CHAPTER 1. The cubic surface

There are several ways of treating the cubic surface $X \subset \mathbb{P}^3$ and its 27 lines, for example, in terms of elementary coordinate geometry in $\mathbb{P}^3$, or as a blowup of $\mathbb{P}^2$ in 6 points, as a blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ in 5 points, etc. The difficulty with approaches in this style is that because they start by finding one or more lines of X, they do not give the full symmetry of the configuration of lines. The symmetry group of the configuration has order

$$51,840 = 2^7 \cdot 3^4 \cdot 5 = 27 \cdot 5! \cdot 2^4 = 27 \cdot 16 \cdot 10 \cdot 6 \cdot 2 = \cdots$$

The configuration of lines does, however, have an entirely symmetric description in terms of a certain lattice—the divisor class group or middle cohomology $\operatorname{Pic} X = H^2(X, \mathbb{Z})$. The lines $L \subset X$ are the solutions in this lattice of the equations $HL = 1$, $L^2 = -1$ where H is the class of the hyperplane section.

As well as discussing the cubic surface, this section introduces and gives examples of the following notions: -1-curve, blowup, conic bundle, $\mathbb{P}^1$-bundle, divisor class group $\operatorname{Pic} X$, intersection numbers, linear system and rational map, del Pezzo surface, normal surface singularity and its resolution.

Summary

The cubic surface $X_3 \subset \mathbb{P}^3$

(1) is rational;
(2) is isomorphic to the plane $\mathbb{P}^2$ blown up at 6 points $\Phi = \{P_1, \dots, P_6\}$;
(3) is the image of a birational map $\varphi\colon \mathbb{P}^2 \to \mathbb{P}^3$ defined by the cubics through Φ;
(4) has a configuration of 27 lines described either in terms of geometry in $\mathbb{P}^3$, or in terms of $\mathbb{P}^2$ blown up in 6 points;
(5) has an associated lattice $A(X) \cong \mathbb{Z}^7$ that can be constructed in terms of the configuration of lines, with scalar product that can be diagonalised to $\operatorname{diag}(1, -1, \dots, -1)$; also $A(X) = \operatorname{Pic} X$ and $H^\perp \subset A(X)$ is isomorphic to the root lattice E_6;
(6) the 27 lines and roots of E_6; double-six and simple roots;
(7) del Pezzo surfaces in general

1.1. Main tricks

Let $X \subset \mathbb{P}^3$ be a nonsingular cubic surface. I'm interested in the lines and the triangles of X. Here a *triangle* is a set of 3 distinct coplanar lines $L_1, L_2, L_3 \subset X$, so that $L_1 + L_2 + L_3 = X \cap H$ is a hyperplane section. The following 5 statements allow me to find and organise all the lines of X.

(0) X contains at least one line L. (This is nontrivial, see [UAG], §7 or Shafarevich [Sh], Chapter I.)
(1) Any two intersecting lines determine a triangle. (Obvious.)
(2) If L_1, L_2, L_3 is a triangle and M a fourth line of X, then M meets exactly one of L_1, L_2, L_3. (Almost obvious.)
(3) Main trick: there are exactly 10 lines meeting L, falling into 5 coplanar pairs L_i, L_i'; the pairs are disjoint, that is, $(L_i \cup L_i') \cap (L_j \cup L_j') = \emptyset$ for $i \neq j$. (See [UAG] or [Sh], Chapter IV or [Beauville], IV.15.)
(4) In particular, there exist two disjoint lines L and M.

1.2. All the lines of X

Here I find and give names to all the lines of X. Fix disjoint lines L and M. By 1.1.3, L takes part in exactly 5 triangles L, L_i, L'_i for $i = 1, \dots, 5$, and M meets exactly one of L, L_i, L'_i for each i. By renumbering, let these be $L_1, \dots, L_5$. Then the 10 lines meeting M are $L''_1, \dots, L''_5$ as in the figure.

Claim. *There are* 10 *further lines* L_{klm} *which meet* L_k, L_l, L_m *and not* L_i, L_j*, where* $\{i, j, k, l, m\} = \{1, 2, 3, 4, 5\}$.

Proof. Notice that if $i \neq j$ then L'_i does not meet M nor L_j, so it must intersect the third line of that triangle, that is, L'_i meets L''_j for $i \neq j$.

