L}) \to k_Q$ is not surjective, so its cokernel $H^1(Z, m_Q\mathcal{L})$ is nonzero. I sidestep the difficulty that this H^1 might come from restriction to a smaller divisor by taking D to be a divisor with $0 < D \leq Z$ which is minimal with $H^1(D, m_Q\mathcal{L}_{|D}) \neq 0$. Minimal means:

$$H^1(D - \Gamma_i, m_Q\mathcal{L}_{|D - \Gamma_i}) = 0 \quad \text{for every component } \Gamma_i < D,$$

so that the nonzero H^1 comes from the sheaf kernel of the restriction map

$$m_Q\mathcal{L} \to m_Q\mathcal{L}_{|D - \Gamma_i}.$$

Outside Q, the kernel is $\mathcal{L} \otimes \mathcal{O}_{\Gamma_i}(-D + \Gamma_i)$ by the dévissage exact sequence of 3.10. To understand what happens at Q, note there are two cases: either $Q \in D - \Gamma_i$, and the kernel is $\mathcal{O}_{\Gamma_i}(\mathcal{L} - D + \Gamma_i)$, since everything vanishing on $D - \Gamma_i$ is automatically in m_Q; or $Q \notin D - \Gamma_i$ (which means that Γ_i is a reduced component of D, and is the unique component of D containing Q, so that the restriction is an isomorphism near Q), and the kernel is $m_Q\mathcal{O}_{\Gamma_i}(\mathcal{L} - D + \Gamma_i)$.

Therefore

$$\begin{aligned} H^1(\Gamma_i, \mathcal{O}_{\Gamma_i}(\mathcal{L} - D + \Gamma_i)) \neq 0 &\quad \text{if } Q \in D - \Gamma_i; \\ \text{and} \quad H^1(\Gamma_i, m_Q\mathcal{O}_{\Gamma_i}(\mathcal{L} - D + \Gamma_i)) \neq 0 &\quad \text{if } Q \notin D - \Gamma_i. \end{aligned}$$

Now quite generally, for a torsion free sheaf $\mathcal{F}$ of rank 1 on an irreducible curve Γ,

$$H^1(\Gamma, \mathcal{F}) \neq 0 \implies \deg_\Gamma \mathcal{F} \leq 2p_a\Gamma - 2. \tag{10}$$

Write $D = \sum a_i\Gamma_i$. Since $K_Y = -Z$ locally, $2p_a\Gamma_i - 2 = (-Z + \Gamma_i)\Gamma_i$. Summing (10) over all components Γ_i of D, and noticing that replacing $\mathcal{L}$ by $m_Q\mathcal{L}$ decreases the degree by 1 for at most one component Γ_i, I get

$$\sum a_i\Gamma_i(\mathcal{L} - D + \Gamma_i) \leq 1 + \sum a_i\Gamma_i(-Z + \Gamma_i).$$

The quadratic terms Γ_i^2 on each side cancel, giving

$$D\mathcal{L} + D(Z - D) \leq 1. \tag{11}$$

But now Z is 2-connected, so that (11) implies that $D = Z$, $\deg \mathcal{L} = 1$. Thus $h^0(\mathcal{L}) = 1$. Moreover, $Q \in \Gamma_i$ holds for a unique irreducible component Γ_i, and the section $\mathcal{O}_Z \to \mathcal{L}$ vanishes at Q only, so that Q supports an effective Cartier divisor of degree 1, which means it is a nonsingular point of Z. This proves the lemma and completes the proof of (I). Q.E.D.

