Minimal models of canonical singularities and their cohomology

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List of examples

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Declaration

Except where otherwise stated, the results of this thesis are, to the best of my knowledge, original.

Summary

Chapter 1 briefly describes the results of the thesis, and reviews the background material used throughout the other chapters.

Chapter 2 gives various formulas for the rank of the local analytic divisor class group of an isolated canonical 3-fold hypersurface singularity, using methods from singularity theory and mixed Hodge structures.

Chapter 3 calculates the number of crepant divisors on a minimal model of a nondegenerate isolated canonical n-dimensional hypersurface singularity. The methods used come from toric geometry.

Chapter 4 is concerned with determining the Betti numbers $b_i(Y)$ of a minimal model Y of an isolated canonical 3-fold singularity (X, x). For $b_2(Y)$ and $b_4(Y)$ we obtain an answer which becomes effective together with the results from the previous chapters. We give a formula for the remaining interesting Betti number $b_3(Y)$ in terms of the 3rd Betti numbers of certain representatives of the crepant valuations of X, and conjecture that this formula has a birational invariant character.

Chapter 1

Introduction

1.1 Motivation and results

Canonical singularities, introduced and studied by Reid in [R1], arise as singularities of the canonical model of a complex projective variety of general type.

A normal variety X has canonical singularities if a multiple of the canonical divisor K_X is Cartier, and if, for any resolution $\varphi\colon Y \to X$, with exceptional divisors E_i, the rational numbers a_i defined by

$$K_Y = \varphi^*(K_X) + \sum a_i E_i$$

are nonnegative. If all the discrepancies a_i are positive, X is said to have terminal singularities. For results about canonical and terminal singularities, see also [R3].

Canonical surface singularities are either Du Val singularities or nonsingular points. In dimension 3 the situation is much harder, the class of canonical 3-fold singularities has yet to reach a complete classification.

Let X be an algebraic 3-fold with canonical singularities. In [R2] Reid proved that there exists a crepant projective morphism $\varphi\colon Y \to X$ from a variety Y with $\mathbb{Q}$-factorial terminal singularities. Recall that a singularity $p \in Y$ is algebraically $\mathbb{Q}$-factorial if, for any Weil divisor D, there exists an integer m such that mD can be defined by one equation near p; in other words, the local algebraic divisor class group $\mathrm{Cl}\,\mathcal{O}_{Y,p}$ is torsion.

When passing from the surface case to 3-folds, various new phenomena occur; one of these is that a $\mathbb{Q}$-factorial terminal minimal model Y of X is not unique in general. Different minimal models of a 3-fold X are closely related, and their study involves finding properties which are independent of choices, that is, finding objects on Y which depend only on X. By a result of Kollár [K1], [K2], the following are among the invariants of X: the collection of analytic singularities of Y, the intersection cohomology groups $IH^k(Y,\mathbb{C})$ together with their Hodge structures, the integer cohomology groups $H^k(Y,\mathbb{Z})$, the Picard group and the divisor class group of Y. This raises the problem of describing these objects.

The topology of X and of its partial resolutions have not been studied in detail since the 1980s [R1], [R2], except in some special cases. This thesis is based on these results, and aims to understand the particular case of an isolated singularity (X,x).

There are 2 local divisor class groups associated to X at its singular point: the algebraic class group $\operatorname{Cl}\mathcal{O}_{X,x}$ and the analytic class group $\operatorname{Cl}\mathcal{O}^{\mathrm{an}}_{X,x}$. We can think about branches of plane curves as an illustration and motivation for many aspects related to local divisor class groups of 3-folds. Namely, if $x\in X$ is an isolated cA_n singularity, i.e., X is given by an equation $f=0$ with

$$f = xy - g(z,t)$$

near its singular point, and g_i are the algebraic (resp. analytic) factors of g, then $\operatorname{Cl}\mathcal{O}_{X,x}$ (resp. $\operatorname{Cl}\mathcal{O}^{\mathrm{an}}_{X,x}$) is generated by $(x=g_i=0)$, with only one dependency relation (see Section 2.2). Even though the inclusion $\operatorname{Cl}\mathcal{O}_{X,x}\hookrightarrow \operatorname{Cl}\mathcal{O}^{\mathrm{an}}_{X,x}$ is not surjective in general, we can use the finite determinancy property of isolated hypersurface singularities, i.e., the fact that $x\in X$ is equivalent to the singularity defined by the kth jet of f for some k and, by a change of coordinates, assume that these groups are equal. For instance, the germ $(X,0)$ defined by $xy+zt+z^n+t^n$ is 2-determined, hence analytically equivalent to the germ given by $xy+zt$. The same is true, in fact, for an arbitrary isolated singularity (X,x), by a finite determinancy theorem of Hironaka [Hi2, Theorem 3.3].

Let Y' be a minimal model of X with (algebraically) $\mathbb{Q}$-factorial terminal singularities. Since Y' has only isolated singularities, its (middle perversity) intersection cohomology groups with $\mathbb{C}$-coefficients are given by

$$IH^k(Y') = \begin{cases} H^k(Y') & \text{for } k>3,\\ \operatorname{Im}(H^k(Y'\setminus \operatorname{Sing}Y') \to H^k(Y')) & \text{for } k=3,\\ H^k(Y'\setminus\operatorname{Sing}Y') & \text{for } k<3\end{cases}$$

(see e.g. [Ki, p. 48]). An important feature of the cohomology groups $IH^k(Y')$ is that, by the Decomposition Theorem (see e.g. [Ki, p. 108]), they are direct summands in $H^k(\widetilde{Y})$, for any resolution of singularities $\widetilde{Y}$ of X. The usual cohomology groups $H^k(Y')$ do not have this birational invariance property; moreover, there is a simple relation between $IH^k(Y')$ and $H^k(Y')$, so once the former is determined, one has enough information about the latter, as well.