Choose some L'_i, for clarity L'_1, and consider the 5 triangles involving L'_1. These are L, L_1, L'_1, and 4 others determined by the intersecting pairs L'_1, L''_j for $j = 2, \dots, 5$. Consider for example the triangle L'_1, L''_2, N. Then by 1.1.3, N is disjoint from the line pairs of the other 4 triangles, in particular N does not intersect L''_3, L''_4, L''_5. Therefore N intersects L_3, L_4 and L_5.

It is easy to see that the lines L, M, L_i, L'_i, L''_i and L_{klm} exhaust all the lines of X.

1.3. The lattice $A(X)$

I define a lattice $A(X)$, a kind of miniature version of the divisor class group of any nonsingular variety. $A(X)$ is defined as an Abelian group with generators and relations. The generators are the 27 lines of X; the relations are simply "triangle = constant". More formally, A is the free Abelian group on the 27 lines, modulo the set of relations $L + L' + L'' = M + M' + M''$ whenever L, L', L'' and M, M', M'' are triangles.

Proposition. $A = \mathbb{Z}^7$*; a basis is* $L_1, \dots, L_4, L'_5, L''_5, L_5$

Proof. The triangles containing L are $L + L_i + L'_i$ for $i = 1, \dots, 5$. Thus I get relation $L + L_i + L'_i = L + L_5 + L'_5$ for $i = 1, \dots, 5$, so that

$$L'_i = L_5 + L'_5 - L_i. \tag{1}$$

Arguing in the same way on the triangles containing M gives $L''_i = L_5 + L''_5 - L_i$.

I showed in 1.2 that $L'_i + L''_j + L_{klm}$ is a triangle when $\{i, j, k, l, m\}$ is a permutation of $\{1, 2, 3, 4, 5\}$, and hence

$$L_{klm} = L + L_i - L''_j. \tag{2}$$

Finally, $L_1 + L_{123} + L_{145}$ is another triangle, so that

$$L + L_1 + L'_1 = L_1 + L_{123} + L_{145} = L_1 + 2L + L_4 - L''_5 + L_2 - L''_3,$$

and therefore

$$L = L'_1 - L_4 + L''_5 - L_2 + L''_3 = 2(L_5 + L'_5 + L''_5) - L_1 - L_2 - L_3 - L_4 - L'_5. \tag{3}$$

Therefore, all the lines, hence everything in $A(X)$ can be written as integral combinations of the seven classes $L_1, \dots, L_4, L'_5, L''_5, L_5$. I prove that these elements are linearly independent in $A(X)$ in the next section by introducing a scalar product on $A(X)$.

1.4. The scalar product

I claim that there is a scalar product $A(X) \times A(X) \to \mathbb{Z}$ such that:

(1) Two distinct lines L, L' have $LL' = 0$ or 1 according as they are disjoint or intersect.
(2) $L^2 = -1$ for any line L.
(3) $L(M + M' + M'') = 1$ for any line L and triangle M, M', M''.

Indeed, formally, I can set LL' and L^2 to be anything I wish, provided that my choice is compatible with the equivalence relation defining A. If M, M', M'' is a triangle and L a line distinct from M, M', M'' then we know that L meets exactly one of M, M', M'', so that (1) implies (3). If on the other hand $L = M$ then $LM' = LM'' = 1$ and $L^2 = -1$ again implies (3). This proves that the scalar product is well defined.

Proposition. *The scalar products of the 7 elements*

$$e_0 = L_5 + L_5' + L_5'', \quad e_1 = L_1, \dots, e_4 = L_4, \quad e_5 = L_5' \quad \textit{and} \quad e_6 = L_5''$$

is given by $e_0^2 = 1$, $e_i^2 = -1$ for $i = 1, \dots, 6$ and $e_ie_j = 0$.

In particular, $e_0, \dots, e_6$ form a basis of $A(X)$.

Proof. $e_1 = L_1, \dots, e_4 = L_4$, $e_5 = L_5'$ and $e_6 = L_5''$ are 6 disjoint lines, and the first 4 are also disjoint from L_5, so that most of the multiplication table follows at once by construction of the scalar product. The only assertions still to prove concerns e_0. Thus

$$e_0e_5 = e_0L_5' = (L_5 + L_5' + L_5'')L_5' = 1 - 1 + 0 = 0,$$

and similarly $e_0e_6 = e_0L_5'' = 0$. Finally

$$e_0^2 = e_0(L_5 + L_5' + L_5'') = e_0L_5 = (L_5 + L_5' + L_5'')L_5 = -1 + 1 + 1 = 1. \quad \text{Q.E.D.}$$

1.5. Symmetric treatment of the lines

Claim. *Write h for the class of a triangle. Then $h^2 = 3$ and $hx \equiv x^2 \bmod 2$ for every $x \in A(X)$.*

Proof. If $L+L'+L''$ is any triangle then $h = L+L'+L''$ and $h^2 = h(L+L'+L'') = 1+1+1 = 3$.