Remark. A minor modification of the same argument proves the following result (see Ex. 4.22): if Z is a 2-connected divisor on a nonsingular surface and $\mathcal{L}$ a line bundle with $\mathcal{L} \otimes \omega_Z^{-1}$ nef and of degree ≥ 1 then $|\mathcal{L}|$ is free unless $\mathcal{L} \cong \omega_Z(Q)$ with $Q \in Z$ a nonsingular point. Catanese and others [CFHR] contains a dual version of the argument, using Serre duality more systematically: $0 \neq H^1(m_Q\mathcal{L}) \overset{\mathrm{d}}{=} \operatorname{Hom}(m_Q\mathcal{L}, \omega_Z)$, so choose a nonzero element $s \in \operatorname{Hom}(m_Q\mathcal{L}, \omega_Z)$, and let $Z = A + B$ be the decomposition such that A is the maximal divisor on which s vanishes. Then s defines an inclusion $m_Q\mathcal{L}_{|B} \hookrightarrow \omega_Z(-A) = \omega_B$. A simple numerical argument based on 2-connectedness and $\mathcal{L} > K_Z$ proves that $A = 0$, and s defines an isomorphism $m_Q\mathcal{L} \cong \omega_Z$, so that $Q \in Z$ is a nonsingular point and $\mathcal{L} \cong \omega_Z(Q)$.

4.24. Proof of (II)

The main object of study of this section, as in 4.18, is the graded ring

$$R(Z, \mathcal{L}) = \bigoplus_{k \geq 0} H^0(Z, \mathcal{L}^{\otimes k}) \tag{12}$$

corresponding to a nef line bundle of degree $d > 0$ over a divisor Z. For $d \geq 3$, the result is as follows.

Theorem. *Let Z be a 2-connected divisor on a nonsingular surface Y for which $\mathcal{O}_Z(K_Y + Z) \cong \mathcal{O}_Z$, and let $\mathcal{L}$ be a nef line bundle on Z with $d = \deg \mathcal{L} \geq 3$. Then the graded ring $R(Z, \mathcal{L})$ is generated by elements of degree $k = 1$. In particular,*

$$H^0(\mathcal{L}) \otimes H^0(\mathcal{L}) \twoheadrightarrow H^0(\mathcal{L}^{\otimes 2})$$

is surjective.

Proof. Because $\mathcal{L}$ is free, a general section $s_1 \in H^0(\mathcal{L})$ does not vanish along any component of Z, so is regular at every point (that is, locally defined by a nonzerodivisor); thus it defines an effective Cartier divisor A of degree d on Z, such that $\mathcal{L} \cong \mathcal{O}_Z(A)$. As in 4.18, I choose a second general section s and use it to identify $\mathcal{L}$ with $\mathcal{O}_Z$ in a neighbourhood of every point of A, and consider the exact sequence

$$0 \to \mathcal{O}_Z \xrightarrow{s_1} \mathcal{L} \to \mathcal{O}_A \to 0. \tag{13}$$

Now A is an Artinian scheme of degree d, and since $h^0(\mathcal{O}_Z) = h^1(\mathcal{O}_Z) = 1$ it follows that $H^0(\mathcal{L}) \to \mathcal{O}_A$ has rank $d - 1$. (I write $\mathcal{O}_A = H^0(\mathcal{O}_A)$ out of laziness.)

Claim 1. *$H^0(\mathcal{L}) \twoheadrightarrow \mathcal{O}_{A'}$ is surjective for every subscheme $A' \subset A$ of length $d - 1$. Geometrically, this means that $\varphi_{\mathcal{L}}$ embeds A in $\mathbb{P}^{d-2}$ as a subscheme of degree d with the property that any subscheme of degree $d - 1$ spans $\mathbb{P}^{d-2}$, like d points forming a projective frame of reference.*

To prove this, write $\mathcal{I}_{A'} \subset \mathcal{O}_Z$ for the ideal sheaf of A', so that $\mathcal{I}_A \subset \mathcal{I}_{A'}$ and $\mathcal{I}_{A'}/\mathcal{I}_A \cong k_Q$ for some point $Q \in Z$. Then $\mathcal{F} = \mathcal{I}_{A'}\mathcal{O}_Z(A)$ is an intermediate torsion free sheaf