The minimal model Y' is a $\mathbb{Q}$-homology manifold if and only if it is locally analytically $\mathbb{Q}$-factorial if and only if, for each $p \in Y'$, the local divisor class groups $\operatorname{Cl}\mathcal{O}_{Y',p}$ and $\operatorname{Cl}\mathcal{O}^{\mathrm{an}}_{Y',p}$ have the same rank. This is not the case in general, but we can slightly modify Y' to make it $\mathbb{Q}$-homology. This is done by taking an analytic $\mathbb{Q}$-factorialisation Y of Y', which exists by [R2], at its singular points. This is a small morphism and, therefore, it leaves the intersection cohomology groups unchanged (see [K1, Corollary 4.12] and [Ki, p. 112]). The variety Y is a $\mathbb{Q}$-homology manifold, hence its intersection cohomology (with rational coefficients) coincides with the usual cohomology. Thus

$$IH^k(Y',\mathbb{Q}) = H^k(Y,\mathbb{Q}).$$

In this thesis we aim to determine the cohomology groups $H^k(Y,\mathbb{Q})$ of a partial resolution of (X,x) with terminal analytically $\mathbb{Q}$-factorial singularities, or, equivalently, the intersection cohomology groups of a minimal model Y' of X with terminal algebraically $\mathbb{Q}$-factorial singularities. The main class of singularities we are interested in is the class of hypersurface singularities $0 \in X : (f=0) \subset \mathbb{C}^4$.

An answer to this question involves calculating:

- the rank $\rho(X)$ of the analytic divisor class group $\operatorname{Cl}\mathcal{O}^{\mathrm{an}}_{X,x}$ of $x \in X$;
- the number of crepant divisors $c(X)$ on a minimal model or resolution of X; this is an invariant of X, that is, it does not depend on choices.

Chapter 2 calculates $\rho(X)$ in the case of an isolated canonical hypersurface singularity $0 \in X : (f=0) \subset \mathbb{C}^4$. For quasihomogeneous singularities an answer is known [Fl]; beyond this class only a few other cases were known, but not a general answer.

In Section 2.1 we recall a result of Flenner [Fl] giving an identification between the local analytic divisor class group $\operatorname{Cl}\mathcal{O}^{\mathrm{an}}_{X,0}$ of X and $H^2(L,\mathbb{Z})$, where L is the link of the singularity $(X,0)$.

Sections 2.2 and 2.3 give interpretations in singularity theory terms of some known results about the class group $\operatorname{Cl}\mathcal{O}^{\mathrm{an}}_{X,0}$. The rest of the chapter is devoted to finding such results for more general singularities.

The isomorphism $\mathrm{Cl}\,\mathcal{O}^{\mathrm{an}}_{X,0} \cong H^2(L,\mathbb{Z})$ is a consequence of the assumption that the singularity $0 \in X$ is canonical. This assumption is used once more in Section 2.4 to relate $\rho(X)$ to other invariants of the singularity. Namely, if F denotes the Milnor fiber and $\Delta(t)$ the characteristic polynomial of the monodromy operator on the homology of F, then $\rho(X)$ coincides with the Hodge number $h_1^{2,2}(F)$; $\rho(X)$ is also equal to the power of $(t-1)$ dividing $\Delta(t)$ (Proposition 2.7). In particular, since $\Delta(t)$ can be calculated from an embedded resolution of X by a result of A'Campo, the same is possible for $\rho(X)$. This is the content of Proposition 2.18. Although embedded resolutions of 3-fold singularities are difficult to control, in simple cases this information is useful (cf. Example 2.19).

Section 2.5 calculates $\rho(X)$ for canonical singularities which are nondegenerate with respect to their Newton diagrams. We use a suitable set of monomials whose residue classes form a basis for the Milnor algebra $M(f)$, which we define to be a regular basis for the Newton filtration on the space Ω^4 of germs at $0 \in \mathbb{C}^4$ of holomorphic 4-forms (Definition 2.10). Making use of results about mixed Hodge structures on the vanishing cohomology from [Sa1], [Sa2], [S2] and [VK], it follows that, for nondegenerate singularities of Newton degree α, the divisor class number $\rho(X)$ can be calculated as

$$\rho(X) = \#\{m \in B : \alpha(m + \mathbf{1}) = 2\},$$

where B is a regular set of monomials whose residue classes give a basis for $M(f)$. This is the main result of Chapter 1; it is an approximation to an idea of Reid that there might exist a certain twisted pairing on $M(f)$, whose kernel has dimension $\rho(X)$. Moreover, the above formula is quite close to the geometry of X; it relates the space of divisor classes to certain differential forms. It would be very interesting to clarify this relation.

Chapter 3 is an application of results from [R1] and [R3] in the case of an isolated canonical hypersurface singularity $0 \in X : (f = 0) \subset \mathbb{C}^{n+1}$, assumed to be nondegenerate with respect to its Newton diagram. We detect the birationally defined crepant divisors and, in particular, derive a formula for their number, $c(X)$. The main application is to the 3-fold case, since, for 3-fold singularities, $c(X)$ is the second invariant we need to compute the 2nd and the 4th Betti numbers of a minimal model.