For the second part, note that for any lattice, $x^2 \bmod 2$ is a *linear* function $A \to \mathbb{F}_2$, because $(x+y)^2 = x^2 + 2xy + y^2 \equiv x^2 + y^2 \bmod 2$. Therefore it is enough to check that $hx \equiv x^2 \bmod 2$ holds for any set of generators of A. But $A(X)$ is generated by lines L, and we know $hL = 1$ and $L^2 = -1$.

To conclude: a cubic surface X has an associated lattice $A(X) \cong \mathbb{Z}^7$, with a scalar product that can be diagonalised to $\operatorname{diag}(1, -1, \dots, -1)$, and an element $h \in A(X)$ such that $h^2 = 3$ and $hx \equiv x^2 \bmod 2$ for every $x \in A$. It is an easy result of lattice theory that the pair $A(X)$ and H is uniquely determined up to isomorphism by the stated properties.

Inside $A(X)$, the lines are the solutions of the equations $hL = 1$ and $L^2 = -1$. This is a completely symmetric description of the configuration of lines.

Exercise. In the basis $e_0, \dots, e_6$ of 1.3–4, show that

$$h = 3(L_5 + L_5' + L_5'') - L_1 - L_2 - L_3 - L_4 - L_5' - L_5'' = 3e_0 - e_1 - \dots - e_6.$$

and that the 27 classes of lines are

$$\begin{aligned}&6 \text{ classes } e_i \text{ for } i = 1, \dots, 6;\\ &15 \text{ classes } e_0 - e_i - e_j \text{ for } i \neq j = 1, \dots, 6;\\ &6 \text{ classes } 2e_0 - e_{i_1} - \dots - e_{i_5} \text{ for 5 distinct } i_1, \dots, i_5 = 1, \dots, 6.\end{aligned}$$

What is $A(X)$? It was derived above in terms of lines, but maybe it has some more natural description in terms of X. One answer is that if $k = \mathbb{C}$, then

$$A(X) = H_2(X, \mathbb{Z}) = H^2(X, \mathbb{Z}).$$

It is not hard to see by topological arguments that this is a lattice of rank 7, with a scalar product defined by cup product of signature $1, -6$ and discriminant 1. Moreover, the hyperplane section $H = X \cap \mathbb{P}^2$ defines a cohomology class $h \in H^2(X, \mathbb{Z})$ with $h^2 = 3$ and $hx \equiv x^2 \bmod 2$ for every $x \in H^2(X, \mathbb{Z})$.

Taking a line L into the homology class $[L]$ defines a map $A(X) \to H_2(X, \mathbb{Z})$, well defined because all hyperplane sections of X are homologous. The proof that this is injective and surjective are rather similar to the arguments of the next section.

1.6. The divisor class group Pic X

I now use the above results on lines on the cubic surface to motivate general ideas of divisors on varieties. A sum of lines $\sum n_i L_i$ is a particular case of a divisor, the relation "all triangles are equal" $L + L' + L'' = M + M' + M''$ a particular case of linear equivalence of divisors, the group $A(X)$ a particular case of the divisor class group Pic X, and the scalar product on $A(X)$ given by $L_1 L_2 = 0, 1$ or -1 a particular case of intersection numbers.

I run briefly through the theory, which is in use throughout these notes; if you have not seen this material before, you should read for example [Sh], Chapter III for more details. Let X be a normal variety (for example, nonsingular). A *prime divisor* of X is an irreducible codimension 1 subvariety $\Gamma \subset X$. A *divisor* $D = \sum n_i \Gamma_i$ is a formal linear combination of prime divisors $\Gamma_i \subset X$ with coefficients $n_i \in \mathbb{Z}$. The group of divisors Div X is the free Abelian group generated by prime divisors. D is *effective*, written $D \geq 0$, if all $n_i \geq 0$.