$$\mathcal{O}_Z \subset \mathcal{F} \subset \mathcal{O}_Z(A),$$

with $\mathcal{F}/\mathcal{O}_Z \cong k_Q$. Now by Serre duality $H^1(\mathcal{F}) \stackrel{\mathrm{d}}{=} \operatorname{Hom}(\mathcal{F}, \mathcal{O}_Z)$. But this group is zero; for a nonzero homomorphism $\mathcal{F} \to \mathcal{O}_Z$ must restrict to a nonzero map $\mathcal{O}_Z \to \mathcal{O}_Z$, necessarily an isomorphism by 2-connectedness, and this would split the exact sequence $\mathcal{O}_Z \to \mathcal{F} \to k_Q$, contradicting $\mathcal{F}$ torsion free. Therefore $H^1(\mathcal{I}_{A'}\mathcal{L}) = 0$, which gives the required surjectivity.

Claim 2. *A subscheme $A_d \subset \mathbb{P}^{d-2}$ with the property stated in Claim 1 imposes linearly independent conditions on quadrics. (This means that the restriction map $H^0(\mathbb{P}^{d-2}, \mathcal{O}(2)) \to \mathcal{O}_{A_d}$ is surjective.)*

The cases $d = 3$ and $d = 4$ of Claim 2 are very easy, see Ex. 4.23. Although the proof involves various bits of mess, in substance it is similar to, and rather easier than [UAG], §1.

Indeed, any Artinian scheme A has a filtration $A_1 \subset \cdots \subset A_{d-1} \subset A$ by subschemes A_i of degree i, and by assumption, each A_i has scheme theoretic linear span $\langle A_i \rangle = \mathbb{P}^{i-1}$. Clearly each A_i imposes i conditions on quadrics (since it does on hyperplanes), and I prove inductively that A_i is a scheme theoretic intersection of quadrics. It follows from this that A_{d-1} imposes $d-1$ conditions on quadrics, and that there is a quadric through A_{d-1} not containing A_d, so that A_d imposes one further condition, which proves Claim 2.

The inductive step is as follows: write

$$\langle A_i \rangle = \mathbb{P}^{i-1} : (x_i = 0) \subset \langle A_{i+1} \rangle = \mathbb{P}^i$$

and $\mathcal{I}_i$ for the ideal sheaf of A_i. Suppose that A_i and A_{i+1} differ only at a point Q. Then it is easy to check that there is an exact sequence

$$0 \to m_Q\mathcal{O}_{\mathbb{P}^i}(1) \xrightarrow{x_i} \mathcal{I}_{i+1}\mathcal{O}_{\mathbb{P}^i}(2) \to \mathcal{I}_i\mathcal{O}_{\mathbb{P}^{i-1}}(2) \to 0. \tag{14}$$

(See Ex. 4.24.) Clearly, to say that A_i is a scheme theoretic intersection of quadrics means exactly that $\mathcal{I}_i\mathcal{O}_{\mathbb{P}^{i-1}}(2)$ is generated by its H^0. Now the right-hand end of (14) is generated by its H^0 by the inductive hypothesis, and the left-hand end obviously so. Therefore so is the middle term. This proves Claim 2.

The proof Theorem 4.24 and Theorem 4.23, (II) is now straightforward. Let $s_1, \dots, s_d$ be a basis of $H^0(Z, \mathcal{L})$, with s_1 defining an effective Cartier divisor A as in (13). Since $s_2, \dots, s_d$ are the homogeneous coordinates in the $\mathbb{P}^{d-2}$ containing A_d, by Claim 2, I can choose a set $\{t_1, \dots, t_d\}$ of quadratic monomials in $s_2, \dots, s_d$ that, viewed as sections of $H^0(Z, \mathcal{O}_Z(2A))$, map to a basis of $\mathcal{O}_A$. For $k \geq 2$, consider the exact sequence

$$0 \to H^0(Z, \mathcal{O}_Z((k-1)A) \xrightarrow{s_1} H^0(Z, \mathcal{O}_Z(kA)) \to \mathcal{O}_A \to 0.$$

Recall that I chose a second section $s \in H^0(Z, \mathcal{L})$ that is a local basis of $\mathcal{L}$ at every point of $\operatorname{Supp} A$, and which I use to identify $\mathcal{L}_{|A}$ with $\mathcal{O}_A$. Then the products $s^{k-2} \cdot \{t_1, \dots, t_d\}$ form a complimentary basis to $s_1 \cdot H^0(\mathcal{O}_Z((k-1)A)$. This proves that the ring $R(Z, \mathcal{L})$ is generated in degree 1.