By a discrepancy calculation from [R3, Section 4.8], it turns out that every crepant valuation is represented by an exceptional divisor on an α-blowup $\varphi_\alpha \colon X(\alpha) \to X$, where α is a weighting satisfying the equation $\alpha(\mathbf{1}) = \alpha(f) + 1$; we call the weightings satisfying this equation crepant, and denote the set of crepant weightings by $W(f)$. This gives a relation between crepant valuations and exceptional prime divisors on weighted blowups with crepant weightings. In order to make this correspondence 1-to-1, we define in Section 3.5 the set $E(f)$ consisting of the irreducible components E of $E(\alpha)$, for some $\alpha \in W(f)$, satisfying $v_E(x_i) = \alpha_i$, where v_E is the valuation of E. After some auxiliary results, we obtain in Section 3.6 a characterisation of the set $E(f)$ (Theorem 3.11). This gives a list of representatives of the crepant valuations, which are certain hypersurfaces in weighted projective spaces. As a consequence, we obtain in Section 3.7 a formula for $c(X) = \#E(f)$ in terms of the Newton diagram $\Gamma(f)$ of f. Namely

$$c(X) = \sum_{\substack{\alpha \in W(f) \\ \dim \Gamma_\alpha = 1}} \operatorname{length} \Gamma_\alpha + \# \left\{ \alpha \in W(f) : \dim \Gamma_\alpha \geq 2 \right\},$$

where Γ_α denotes the face of $\Gamma(f)$ corresponding to α.

Chapter 4 is devoted to studying how the invariants of an isolated canonical singularity (X, x) change under a proper birational morphism. The basic tool for comparing the integer cohomology groups is a Mayer–Vietoris type sequence of that morphism (see (4.1)).

Fix a representative X of the germ (X, x) which is Stein and contractible, and let $\varphi\colon Y \to X$ be a minimal model, assumed throughout this chapter to have terminal analytically $\mathbb{Q}$-factorial singularities. Denote by $b_i(Y) = \dim H^i(Y, \mathbb{R})$ the Betti numbers of Y, and let E_i for $1 \leq i \leq c(X)$ be any projective nonsingular representatives of the crepant valuations of X. Under certain strong assumptions, we show in Section 4.3 that

$$b_3(Y) = \sum_{i=1}^{c(X)} b_3(E_i)$$

(Theorem 4.4). This is the "ideal case", on which we hope to model the general situation. Although the assumptions are too strong, there are examples satisfying them, so we can calculate b_3 for some cases (see Section 4.7).

Section 4.4 contains a collection of lemmas, leading to a simplification of the Mayer–Vietoris sequence (4.1). These results are also used in Section 4.5, which calculates the 2nd and 4th Betti numbers of a minimal model (Theorem 4.12). Together with the formulas from Chapter 2 and Chapter 3 about $\rho(X)$ and $c(X)$, we thus obtain explicit answers about $b_i(Y)$, $i \neq 3$, for a large class of isolated canonical singularities $0 \in X : (f = 0) \subset \mathbb{C}^4$.

The rest of this chapter is concerned with calculating the remaining Betti number of Y, namely b_3. After working out some examples in Sections 4.7 and 4.8, we prove in Section 4.9 an approximation of the "ideal case" from

Theorem 4.4 for general singularities. The approach is now geometrical, not topological, namely we study divisor class groups rather than cohomology groups. This is done in two steps, first for a morphism $\varphi_i \colon X_i \to X_{i-1}$ between two (globally) $\mathbb{Q}$-factorial partial resolutions of X contracting precisely one crepant divisor E_i (Proposition 4.20), and second for an analytic $\mathbb{Q}$-factorialisation of X_i at the set of its dissident points (Proposition 4.21). Combining the two, we obtain the relation (4.15) involving the 3rd Betti numbers of X_i, X_{i-1} and a certain transformation $\widetilde{E}_i$ of E_i. This leads to

$$b_3(Y) = \sum_{i=1}^{c(X)} b_3(\widetilde{E}_i),$$

which is the conclusion of Theorem 4.22.

We conjecture that $b_3(\widetilde{E}_i) = b_3(\widetilde{E}'_i)$, where $\widetilde{E}'_i$ is any nonsingular projective surface birational to E_i, in other words, the "ideal case" is, in fact, general. Since each $\widetilde{E}'_i$ is rational or ruled [R1, Corollary 2.14], this conjecture implies that

$$b_3(Y) = 2 \sum g(C_i),$$

where the summation is taken over the surfaces $\widetilde{E}'_i$ which are ruled over C_i. In particular, for nondegenerate hypersurface singularities, this would give a formula for $b_3(Y)$ in terms of the Newton diagram of the defining equation (Remark 4.26).

Finally, Remark 4.25 gives a partial reason for the conjecture.

1.2 Preliminaries

This section is a review of some fundamental concepts and results about the topology of isolated hypersurface singularities, mixed Hodge structures and singularities in higher dimensional algebraic geometry. All the results presented in this section are known and constitute the foundational material for the later chapters of this thesis.

The first three sections are devoted to the local study of singularities and most of the results are due to Milnor [M]. For these and many other aspects related to the topology of hypersurface singularities see also [Di] and [N]. The fourth section contains a construction of Khovanskiĭ, leading to desingularisations of hypersurface singularities satisfying a certain generality condition. Section five is devoted to canonical singularities, which are the central objects of this thesis. The main references are [R1], [R2] and [R3].

Finally, the results in section six, together with subsequent references, can be found in [AGVII].

1.2.1 The Milnor fibration

Let $0 \in X : (f = 0) \subset \mathbb{C}^{n+1}$ be an isolated hypersurface singularity, that is, a germ $(X, 0) \subset (\mathbb{C}^{n+1}, 0)$ defined by an analytic function $f \colon \mathbb{C}^{n+1} \to \mathbb{C}$ having an isolated critical point at the origin. If we denote by $\mathcal{O}_{n+1} = \mathbb{C}\{x_1, \dots, x_{n+1}\}$ the $\mathbb{C}$-algebra of analytic function germs, and by $J(f) = (\partial f / \partial x_i)_{1 \le i \le n+1}$ the Jacobian ideal of f, then the assumption that 0 is an isolated singularity is equivalent to the condition that the Milnor algebra of f, $M(f) = \mathcal{O}_{n+1}/J(f)$, is finitely dimensional (as $\mathbb{C}$-vector space). Its dimension, $\mu(f) = \dim_{\mathbb{C}} M(f)$, is called the *Milnor number*. Since isolated singularities are finitely determined, we can always take f to be a polynomial.