Recall that if $\Gamma \subset X$ is a prime divisor then $\mathcal{O}_{X,\Gamma}$, the *local ring* at Γ, is the subring of $f \in k(X)$ regular at some point of Γ (therefore regular on a dense open set); $\mathcal{O}_{X,\Gamma}$ is a discrete valuation ring, that is, there is a valuation $v_\Gamma \colon k(X) \setminus 0 \to \mathbb{Z}$ such that $\mathcal{O}_{X,\Gamma} = \{f \mid v_\Gamma(f) \geq 0\} \cup \{0\}$. Here for $0 \neq f \in k(X)$, we say that

$$\begin{aligned}&v_\Gamma(f) > 0 \text{ if and only if } f \text{ has a zero along } \Gamma \text{ of order } v_\Gamma(f);\\ &v_\Gamma(f) < 0 \text{ if and only if } f \text{ has a pole along } \Gamma \text{ of order } -v_\Gamma(f);\\ &v_\Gamma(f) = 0 \text{ if and only if } f \text{ and } f^{-1} \text{ are regular along } \Gamma\end{aligned}$$

If $0 \neq f \in k(X)$, the expression $\operatorname{div} f = \sum_\Gamma v_\Gamma(f)\Gamma$ defines the divisor of f; the sum runs over all prime divisors, but is finite, since both f and f^{-1} are regular on dense open sets of X. A *principal divisor* is a divisor of the form $\operatorname{div} f$. Two divisors D and D' are *linearly equivalent*, written $D \overset{\text{lin}}{\sim} D'$ or $D \sim D'$ if $D - D'$ is a principal divisor, that is,

$$D - D' = \operatorname{div} f \qquad \text{for some } 0 \neq f \in k(X).$$

If X is a nonsingular variety, its *divisor class group* $\operatorname{Pic} X$ is the group of divisors modulo linear equivalence, $\operatorname{Pic} X = \operatorname{Div} X/\sim$.

Beware that there are two or three other equivalence relations on divisors (and many others on algebraic cycles) in common use. To get the idea of linear equivalence, suppose that D, D' are effective with no common components and $D - D' = \operatorname{div} f$. This means that f has zeros on D and poles on D'. I can view f as a rational map $f\colon X \dashrightarrow \mathbb{P}^1$, and $D = f^{-1}(0)$, $D' = f^{-1}(\infty)$ (the locus of indeterminacy of f is a subset of X of codimension ≥ 2). Thus $D_t = f^{-1}(t)$ is a divisor that moves with $t \in \mathbb{P}^1$ from D at $t = 0$ to D' at $t = \infty$.

Example. Let $X \subset \mathbb{P}^3$ be a nonsingular cubic surface and $L+L'+L''$ and $M+M'+M''$ two triangles cut out on X by the planes $A = 0$ and $B = 0$ where A and B are linear forms on $\mathbb{P}^3$. Then $A/B \in k(X)$ and $\operatorname{div}(A/B) = L+L'+L''-M-M'-M''$, so that any two triangles are linearly equivalent. Since all the defining relations for A were of this form, this implies that there is a well-defined map $\alpha\colon A(X) \to \operatorname{Pic} X$.

I will prove in the next two sections that α is injective and surjective.

1.7. Intersection numbers

Here are some basic facts about divisors on a projective nonsingular surface Y, which are discussed in more detail in Chapter A below. One can define intersection numbers of divisors D_1D_2 such that

(1) D_1D_2 is bilinear in each factor and symmetric;
(2) D_1D_2 only depends on D_1, D_2 up to linear equivalence: that is,

$$\mathcal{O}_Y(D_1) \cong \mathcal{O}_Y(D_1') \implies D_1D_2 = D_1'D_2;$$

(3) if $D_1, D_2 \geq 0$ and have no common components then $D_1D_2 = \sum_P (D_1D_2)_P$, where the sum runs over all $P \in D_1 \cap D_2$ and

$$(D_1D_2)_P = \dim_k \mathcal{O}_{Y,P}/(I_{D_1} + I_{D_2})\mathcal{O}_{Y,P} = \dim_k \mathcal{O}_{Y,P}/(f_1, f_2),$$

where D_1, D_2 are locally defined by f_1, f_2;
(4) if C is an irreducible curve then $CD = \deg_C \mathcal{O}_C(D)$.