The final part of the proof of (II) is the same as in 4.18. Since

$$0 \to f_*\mathcal{O}_Y(-(k+1)Z) \to f_*\mathcal{O}_Y(-kZ) \to H^0(Z, \mathcal{O}_Z(-kZ)) \to 0$$

is exact, any $g \in f_*\mathcal{O}_Y(-kZ)$ can be written modulo $f_*\mathcal{O}_Y(-(k+1)Z)$ as a polynomial of degree k in elements of m_P. By induction,

$$f_*\mathcal{O}_Y(-kZ) \subset m_P^k + f_*\mathcal{O}_Y(-lZ) \quad \text{for any } l > k.$$

But since the $\mathcal{O}_X$-algebra $\mathcal{R}(Y, \mathcal{O}_Y(-Z))$ is finitely generated, the argument at the end of 4.18 proves that $f_*\mathcal{O}_Y(-lZ) \subset m^k$ for any fixed k and $l \gg 0$. Therefore $f_*\mathcal{O}_Y(-kZ) = m_P^k$, as requested. Q.E.D.

4.25. The special cases $d = 2, 1$

The results for $d = \deg \mathcal{L} = 2$ or 1 are similar in spirit, but the statements must involve weighted homogeneous rings because the ring $R(Z, \mathcal{L})$ is not generated in degree 1.

Theorem. *For* $d = 2$:

$$R(Z, \mathcal{L}) = k[x_1, x_2, y]/f, \quad \textit{where } \deg x_1, x_2, y = 1, 1, 2,$$

and f is homogeneous of degree 4, of the form $f = y^2 +$ terms involving x_1, x_2.

For $d = 1$:

$$R(Z, \mathcal{L}) = k[x, y, z]/f, \quad \textit{where } \deg x, y, z = 1, 2, 3,$$

and f is homogeneous of degree 6, of the form $f = z^2 + y^3 +$ terms involving x.

Proof. In either case, there exists a section $x \in H^0(Z, \mathcal{L})$ defining a Cartier divisor A of degree d, and therefore an exact sequence

$$0 \to \mathcal{O}_Z \to \mathcal{L} \to \mathcal{O}_A \to 0.$$

When $d = 2$, I set $x = x_1$, and I can find a second section x_2 which is a local basis of $\mathcal{L}$ near $\operatorname{Supp} A$. Now choose an element $y \in H^0(Z, \mathcal{L}^{\otimes 2})$ so that y, x_2^2 map to a basis of $\mathcal{O}_A$. Arguing as usual on the restriction exact sequence

$$0 \to H^0(Z, \mathcal{O}_Z((k-1)A)) \to H^0(Z, \mathcal{O}(kA)) \to \mathcal{O}_A \to 0,$$

we see that $x_2^k, x_2^{k-2}y$ forms a complimentary basis to $x_1 \cdot H^0(Z, \mathcal{O}_Z((k-1)A))$ in every degree $k \geq 2$. It's very easy to see from this that

$$S^k(x_1, x_2), S^{k-2}(x_1, x_2)y \quad \text{bases} \quad H^0(Z, \mathcal{L}^{\otimes k}).$$

In particular, x_1, x_2, y generate the ring, and there is a unique relation expressing y^2 in terms of the stated basis.

The case $d = 1$ is similar, and is left as an easy exercise. [Hint: You just have to pick elements y, z in degrees $2, 3$ that map to a basis of $\mathcal{O}_A = k_Q$.] Q.E.D.