Let $B_\varepsilon = \{x \in \mathbb{C}^{n+1} : ||x|| \le \varepsilon\}$ be the ball of radius ε about the origin of $\mathbb{C}^{n+1}$, and $S_\varepsilon = \partial B_\varepsilon \cong S^{2n+1}$ its boundary. Choose $\varepsilon > 0$ so that, for every ε' with $0 < \varepsilon' \le \varepsilon$, the sphere $S_{\varepsilon'}$ intersects X transversely. The intersection $L := S_\varepsilon \cap X$ is called the *link* of f. It is a compact $(2n-1)$-dimensional, $(n-1)$-connected submanifold $L \subset S^{2n+1}$. Near the singular point, the hypersurface X has a cone structure; namely, if ε is small enough, there is a homeomorphism of pairs

$$(B_\varepsilon, B_\varepsilon \cap X) \simeq (\mathrm{Cone}(S_\varepsilon), \mathrm{Cone}(L)). \tag{1.1}$$

For ε sufficiently small, the map

$$\varphi \colon S_\varepsilon \setminus L \to S^1, \text{ given by } \varphi(x) = \frac{f(x)}{|f(x)|} \tag{1.2}$$

is a smooth locally trivial fibration [M, Theorem 4.8]. Also, if $D_\delta = \{t \in \mathbb{C} : |t| < \delta\}$ is a small disk of radius δ, and $D_\delta^* = D_\delta \setminus 0$ is the punctured disk, then, for any $0 < \delta \ll \varepsilon$ small enough, the map

$$B_\varepsilon \cap f^{-1}(D_\delta^*) \to D_\delta^*, \ x \mapsto f(x) \tag{1.3}$$

is a smooth locally trivial fibration [Lê].

The fibrations (1.2) and (1.3) restricted to ∂D_δ are isomorphic as C^∞ fiber bundles. They are both called the *Milnor fibration* of the hypersurface singularity. The fiber $F = f^{-1}(t)$ is called the *Milnor fiber*. It is a $2n$-dimensional manifold with boundary. When restricted to S_ε, the fibration

(1.3) extends to a trivial fibration over the whole disk; in particular ∂F is diffeomorphic to $S_\varepsilon \cap X$, which is L by definition.

The fiber F has the homotopy type of a bouquet of n-spheres $\bigvee S^n$. The number of spheres in this bouquet equals $\mu(f)$, the Milnor number of f. In particular, the reduced homology of F satisfies

$$\tilde{H}_k(F,\mathbb{Z}) = \begin{cases} \mathbb{Z}^\mu & \text{if } k = n, \\ 0 & \text{otherwise.} \end{cases} \tag{1.4}$$

1.2.2 Monodromy

To any C^∞ fiber bundle over the punctured disk D^*_δ, with fiber F, one can associate a gluing diffeomorphism $m_g \colon F \to F$, well defined up to isotopy, called the *geometric monodromy* of the fibration. In the case of the fiber bundle (1.3), we can assume that m_g is the identity on $\partial F = L$, because the fibration extends over the disk D_δ. At the homology level, m_g induces an automorphism

$$h = (m_g)_* \colon H_n(F,\mathbb{Z}) \to H_n(F,\mathbb{Z}),$$

called the *(homology) monodromy operator*, which is uniquely determined.

Let $\Delta(t) = \det(h - t \cdot \mathrm{id})$ be its characteristic polynomial and

$$H_n(F)_1 = \{v \in H_n(F) : \exists m \in \mathbb{N} \text{ s.t. } (h - \mathrm{id})^m v = 0\}$$

the generalised eigenspace of the monodromy belonging to the eigenvalue 1. Denote by h_1 the restriction $h|H_n(F)_1$.

A basic result is the following Monodromy Theorem:

Theorem 1.1 *The monodromy operator h satisfies the following properties:*

(1) *all the eigenvalues of h are roots of unity;*

(2) *the dimension of any Jordan block of h is at most $n+1$;*

(3) *the dimension of any Jordan block of h_1 is at most n.*

Part (1) is proved in [B] and parts (2) and (3) in [S1], using mixed Hodge structures, which is closely related to the study of the monodromy (see Section 1.2.6).

Suppose $\Delta(t) = \prod_{i=1}^{\mu(f)}(t-\zeta_i)$, where ζ_i are the eigenvalues of h. A useful way of encoding the information about the eigenvalues of h is

$$\mathrm{Div}(f) := \sum_{i=1}^{\mu(f)} (\zeta_i) \in \mathbb{Z}[S^1].$$

1.2.3 The homology of L

Let $(\,,): H_n(F,\mathbb{Z})\otimes H_n(F,\mathbb{Z}) \to \mathbb{Z}$ be the intersection form induced by the algebraic intersection of n-cycles. Since $H_k(F,L,\mathbb{Z}) \cong H^{2n-k}(F,\mathbb{Z})$ by Lefschetz duality (see [Sp, p. 298]) and $\tilde{H}^k(F,\mathbb{Z}) = 0$ unless $k = n$ by (1.4), the long exact sequence of the pair (F,L) gives:

$$0 \to H_n(L,\mathbb{Z}) \to H_n(F,\mathbb{Z}) \to H_n(F,L,\mathbb{Z}) \to H_{n-1}(L,\mathbb{Z}) \to 0.$$

This sequence, together with the identification

$$H_n(F,L,\mathbb{Z}) \cong H_n(F,\mathbb{Z})^*,$$

leads to the first useful description of the (co)homology of L :

$$H_n(L,\mathbb{Z}) \cong \ker(\,,) \subset H_n(F,\mathbb{Z}). \tag{1.5}$$

The kernel of the intersection form is not easy to describe. An answer to this problem can be formulated in terms of mixed Hodge structures (see Section 1.2.6).