The properties (1–3) uniquely define the intersection number as a bilinear pairing $\operatorname{Div} Y \times \operatorname{Div} Y \to \mathbb{Z}$ (see, for example, Hartshorne [H1], Chapter V). The point of (2) is that D_1D_2 is well defined on $\operatorname{Pic} Y$, not just on $\operatorname{Div} Y$. The bilinear pairing I gave on $A(S)$ satisfies (1–3) so is the intersection pairing on $\operatorname{Pic} S$ (under the map α defined at the end of 1.6).

Notice the useful fact that for irreducible curves C, C', the only way $CC' < 0$ is possible is for $C = C'$. The selfintersection C^2 can be interpreted as the degree of the normal bundle $N_{C/Y}$.

1.8. Conic bundles and the cubic surface

Let $L \subset S$ be a line and $\varphi = \varphi_L \colon S \dashrightarrow \mathbb{P}^1$ the projection away from L. I can realise this map as follows. Take $M = \mathbb{P}^1 \subset \mathbb{P}^3$ disjoint from L. If $P \in S \setminus L$ then P and L span a unique plane $\Pi = \mathbb{P}^2 \subset \mathbb{P}^3$. Say $\Pi \cap M = \pi \in \mathbb{P}^1$ and define $\varphi(P) = \pi$. φ is not *a priori* defined at points in L but in fact I have

Lemma. *$\varphi \colon S \setminus L \to \mathbb{P}^1$ extends to a morphism $\varphi \colon S \to \mathbb{P}^1$.*

I give two proofs.

First Proof. Let $L = (x = y = 0)$, so that S is defined by $Ax + By = 0$, where A and B are forms of degree 2 in $\{x, y, z, t\}$ which have no common zero on L (by nonsingularity). So the ratio $x : y = B : A$ is everywhere well defined. This ratio defines a morphism to $\mathbb{P}^1$ and is clearly just the map described above. Q.E.D.

Second Proof. This proof is by intersection numbers. Let $H = L + F$ be a hyperplane section of S through L. Moving H to a linearly equivalent section I can see that $HL = 1$ and $FL = FH = 2$ so $H^2 = 3$ and $F^2 = F(H - L) = 0$. Since the F are effective and distinct they must be disjoint so φ is well defined. Q.E.D.

By construction, the fibres of φ are the plane conics residual to L. By the Main Trick 1.1.3, five of these are line pairs, the remainder nonsingular conics. Also, $F^2 = 0$, as we saw in the second proof. In particular, if $F = L_1 + L_2$, then $L_1 L_2 = 1$, so that $L_i^2 = -1$, which agrees with the choice made in 1.4.2.

Remark. L is not a section of the bundle. Since $LF = 2$, it cuts a generic fibre twice.

1.9. Other birational models for S

The conic bundle structure described in 1.8 can be constructed starting from any line L lying on S. I have two disjoint lines L and M lying on S, provided by 1.1.4, so I can construct

$$\varphi = \varphi_L \times \varphi_M \colon S \to \mathbb{P}^1 \times \mathbb{P}^1.$$

Since φ_L is the conic bundle obtained as the linear projection away from L, it is clear from 1.1.3 that it takes the 5 line pairs meeting L to 5 distinct points of $\mathbb{P}^1$. Thus both φ_L and φ_M are constant on the 5 lines $L_1, \dots, L_5$ meeting both of them, so that $\varphi = \varphi_L \times \varphi_M$ contracts these 5 lines $L_1, \dots, L_5$ to points $Q_1, \dots, Q_5$ of $\mathbb{P}^1 \times \mathbb{P}^1$.

In fact S is $\mathbb{P}^1 \times \mathbb{P}^1$ blown up in five distinct points (see below for blowup).

If I identify $\mathbb{P}^1 \times \mathbb{P}^1$ with the quadric $Q \subset \mathbb{P}^3$, and compose the map $\varphi \colon S \to \mathbb{P}^1 \times \mathbb{P}^1$ with the linear projection $Q \dashrightarrow \mathbb{P}^2$ from Q_5 then I obtain a morphism $\psi \colon S \to \mathbb{P}^2$ that contracts $L_1, \dots, L_4, L_5', L_5''$ to 6 points of $\mathbb{P}^2$. See Ex. 1.7 for this map in coordinates.