Corollary. *If $d = 2$ then $\operatorname{mult}_P X = 2$ and $\operatorname{emb\,dim}_P X = 3$, that is, $P \in X$ is a hypersurface double point. The linear system $|-Z|$ on Y is free; that is, $Z = Z_{\mathrm{f}}$, and $m_P \mathcal{O}_Y = \mathcal{O}_Y(-Z)$. Moreover, $f_* \mathcal{O}_Y(-2Z) = m_P^2 + (y)$ for some element $y \in m_P \setminus m_P^2$, and the $\mathcal{O}_X$-algebra $\mathcal{R}(Y, K_Y) = \bigoplus_{k \geq 0} f_* \mathcal{O}_Y(-kZ)$ is generated by m_P in degree 1 and y in degree 2 (together with m_P^2).*

If $d = 1$ then $\operatorname{mult}_P X = 2$ and $\operatorname{emb\,dim}_P X = 3$, that is, $P \in X$ is a hypersurface double point. The linear system $|-Z|$ on Y has a single base point $Q \in Z$, which is a nonsingular point of Z; that is, $Z = Z_{\mathrm{f}}$, and $m_P \mathcal{O}_Y = m_Q \mathcal{O}_Y(-Z)$. The algebra $\mathcal{R}(Y, K_Y) = \bigoplus_{k \geq 0} f_* \mathcal{O}_Y(-kZ)$ is generated by m_P in degree 1, and elements $y \in m_P \setminus m_P^2$ in degree 2 and $z \in m_P \setminus (m_P^2, y)$ in degree 3.

Remark. In either case, the system of ideals

$$I_k = f_* \mathcal{O}_Y(-kZ) \subset \mathcal{O}_X \quad \text{for } k \geq 0$$

defines a filtration of $\mathcal{O}_X$, and the relative canonical model $Y \to \overline{Y} \to X$, defined as $\overline{Y} = \operatorname{Proj} \mathcal{R}(Y, K_Y)$, is obtained by blowing up X along this filtration. In either case, the ideals I_k can be defined by a weighting of the ambient space $\mathbb{A}^3$. That is, give

$$\begin{aligned} &y, x_1, x_2 \text{ weights } (2,1,1) \text{ in case } d = 2, \\ &z, y, x \text{ weights } (3,2,1) \text{ in case } d = 1. \end{aligned}$$

then I_k is the ideal generated by monomials of weighted degree $\geq k$. Taking $\operatorname{Proj} \bigoplus_{k \geq 0} I_k$ can also be described in toric geometry as the weighted blowup.

When $d = 2$, the relative canonical model can also be described as the blowup of X at P followed by normalisation.

Exercises to Chapter 4

1. Let $\mathbb{Z}/2$ act on $\mathbb{A}^2$ by $(u, v) \mapsto (-u, -v)$. Prove that the subring of $k[u, v]$ of invariant polynomials is generated by $x = u^2, y = uv, z = v^2$, and that the morphism $\mathbb{A}^2 \to \mathbb{A}^3$ given by $(u, v) \mapsto (x, y, z)$ defines an embedding of the orbit space $\mathbb{A}^2/(\mathbb{Z}/2)$ as the ordinary quadratic cone $X : (xz = y^2) \subset \mathbb{A}^3$ of 4.1.

2. Let $V \to \mathbb{A}^2$ be the blowup of the origin; show that the action of $\mathbb{Z}/2$ of Ex. 4.1 extends to an action on V which fixes the -1-curve pointwise, and that the quotient space $V/(\mathbb{Z}/2)$ is isomorphic to the blowup $Y \to X$, and fits into a commutative diagram

$$\begin{array}{ccc} V & \to & \mathbb{A}^2 \\ \downarrow & & \downarrow \\ Y & \to & X. \end{array}$$

Calculate the canonical class K_V and its relation with $K_{\mathbb{A}^2}$ and K_Y.