The Wang exact sequence associated to the fibration (1.2) gives:

$$0 \to H_{n+1}(S_\varepsilon \setminus L) \to H_n(F) \xrightarrow{h-\mathrm{id}} H_n(F) \to H_n(S_\varepsilon \setminus L) \to 0.$$

By Alexander and Poincaré duality

$$\begin{aligned} H_k(S_\varepsilon \setminus L) &\cong H^{2n-k}(L,\mathbb{Z}) \\ &\cong H_{k-1}(L,\mathbb{Z}), \end{aligned}$$

for $n \geq 2$ this sequence becomes:

$$0 \to H_n(L) \to H_n(F) \xrightarrow{h-\mathrm{id}} H_n(F) \to H_{n-1}(L) \to 0.$$

This gives, in particular, the second useful description of the homology of L:

$$H_n(L,\mathbb{Z}) \cong \ker(h-\mathrm{id}) \subset H_n(F,\mathbb{Z}). \tag{1.6}$$

Similarly, for $n = 1$, the reduced cohomology of L satisfies

$$\widetilde{H}^0(L) \cong \ker(h - \mathrm{id}). \tag{1.7}$$

Denote by $b_n(F)_1$ the exponent of $(t-1)$ in $\Delta(t)$, the characteristic polynomial of h. Then, from (1.6), the interesting Betti numbers of L satisfy

$$b_{n-1}(L) = b_n(L) = \dim\ker(h - \mathrm{id}) \le \dim H_n(F)_1 = b_n(F)_1, \tag{1.8}$$

with equality if and only if the Jordan matrix of the monodromy operator h belonging to the eigenvalue 1 is diagonal. As it will turn out, canonical 3-fold singularities have this property (see Section 2.4).

1.2.4 Nondegenerate singularities

Recall standard toric terminology. Let M be the free Abelian group $\mathbb{Z}^{n+1}$, $N = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ its dual, $M_k = M \otimes_{\mathbb{Z}} k$ and $N_k = N \otimes_{\mathbb{Z}} k$ the vector spaces obtained by extending scalars to k, where $k = \mathbb{Q}$ or $\mathbb{R}$. By identifying $m \in M$ with $x^m = x_1^{m_1} \cdots x_{n+1}^{m_{n+1}}$, we can refer to M as the *lattice of monomials* and to N as the *lattice of weightings*. A weighting $\alpha = (\alpha_1, \dots, \alpha_{n+1})$ is *primitive* if the α_i have no common factor.

Write e_i for the standard basis of $N_{\mathbb{R}}$, $\sigma = \sum_{i=1}^{n+1} \mathbb{R}_{\ge 0}\, e_i \subset N_{\mathbb{R}}$ for the positive quadrant, and $\sigma^\vee = \sum_{i=1}^{n+1} \mathbb{R}_{\ge 0}\, e_i^*$ for the dual quadrant. It is convenient to write $m \in f$ if $f = \sum_{m \in M} a_m x^m$ with $a_m \ne 0$; define then, for $\alpha \in N \cap \sigma$, $\alpha(m) = \sum_{i=1}^{n+1} \alpha_i m_i$ and

$$\alpha(f) = \min\{\alpha(m) : m \in f\}.$$

Toric geometry (see e.g. [Fu]) provides a dictionary between the geometry of toric varieties and combinatorial data of polyhedral sets. Even when working with non-toric varieties, it can happen that a toric resolution of the ambient space leads to a desingularisation. This property defines a large class of hypersurface singularities.

Definition 1.2 *Let $X : (f = 0) \subset \mathbb{C}^{n+1}$ be a hypersurface.*

(*i*) *The* Newton polyhedron *of f, $\Gamma_+(f)$, is the convex hull in $M_{\mathbb{R}}$ of the set $\bigcup_{m \in f}(m + \sigma^\vee)$; the union of all its compact facets, $\Gamma(f)$, is called the* Newton diagram *of f.*

(*ii*) *For any face $\Gamma \prec \Gamma(f)$ denote $f_\Gamma = \sum_{m \in \Gamma \cap M} a_m x^m$; f is* nondegenerate *with respect to its Newton diagram if, for each face Γ of $\Gamma(f)$, the hypersurface defined by $f_\Gamma = 0$ is nonsingular on the open stratum $T_\Gamma = (\mathbb{C}^*)^{\dim(\Gamma)+1}$.*

Since the nondegenerate hypersurfaces form an open dense set in the class of all hypersurfaces with a given Newton diagram [Kh], this concept is not very restrictive. One of the main features of nondegenerate singularities is that, as in the case of toric varieties, it is often possible to express their invariants in a purely combinatorial way.

Let $X : (f = 0) \subset \mathbb{C}^{n+1}$ be a nondegenerate hypersurface singularity. Recall a construction of Khovanskiĭ [Kh], leading to a desingularisation of X.

The function $g\colon \sigma \to \mathbb{R}_+$ defined by

$$g(\alpha) = \min_{m \in \Gamma_+(f)} \alpha(m)$$

is called the *supporting function of the polyhedron* $\Gamma_+(f)$. For any $\alpha \in \sigma$, denote

$$\Gamma_\alpha = \{m \in \Gamma_+(f) : \alpha(m) = g(\alpha)\}.$$

Define an equivalence relation $\sim$ on σ by $\alpha \sim \beta$ iff $\Gamma_\alpha = \Gamma_\beta$. The closure of any equivalence class is a cone in σ. The collection of all cones obtained in this way turns out to be a fan $\Delta(f)$. We call it the *fan associated to* f.

The required refinement of $\Delta(f)$ is obtained in two steps:

(i) by subdividing $\Delta(f)$ (in an arbitrary way) until it becomes *simplicial*, i.e., all its cones are generated by linearly independent vectors;

(ii) by subdividing the simplicial fan further until it becomes *nonsingular*, i.e., every cone in it is generated by a part of a basis for the lattice N.