1.10. Blowup

We're going to see that S is isomorphic to $\mathbb{P}^2$ blown up in 6 points "in general position", with the basis of $A(S)$ corresponding to $e_0 =$ a line in $\mathbb{P}^2$ (with $e_0^2 = 1$), and the 27 classes of lines given as:

e_i the inverse image of the blowup of P_i;
$e_0 - e_i - e_j$ the line through P_i, P_j;
$2e_0 - e_1 - \cdots - e_5$ the conic through $P_1, \ldots, P_5$.

For this I need to say what blowup means. (Compare also [Sh], Chapter II, §4.)

Let $P \in S$ be a point of a nonsingular surface. Then there exists a nonsingular surface and a morphism $\sigma\colon S_1 \to S$ such that

(i) σ is an isomorphism $S_1 \setminus \sigma^{-1}(P) \cong S \setminus P$;

(ii) $\sigma^{-1}(P) \cong \mathbb{P}^1$.

S_1 and σ are in fact uniquely determined by these properties.

Construction. Suppose first that $S = \mathbb{A}^2$ with coordinates x, y, and $P = (0,0)$. Define $S_1 \subset \mathbb{A}^2 \times \mathbb{P}^1$ to be the closed graph of the rational map $(x, y) \mapsto (x : y)$. If $(u : v)$ are homogeneous coordinates of $\mathbb{P}^1$ then S_1 is defined by $x/u = y/v$, that is $xv = yu$.

So if $(x, y) \neq (0,0)$, the ratio $(u : v)$ is well defined, which gives (i). Also $\sigma^{-1}(0,0)$ is clearly the whole of $\mathbb{P}^1$. Finally, to see that S_1 is nonsingular, note that since $(u : v)$ are homogeneous coordinates of $\mathbb{P}^1$, if $u \neq 0$ then I can assume $u = 1$, so that the open set $(u \neq 0) \subset S_1$ is the surface in $\mathbb{A}^3$ with coordinates (x, y, v) defined by $y = xv$.

More generally, if $P \in S$ is any nonsingular point of a surface, let x, y be local coordinates at P; by passing to a small enough open set, I can assume that S is affine, and $m_P = (x, y)$ (where m_P is the maximal ideal of functions vanishing at P in the coordinate ring of S). The same construction works. The curve C that comes out of blowing up a nonsingular point on a surface satisfies $C \cong \mathbb{P}^1$ and $C^2 = -1$. I call such a curve a -1-*curve*.

Castelnuovo's criterion. *Let Y be a nonsingular surface and C a curve in Y. Then there is a morphism $\varphi\colon Y \to X$ to a nonsingular surface X contracting C to a point and an isomorphism outside C if and only if C is a -1-curve.*

One proof of this is given in Contraction Theorem 4.15 below.

Remark. A blowup is also called a monoidal transformation, sigma process, etc.; blowups play a star role in birational geometry of surfaces and in resolution of singularities. For example, if $C \subset S = \mathbb{A}^2$ is a curve which has a singularity of multiplicity m at the origin, $\sigma\colon S_1 \to S$ the blowup, then $\sigma^{-1}(C) = L \cup C'$ where $C' \subset S_1$ is "less singular" than C.

Exercise. Show that $C : (x^2 = y^3) \subset \mathbb{A}^2$ is resolved by a single blowup, whereas $(x^2 = y^5)$ needs two.

1.11. The cubic surface as $\mathbb{P}^2$ blown up 6 times

Theorem. *Suppose $\Phi = \{P_1, \ldots, P_6\} \subset \mathbb{P}^2$ is a set of 6 points with the following "general position" properties: (i) no 2 points coincide; (ii) no 3 collinear; (iii) not all 6 on a conic. Let S be the blowup of $\mathbb{P}^2$ at the 6 points Φ.*

Then the vector space of cubic forms on $\mathbb{P}^2$ vanishing at Φ is 4-dimensional, and if $F_1, \ldots, F_4$ is a basis then the rational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^3$ defined by $P \mapsto$

$(F_1(P) : \cdots : F_4(P))$ induces an isomorphism of S with a nonsingular cubic surface $S \cong S_3 \subset \mathbb{P}^3$.

This construction is a 2-sided inverse to the map ψ described at the end of 1.9 (see also Ex. 1.7).

The fact that $S_3(P_1, \ldots, P_6)$ is 4-dimensional is proved in [UAG], §1. The theorem can be proved by similar arguments, but I omit it for lack of time.