3. Carry out blowups to resolve the Du Val singularities

$$\begin{aligned} A_n : &\quad x^2 + y^2 + z^{n+1} = 0, \\ D_n : &\quad x^2 + y^2 z + z^{n-1} = 0, \\ E_6 : &\quad x^2 + y^3 + z^4 = 0, \\ E_7 : &\quad x^2 + y^3 + yz^3 = 0, \\ E_8 : &\quad x^2 + y^3 + z^5 = 0 \end{aligned}$$

in the spirit of 4.2. [Hint: In 4.2, I only calculated one affine piece of the blowup and stated that nothing interesting happens on the others. In general, one has to consider more than one affine piece.] Check that the exceptional locus is a bunch of -2-curves with intersections given by the Dynkin diagram.

4. Calculate the numerical cycle Z_{num} for each of the Du Val singularities. Check that $Z_{\text{num}}^2 = -2$. [Hint: Start from $Z_0 = \sum \Gamma_i$, and successively increase the coefficients of Γ_i only if $Z_j \Gamma_i > 0$.]

5. If $\bigcup \Gamma_i$ is a connected bunch of -2-curves with negative definite intersection matrix $(\Gamma_i \Gamma_j)$, check that the configuration is given by one of the Dynkin diagrams A_n, D_n, E_6, E_7, E_8. [Hint: If the configuration is more complicated, you can write down a combination $D = \sum a_i \Gamma_i$ with $D^2 \geq 0$. All the completed Dynkin diagrams $\widetilde{A}_n, \ldots, \widetilde{E}_8$ appear as logical ends of the argument. For example, on the graph

$$\widetilde{D}_4: \qquad \begin{array}{ccccc} & & \bigcirc\,1 & & \\ & & | & & \\ 1 & & 2 & & 1 \\ \bigcirc & - & \bigcirc & - & \bigcirc \\ & & | & & \\ & & \bigcirc\,1 & & \end{array},$$

$D = \sum a_i \Gamma_i$ (with a_i as indicated) has $D^2 = 0$, which cannot happen in a negative definite graph G, so that G must have valency ≤ 3. You have to prove that G does not have a loop, or 2 vertexes of valency 3, or 1 vertex of valency 3 and very long arms out of it, etc.]

6. Let $\mathbb{Z}/3$ act on $\mathbb{A}^2$ by $(u, v) \mapsto (\varepsilon u, \varepsilon^2 v)$, where ε is a primitive 3rd root of 1. Study the quotient

$$\mathbb{A}^2/(\mathbb{Z}/3) = X : (xz = y^3) \subset \mathbb{A}^3$$

as in Ex. 4.1 in terms of the quotient map $\mathbb{A}^2 \to \mathbb{A}^3$ defined by

$$x = u^3, y = uv, z = v^3.$$

Let $Y \to X$ be the minimal resolution. Show how to construct a $\mathbb{Z}/3$ cover $V \to Y$ by first blowing up the origin in $\mathbb{A}^2$, then blowing up the two points in which the -1-curve meets the coordinate axes.

The point of the question is that the minimal resolution Y does not have a finite $\mathbb{Z}/3$ cover by a nonsingular blowup of $\mathbb{A}^2$.

7. Do the same as in Ex. 4.6 for $\mathbb{Z}/4$ acting by $(u, v) \mapsto (iu, i^3 v)$, where $i^2 = -1$.

8. Let ε be a primitive 5th root of 1. Make $\mathbb{Z}/5$ act on $\mathbb{A}^2$ by

$$(u, v) \mapsto (\varepsilon u, \varepsilon^2 v).$$

Show that the subring of $k[u, v]$ of invariant polynomials is generated by 4 monomials, so that the quotient space $\mathbb{A}^2/(\mathbb{Z}/5) \cong X \subset \mathbb{A}^4$. (This is the quotient singularity of type $(5, 2)$ in Hirzebruch's terminology, or of type $\frac{1}{5}(1, 2)$ in that of [YPG].) Show that the ideal of relations holding between the invariant monomials is generated by 3 relations that can be written as the maximal minors of a 2×3 matrix. Show how to resolve the singularity by an explicit blowup, and describe the exceptional curves.