The procedure described above gives a subdivision of the positive quadrant σ, hence a proper birational morphism

$$\psi\colon \widetilde{\mathbb{C}}^{n+1} \to \mathbb{C}^{n+1}.$$

By [Kh], the proper transform $\tilde{X}$ of X with respect to ψ is nonsingular.

1.2.5 Canonical singularities

In higher dimensional algebraic geometry it is not possible in general to select a nonsingular representative with good global properties in a given birational class, so one has to allow certain singularities. Choosing the correct class is fundamental. These are the canonical and terminal singularities introduced and studied by Reid in [R1].

Definition 1.3 *A variety Z is said to have* canonical singularities *if it is normal and the following two conditions are satisfied:*

(i) the Weil divisor K_Z (which is well defined by the normality assumption) is $\mathbb{Q}$-Cartier;

(ii) for any resolution of singularities $\varphi\colon \tilde{Z} \to Z$, with exceptional divisors $E_i \subset \tilde{Z}$, the rational numbers a_i satisfying

$$K_{\tilde{Z}} = \varphi^*(K_Z) + \sum a_i E_i$$

are nonnegative. The numbers a_i are called the discrepancies *of φ at E_i; if they are all positive, Z is said to have* terminal singularities.

Let X be a variety with canonical singularities. A *partial resolution* of X is a proper birational morphism $\varphi\colon Z \to X$ from a normal variety; the morphism φ is said to be *crepant* if $K_Z = \varphi^*(K_X)$ [R2].

A *contraction* is a proper surjective morphism with connected fibers $\psi\colon Z' \to Z$ between normal irreducible varieties, with the property that $\psi_*\mathcal{O}_{Z'} = \mathcal{O}_Z$. The contraction ψ is said to be *crepant* if it is birational and satisfies $K_{Z'} = \psi^* K_Z$.

A normal variety X has *rational singularities* if $R^i\varphi_*\mathcal{O}_Y = 0$ for some resolution $\varphi\colon Y \to X$ and all $i > 0$. Canonical singularities are rational; moreover, for Gorenstein varieties, canonical is equivalent to rational (see [R3, (3.8)] for discussions and related references).

The following result, due to Reid, [R3] is a characterisation of canonical and terminal singularities, giving necessary conditions for a hypersurface singularity to be canonical or terminal, in terms of the position of $\mathbf{1} = (1, \dots, 1)$ with respect to the Newton diagram of f. These conditions are also sufficient provided that the singularity is nondegenerate.

Theorem 1.4 *Let $0 \in X : (f = 0) \subset \mathbb{C}^{n+1}$ be a hypersurface singularity. Then*

(1) *if $0 \in X$ is canonical, then $\alpha(\mathbf{1}) \geq \alpha(f) + 1$ for all $\alpha \in \sigma \cap N$ primitive, i.e., $\mathbf{1} \in \operatorname{Int} \Gamma_+(f)$;*

(2) *if X is nondegenerate, then the converse of* (1) *also holds.*

The same holds with canonical replaced by terminal and $\geq$ by $>$.

Let $\operatorname{WDiv} Z$ and $\operatorname{CDiv} Z$ denote the group of Weil divisors and Cartier divisors on a variety Z. If Z has at most rational singularities, then the Abelian group $\operatorname{WDiv} Z / \operatorname{CDiv} Z$ is finitely generated [Ka]; denote by $\sigma(Z)$ its rank. The same holds if we replace Z by a pair (Z, W) consisting of an analytic space and its compact subset; denote in this case

$$\sigma(Z, W) = \operatorname{rank} \varinjlim \operatorname{WDiv} U / \varinjlim \operatorname{CDiv} U,$$

where the limit is taken over all open neighbourhoods U of W. In particular, for a germ (Z, p) with $p \in Z$ a point, we denote by $\rho(Z)$ or $\rho(p)$ the number $\sigma(Z, p)$.

An algebraic (or analytic) 3-fold Z is *(globally analytically) $\mathbb{Q}$-factorial* if $\operatorname{WDiv} Z \subset (\operatorname{CDiv} Z)_{\mathbb{Q}}$. An analytic variety is said to be *locally analytically $\mathbb{Q}$-factorial* if the germ (Z, p) is $\mathbb{Q}$-factorial, for any point $p \in Z$. This is the same as saying that the local analytic divisor class group $\operatorname{Cl}_p Z = \operatorname{Cl} \mathcal{O}^{\mathrm{an}}_{Z,p}$ is torsion, or that the local ring $\mathcal{O}^{\mathrm{an}}_{Z,p}$ is almost factorial, for any $p \in Z$. As remarked in [Ka] and [KM, Example 2.17], these notions are different, namely

$$\begin{aligned} \text{algebraic } \mathbb{Q}\text{-factoriality} &\not\Rightarrow \text{global analytic } \mathbb{Q}\text{-factoriality} \\ &\not\Rightarrow \text{local analytic } \mathbb{Q}\text{-factoriality.} \end{aligned}$$

For example, the germ $(X, 0)$ given by $xy + zt + z^n + t^n$ is algebraically $\mathbb{Q}$-factorial but not analytically $\mathbb{Q}$-factorial, since it is 2-determined, hence analytically equivalent to the germ given by $xy + zt$. See Example 2.15 and the examples in Section 4.8 and Section 4.9 for further discussions and examples related to this.

Let X be a 3-fold with canonical singularities. The following theorem of Reid [R2] proves the existence of a minimal model with terminal singularities of X.