1.12. Final remarks on cubic surface

All the topics below have been extensively studied:

1. Weyl group of E_6 and Galois theory of cubic surfaces. If S is defined over an algebraically nonclosed field K, the 27 lines will not in general be defined over K, but will usually require a field extension L/K. Then $\mathrm{Gal}(L/K)$ acts by permuting the 27 lines, so defines a homomorphism $\varphi\colon \mathrm{Gal}(L/K) \to W(E_6)$ to the symmetry of the configuration of lines, which by what I said in 1.5 is the Weyl group $W(E_6)$ of the root system in the lattice $h^\perp \subset A(X)$ (compare Ex. 1.4). For example if $K = \mathbb{R}$ and $L = \mathbb{C}$ then complex conjugation interchanges some of the lines; the lines of the real locus $S(\mathbb{R}) \subset S(\mathbb{C})$ are just the fixed lines.

The image of φ is a measure of how complicated the arithmetic properties of S are. The question of whether S is rational over the given field K (that is, does there exist birational map $S \dashrightarrow \mathbb{P}^2$ defined over K?) can be discussed in terms of $\operatorname{im}\varphi$.

2. Monodromy. If $\{S_t\}_{t \in T}$ is a family of cubic surfaces, say with S_t nonsingular for every $t \in T_0 \subset T$ then the 27 lines of S_t define a 27-to-1 cover $L \to T_0$. Moving round a closed loop in T_0 permutes the lines among themselves, which defines a homomorphism $\psi\colon \pi_1(T_0, t_0) \to W(E_6)$.

3. Singular cubic surfaces. If S is a cubic surface with an ordinary double point $P \in S$ as its only singularity, it turns out that it still has 15 lines not through P, and 6 lines through P (which should be "counted with multiplicity 2"). S can still be obtained from a linear system of cubics in $\mathbb{P}^2$ through 6 points, but these must be in "special position", for example 3 collinear: then the line joining these 3 is contracted to the singularity of S (see Ex. 1.12).

4. Cubic surfaces and del Pezzo surfaces in general. Cubic surfaces are the best known of the series of del Pezzo surfaces. A *del Pezzo surface* is a nonsingular projective surface S with $-K_S$ ample. In addition to the cubic S_3, examples include the complete intersection of two quadrics $S_4 = Q_1 \cap Q_2 \subset \mathbb{P}^4$, the double plane $S_2 \to \mathbb{P}^2$ with quartic branch curve, and $S_6 \subset \mathbb{P}^6$, the section of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^7$ by a general hyperplane $\mathbb{P}^6$. Just as the cubic is obtained by blowing up $\mathbb{P}^2$ in 6 general points (then making the anticanonical embedding), the other del Pezzo surfaces are obtained by blowing up $\mathbb{P}^2$ in k general points for some $k = 0, \ldots, 8$ (except $\mathbb{P}^1 \times \mathbb{P}^1$, which is also a del Pezzo).

Del Pezzo surfaces have been studied in many contexts of geometry, singularity theory and number theory (see for example [Manin], Manin and Tsafsman [M–Ts]). They have recently acquired extra significance: in Mori theory (of minimal models, compare C.3 and Chapter D), a principal role is played by *Mori fibre spaces* $X \to Y$, that is, morphisms whose fibres S are varieties with ample $-K_S$. Thus del Pezzo surfaces and 1-parameter families of del Pezzo surfaces appear in a fundamental way in the classification of surfaces and 3-folds.

Exercises to Chapter 1

1. Determine the entire configuration of lines in terms of the notation L, M, L_i, L'_i, L''_i, L_{klm}. Compare [UAG], §7.

2. A *double six* is two sextuples of disjoint lines $l_1, \dots, l_6$ and $m_1, \dots, m_6$ such that

$$l_i \cap m_j = \begin{cases} \text{one point} & \text{if } i \neq j, \\ \emptyset & \text{otherwise.} \end{cases}$$

Let $l_1 = L_1, \dots, l_4 = L_4, l_5 = L'_5, l_6 = L''_5$. Find $m_1, \dots, m_6$ forming a double six.

3. If $l_1, \dots, l_6, m_1, \dots, m_6$ form a double six, and have classes $l_i = e_i$ and $m_i = 2e_0 -$ (the five the other e_j), show how to index the remaining 15 lines so that n_{ij} meets l_i, l_j, m_i, m_j. When does n_{ij} meet n_{kl}?