9. Let ε be a primitive $2n$th root of 1 and consider the two matrixes

$$A = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Show that they generate a binary dihedral subgroup $\mathrm{BD}_n \subset \mathrm{SL}(2, \mathbb{C})$ of order $4n$. Find the ring of invariants of BD_n acting on $\mathbb{C}^2$ and prove that $\mathbb{C}^2 / \mathrm{BD}_n$ is isomorphic to the Du Val singularity D_{n+2}. [Hint: Find first the ring of invariants of $\langle A \rangle \cong \mathbb{Z}/2n$, and show that B acts by an involution on it.]

10. In Ex. 4.9 you saw that the inclusion $\mathbb{Z}/2n \triangleleft \mathrm{BD}_n$ defines a double cover $A_{2n-1} \to D_{n+2}$ between Du Val singularities. Show also that $\mathrm{BD}_n \subset \mathrm{BD}_{2n}$ is a normal subgroup of index 2 and that it defines a double cover $D_{n+2} \to D_{2n+2}$ between Du Val singularities.

11. Prove Proposition 4.12, (2). [Hint: Several methods are possible. For example, you can prove that $h^0(\mathcal{O}_D(-Z_{\text{num}})) \neq 0$, so that $h^0(\mathcal{O}_{D+Z}) \geq 2$ for every $D > 0$. Or you can calculate $\chi(\mathcal{O}_{D+Z})$ using the numerical games of (1).]

12. Prove Proposition 4.12, (3). [Hint: If $Z_{\text{num}} = D_1 + D_2$, write out $p_a(Z_{\text{num}}) = 0$, $p_a(D_1), p_a(D_2) \leq 0$ in terms of the adjunction formula 3.6 or 4.10. Compare the method of 4.21, (7–8)]

13. Consider the codimension 2 singularity $P \in X \subset \mathbb{C}^4$ defined by the two equations $x_1^2 = y_1^3 - y_2^3$ and $x_2^2 = y_1^3 + y_2^3$. Prove that $P \in X$ has a resolution $f\colon Y \to X$ such that $f^{-1}P$ is a nonsingular curve C of genus 2 and $C^2 = -1$, but with $H^0(\mathcal{O}_C(-C)) = 0$. Prove that x_1, x_2 vanish along C with multiplicity 2 and y_1, y_2 with multiplicity 3. Hence $Z_{\text{f}} = 2C$, although obviously $Z_{\text{num}} = C$.

14. If $(\Gamma_i \Gamma_j)$ is negative definite, prove that $H^0(\mathcal{O}_D(D)) = 0$ for every effective divisor supported on Γ_i. More generally, if $\mathcal{L}$ is a line bundle on D such that $\deg_{\Gamma_i} \mathcal{L} \leq 0$ for all i then $H^0(\mathcal{O}_D(D) \otimes \mathcal{L}) = 0$. [Hint: Since $D^2 < 0$, any section $s \in H^0(\mathcal{O}_D(D))$ must vanish on some components of D. Now use the argument of Lemma 3.10 and Ex. 3.27.]

15. Let $Y \to X$ be a resolution of a normal surface singularity $P \in X$, and $\mathcal{L}$ a line bundle on Y such that $\mathcal{L}\Gamma_i \geq K_Y \Gamma_i$ for each exceptional curve Γ_i. Prove that $H^1(D, \mathcal{L}_{|D}) = 0$ for every exceptional divisor D, and deduce that $H^1(Y, \mathcal{L}) = R^1 f_* \mathcal{L} = 0$. This is related to the vanishing theorems of Kodaira and Grauert–Riemenschneider, but works in all characteristic.

16. Suppose that X is projective with ample H, and $f\colon Y \to X$ as usual. Let $Z = \sum a_i \Gamma_i$ be an effective exceptional divisor with $Z\Gamma_i < 0$. Prove that $nf^*H - Z$ is ample on Y for $n \gg 0$. [Hint: Some section of nH on X vanishes on Z, so that $nf^*H - Z$ is effective for some n. Now prove that by taking a larger n if necessary, $(nf^*H - Z)C > 0$ for every curve $C \subset Y$ and $(nf^*H - Z)^2 > 0$. The result follows by the Nakai–Moishezon ampleness criterion, compare footnote to 4.8 and [H1], Chapter V, 1.10.]