Theorem 1.5 *Let X be an algebraic (resp. analytic) 3-fold with canonical singularities. Then there exists a crepant projective (resp. locally projective) birational morphism $\psi\colon Y \to X$ from a 3-fold Y with at most terminal singularities.*

The existence of a $\mathbb{Q}$-factorial small partial resolution of a variety with terminal singularities was proved by Reid [R2]. This result was extended by Kawamata [Ka] to a variety with canonical singularities. The theorem is the following:

Theorem 1.6 *Let X be an algebraic (resp. analytic) 3-fold with canonical singularities. Then there exists a birational projective (resp. locally projective) morphism $\psi\colon Y \to X$, which is an isomorphism in codimension 1, from a 3-fold Y with $\mathbb{Q}$-factorial terminal singularities.*

Definition 1.7 *Let X be a 3-fold with canonical singularities. A* minimal model *of X is a 3-fold Y with analytically $\mathbb{Q}$-factorial terminal singularities, together with a crepant birational morphism $\varphi\colon Y \to X$.*

1.2.6 Mixed Hodge structures

The two filtrations

Let $0 \in X : (f = 0) \subset \mathbb{C}^{n+1}$ be an isolated singularity and F its Milnor fiber. The geometric monodromy m_g induces an action $T = (m_g^*)^{-1}$ on the cohomology group $H^n(F)$, called the *cohomological monodromy operator.*

A *mixed Hodge structure* on $H^n(F)$ consists of an increasing weight filtration $W_\bullet$ on $H^n(F, \mathbb{Q})$ and a decreasing Hodge filtration $F^\bullet$ on $H^n(F, \mathbb{C})$ such that the filtration induced by $F^\bullet$ on the lth graded quotient $\mathrm{Gr}_l^W = W_l/W_{l-1}$ of the weight filtration gives a pure Hodge structure of weight l, i.e.,

$$\mathrm{Gr}_l^W H^n(F) = F^k \mathrm{Gr}_l^W H^n(F) \oplus \overline{F^{l-k+1} \mathrm{Gr}_l^W H^n(F)},$$

for all l and all $0 \leq k \leq l+1$.

A mixed Hodge structure on the vanishing cohomology $H^n(F)$ was constructed by Steenbrink [S1], using the resolution of the singularity and alternatively by Varchenko [V2], using the asymptotic expansion of integrals over vanishing cycles.

If we decompose T into the product $T_s T_u$ of its semisimple and unipotent parts, then T_s preserves the filtrations $F^\bullet$ and $W_\bullet$, hence T_s acts on $\mathrm{Gr}_F^p = F^p/F^{p+1}$ and $\mathrm{Gr}_{p+q}^W \mathrm{Gr}_F^p H^n(F, \mathbb{C})$. For any eigenvalue λ of T, we write $(\mathrm{Gr}_F^p)_\lambda$ and $H_\lambda^{p,q}$ for the corresponding eigenspaces, and $h_\lambda^{p,q} = \dim_{\mathbb{C}} H_\lambda^{p,q}$ for the *Hodge numbers.*

For any $r \in \mathbb{Q}$, let $\lambda = \exp(-2\pi i r)$ and define the integers n_r in terms of the Hodge numbers $h_\lambda^{p,q}$ corresponding to the eigenvalue λ by

$$n_r = \sum_q h_\lambda^{n+1+[-r],q}.$$

One way of encoding the relation between the semisimple part of the monodromy and the Hodge filtration is the invariant

$$\mathrm{Sp}(f) = \sum n_r(r),$$

defined as an element of the free Abelian group on $\mathbb{Q}$. This is essentially a more precise version of the invariant $\mathrm{Div}(f)$ defined in Section 1.2.2; the relation between them is given by taking exponentials $r \mapsto \lambda = \exp(-2\pi i r)$. In the literature $\mathrm{Sp}(f)$ is referred to either as the *exponents* of the singularity by M. Saito (see [Sa1], [Sa2]), or, if shifted by 1 to the left, as the *spectrum* the singularity. We call $\mathrm{Sp}(f)$ the spectrum of the singularity, the rational numbers r the spectral numbers, and the integers n_r their multiplicities.

Symmetries and range

(1) Let $N = \log T_u$ be the logarithm of the unipotent part of the monodromy operator. The Hodge numbers enjoy the following symmetries arising from complex conjugation and the isomorphisms given by the powers of N ([S1]):

$$\begin{aligned} h^{p,q}_\lambda &= h^{q,p}_{\overline{\lambda}};\\ h^{p,q}_1 &= h^{n+1-q,n+1-p}_1;\\ h^{p,q}_\lambda &= h^{n-q,n-p}_\lambda \text{ if } \lambda \neq 1. \end{aligned} \tag{1.9}$$

(2) In particular, the spectrum $\mathrm{Sp}(f)$, regarded as a subset of $\mathbb{R}$, is symmetric under reflection in $(n+1)/2$, i.e.,

$$n_r = n_{n+1-r}.$$

(3) The spectrum $\mathrm{Sp}(f)$, regarded as a subset of $\mathbb{R}$, is contained in the interval $(0, n+1)$,

$$\mathrm{Sp}(f) \subset (0, n+1).$$

(4) If f defines a canonical singularity, Saito proved in [Sa1] that

$$\mathrm{Sp}(f) \subset (1, n). \tag{1.10}$$

Intersection form

Let $(\,,)$ be the intersection form on the nth homology group $H_n(F, \mathbb{R})$, and $\mu_0 = \dim\ker(\,,)$. This invariant can be expressed in terms of the Hodge numbers of f ([S1, Theorem 4.11]) as

$$\mu_0 = \sum_{p+q\leq n+1} h^{p,q}_1 - \sum_{p+q\geq n+3} h^{p,q}_1. \tag{1.11}$$

1.2.7 Thom–Sebastiani type results

These are results relating the invariants of a direct sum (join) of singularities to the corresponding invariants of the components.