4. Prove that the six classes $f_1 = L_1 - L_2, f_2 = L_2 - L_3, f_3 = L_3 - L_4, f_4 = L_4 - L'_5, f_5 = L'_5 - L''_5$ and $f_6 = L_4 - L_5$ base the orthogonal complement of H in $A(X)$, where H is the hyperplane section (that is, the class of a triangle), and that $f_i^2 = -2$ and $f_i f_j = 0$ or 1 for $i \neq j$. (If we define a graph with nodes $\{f_i\}$ and an edge joining f_i and f_j if and only if $f_i f_j = 1$ then we obtain the graph E_6, a Dynkin diagram which appears in many areas of math.)

5. This is an exercise on $\mathbb{P}^1 \times \mathbb{P}^1$. Show that $\operatorname{Pic}(\mathbb{P}^1 \times \mathbb{P}^1) = \mathbb{Z} \oplus \mathbb{Z}$. Writing an element of $\operatorname{Pic}(\mathbb{P}^1 \times \mathbb{P}^1)$ as (a, b), work out an intersection formula for two curves lying on $\mathbb{P}^1 \times \mathbb{P}^1$.

6. Suppose that $L : (x = y = 0)$, $M : (z = t = 0)$ and $L_5 : (y = t = 0)$ lie on a nonsingular cubic $X \subset \mathbb{P}^3$, and let L'_5, L''_5 be as in 1.2–3. Write $F = L_5 + L'_5 + L''_5$. Let $\mathcal{L}(F) \subset k(X)$ be the vector space of rational functions whose divisor of poles is at most F. Prove that $\{\frac{x}{y}, \frac{z}{t}, 1\}$ is a basis of $\mathcal{L}(F)$. Write out in coordinates the rational map $\alpha : X \dashrightarrow \mathbb{P}^2$ to the projective plane that this defines.

7. As in the preceding question, suppose that $L : (x = y = 0)$, $M : (z = t = 0)$ and $L_5 : (y = t = 0)$ lie on a nonsingular cubic $X \subset \mathbb{P}^3$. Prove that

$$\alpha : X \dashrightarrow \mathbb{P}^2, \quad \text{defined by } (x, y, z, t) \mapsto (xt : yz : yt)$$

extends to a morphism. [Hint: $(x : y)$ is well defined at any point of L and $(y : t)$ is well defined at any point of L_5.]

8. Find three lines M_1, M_2, M_3 such that

$$L_1 + L_2 + M_3 = L_2 + L_3 + M_1 = L_1 + L_3 + M_2 = 2H - L - M - M_5$$

in $A(X)$. Prove that

$$(L_1 + L_2 + M_3) \cap (L_2 + L_3 + M_1) \cap (L_1 + L_3 + M_2) = \emptyset,$$

and use this to give an alternative proof of Ex. 1.7.

9. Show that α in Ex. 1.7 contracts the six lines $L_1, \dots, L_4, L'_5, L''_5$ to points and is an isomorphism outside these.

10. Let L be a line in $\mathbb{P}^2$ and P_1, P_2, P_3 noncollinear points. Describe the map to projective space determined by the linear system $|2L - P_1 - P_2 - P_3|$ and its inverse in terms of blowing up points and contracting -1-curves. This map is called the *elementary quadratic transformation.*

11. Let S_m be a nonsingular surface of degree m in $\mathbb{P}^3$. Generalise the proof of 1.8 to show that if $L = \mathbb{P}^1$ lies on S_m then $L^2 = 2 - m$. Compare this to what you know on S_2 and S_3. The surface S_4 is an example of a K3 surface, and is studied in greater detail later.

12. Suppose that $\Phi = \{P_1, \dots, P_6\} \subset \mathbb{P}^2$ is a set 6 points which is in general position except that P_1, P_2, P_3 are collinear, on a line L. Prove that cubics through Φ define a birational map $\varphi\colon \mathbb{P}^2 \dashrightarrow S \subset \mathbb{P}^3$ to a singular cubic surface. It blows up the 6 points of Φ, then contracts the line L to a singular point $P \in S$, and is otherwise an isomorphism.

13. Let $S \subset \mathbb{P}^3$ be a cubic surface with an ordinary double point $P \in S$. Study the projection map $\pi\colon S \dashrightarrow \mathbb{P}^2$ from P, and show that it contracts 6 lines of S through P to a set of 6 points $\Phi \subset \mathbb{P}^2$ lying on a conic. Study the inverse map $\varphi\colon \mathbb{P}^2 \dashrightarrow S$ of π as in the preceding question.