17. Show how to resolve the elliptic Gorenstein singularity

$$T_{p,q,r} \quad \text{defined by} \quad xyz + x^p + y^q + z^r \quad \text{for} \quad \frac{1}{p} + \frac{1}{q} + \frac{1}{r} \leq 1,$$

and describe the configuration of exceptional curves. [Hint: Start with the cases $p, q, r > 3$. Blowing up the maximal ideal gives a surface X_1 meeting the exceptional $\mathbb{P}^2$ in the tangent cone, which is the triangle $xyz = 0$, and having Du Val singularities at the 3 corners.]

18. Show how to resolve the elliptic Gorenstein singularities of degree 1 given by

$$x^2 + y^3 + z^k = 0 \quad \text{for } k = 6, 7, \dots, 11, \quad \text{and} \quad x^2 + y^3 + yz^l = 0 \quad \text{for } l = 5, 7.$$

[Hint: Use the weighted blowup with weights $3, 2, 1$, followed by the resolution of Du Val singularities.]

19. Ex. 4.8 is the special case $p = 2, q = r = 1$ of the rational triple points of Example 4.12. If you enjoyed it and want more of the same, carry out the calculations to verify the assertions made in Example 4.12.

20 (harder). The possible configurations of exceptional curves of a resolution $Y \to X$ of a rational triple point $P \in X$ are listed in [A2], p. 135. Verify this list following the hint given there. Find equations for some of these following the ideas of Example 4.12.

21 (harder). Rational surface singularities include the quotient singularities $\mathbb{C}^2/G$ where $G \subset \mathrm{GL}(2, \mathbb{C})$ is a finite group not containing any quasireflections. Prove that $P \in X$ is a quotient singularity if and only if it is a rational singularity, and the minimal resolution $Y \to X$ has $-K_Y \overset{\text{num}}{\sim} \sum a_i \Gamma_i$ with $a_i \in \mathbb{Q}$, $a_i < 1$. List all the cases. [Hint: Because $P \in X$ is rational, some multiple of K_X is a Cartier divisor. Then $P \in X$ has a cyclic cover which is a Du Val singularity. The lists can be found in Alexeev [Utah2], Chapter 3.]

22. Let Z be a 2-connected divisor on a nonsingular surface and $\mathcal{L}$ a line bundle with $\mathcal{L} \otimes \omega_Z^{-1}$ nef and of degree ≥ 1. Modify the proof of Lemma 4.23 to prove that if $|\mathcal{L}|$ is not free at Q then $Q \in Z$ is a nonsingular point, and $\mathcal{L} \cong \omega_Z(Q)$.

23. State and prove 4.24, Claim 2 for $d = 3$ and $d = 4$. [Hint: For $d = 4$, compare [UAG], §1.]

24. Check the exact sequence 4.24, (14). [Hint: Step 1. There is a restriction map $\mathcal{I}_{i+1} \to \mathcal{I}_i$. The kernel is obviously contained in $\mathcal{O}_{\mathbb{P}^i}(-1)$. Step 2. Because $A_{i+1} \not\subset \mathbb{P}^{i-1}$, at Q, the kernel is equal to $m_Q \mathcal{O}_{\mathbb{P}^i}(-1)$. Step 3. It's tricky to see that $\mathcal{I}_{i+1} \to \mathcal{I}_i$ is surjective by local considerations at Q in the crucial case $Q \in \operatorname{Supp} A_i$; however, there is a very easy proof by calculating χ of the 3 sheaves in the exact sequence

$$0 \to m_Q \mathcal{O}_{\mathbb{P}^i}(-1) \to \mathcal{I}_{i+1} \to \mathcal{I}_i.]$$