Let $g \oplus h$ be the *direct sum* of the isolated singularities $g\colon (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ and $h\colon (\mathbb{C}^{m+1}, 0) \to (\mathbb{C}, 0)$, i.e., $g \oplus h$ is defined to be the singularity of $g(y) + h(z)\colon (\mathbb{C}^{m+n+2}, 0) \to (\mathbb{C}, 0)$. Then $g \oplus h$ is isolated and satisfies:

(1) $M(g \oplus h) = M(g) \otimes M(h)$; in particular $\mu(g \oplus h) = \mu(g)\mu(h)$;

(2) $H_{n+m+1}(F_{g\oplus h}, \mathbb{Z}) = \tilde{H}_n(F_g, \mathbb{Z}) \otimes \tilde{H}_m(F_h, \mathbb{Z})$;

(3) $h_{g\oplus h} = h_g \otimes h_h$;

(4) $\mathrm{Div}(g \oplus h) = \mathrm{Div}(g)\,\mathrm{Div}(h)$;

(5) $\mathrm{Sp}(g \oplus h) = \mathrm{Sp}(g)\ \mathrm{Sp}(h)$.

Parts (1), (2) and (3) are proved in [ST], Part (4) in [MO]. Part (5) was conjectured by Steenbrink [S1] and proved for isolated singularities by Varchenko [V2].

Chapter 2

The divisor class group

2.1 The identification with $H^2(L, \mathbb{Z})$

Let $0 \in X : (f = 0) \subset \mathbb{C}^{n+1}$ be an isolated hypersurface singularity, and denote by $\mathrm{Cl}\, X$ its local analytic divisor class group (see Section 1.2.5). This is a very interesting invariant of the singularity, which is related to the geometry and topology of X and of its partial resolutions. The key towards computing $\mathrm{Cl}\, X$ is the following result of Flenner.

Proposition 2.1 ([Fl]) *If $0 \in X$ is as above, then*

(1) *the first Chern class map*

$$c_1 \colon \mathrm{Pic}(X \setminus 0) \to H^2(X \setminus 0, \mathbb{Z})$$

is an injection;

(2) *if, moreover, $0 \in X$ is canonical, then c_1 is an isomorphism.*

Thus, via the retraction $(X \setminus 0) \searrow L$ coming from the cone structure of analytic sets (1.1), one has, in the canonical case, an isomorphism

$$\mathrm{Cl}\, X \cong H^2(L, \mathbb{Z}).$$

If $n \geq 4$, it follows from the above identification that X is (locally analytically) factorial, since L is 2-connected in this case. This is just a particular case of a result of Grothendieck [Gr1, Exp. XI]. In the surface case, canonical singularities are characterised by having a finite local class group, and these groups are well known (see e.g. [Du] for a description of the invariants of Du Val singularities).

Therefore, the interesting case is $n = 3$. If $(X, 0)$ is a 3-fold singularity, then $\mathrm{Cl}\, X$ is a finitely generated free Abelian group; set $\rho(X) := \operatorname{rank} \mathrm{Cl}\, X$. For certain results about the divisor class group of a canonical 3-fold singularity and connections between $\mathrm{Cl}\, X$ and the geometry of X see [K2, Sections 2.2 and 6.1].

The aim of this chapter is to study the divisor class group $\mathrm{Cl}\, X$ in the 3-dimensional case. The divisor class number $\rho(X)$ has been calculated for several classes of singularities:

- singularities defined by $f = xy - g(z, t)$;
- Brieskorn–Pham singularities [BS], [Sto];
- arbitrary quasihomogeneous singularities [Fl].

In Sections 2.2 and 2.3 we show how to derive formulas for $\rho(X)$ directly from Proposition 2.1, if X is one of the singularities above. The rest of this chapter is devoted to obtaining such answers in more general circumstances.

2.2 Singularities of the form $xy = g(z, t)$

Suppose $f = xy - g(z, t)$. In this case F_f is a double suspension of F_g and, since the monodromy of xy is trivial, by (1.6), (1.7) and Section 1.2.7, we can write

$$\begin{aligned} H_3(L_f) &= \ker(h_f - \mathrm{id}) \\ &= \ker(h_g \otimes h_{xy} - \mathrm{id}) \\ &= \ker(h_g - \mathrm{id}) \\ &= \widetilde{H}^0(L_g). \end{aligned}$$

If r denotes the number of branches of g, we have $L_g = \bigsqcup_r S^1$ and, since f defines a canonical singularity,

$$\begin{aligned} \rho(X) &= \operatorname{rank} \widetilde{H}^0(L_g) \\ &= r - 1. \end{aligned}$$

In fact, this class of singularities is very special since, in this case, it is possible to write explicitly generators of $\mathrm{Cl}\, X$ in terms of a factorisation. Namely, if g_i are the irreducible components of g and $D_i = (x = g_i = 0)$ for $1 \le i \le r$, then it is known that

$$\mathrm{Cl}\, X = \frac{\bigoplus \mathbb{Z}[D_i]}{(\sum D_i)}$$

(see [K2, Proposition 2.2.6]).

2.3 Quasihomogeneous singularities

Definition 2.2 *A hypersurface $X : (f = 0) \subset \mathbb{C}^{n+1}$ is* quasihomogeneous *if there exists $\alpha \in N_{\mathbb{Q}}$ such that $\alpha(m) = 1$ for every $m \in f$.*

We write $\alpha = (\alpha_1, \dots, \alpha_{n+1}) \in N_{\mathbb{Q}}$ and $\alpha_i = u_i/v_i$ with $\operatorname{hcf}(u_i, v_i) = 1$. In the three-dimensional canonical case, formulas for the divisor class number of singularities of Brieskorn–Pham type were obtained in [BS] and for quasihomogeneous singularities in [Fl]. This section contains a way of deriving such formulas from Proposition 2.1, via a result of Milnor and Orlik [MO, Theorem 4].

With the notation

$$\Lambda_m := \sum_{k=1}^{m} (e^{2\pi i k/m}) \in \mathbb{Z}[S^1] \quad \text{for} \quad m \geq 1,$$

the characteristic polynomial of a quasihomogeneous singularity is given, in terms of its weights, by the following result.

Theorem 2.3 ([MO]) *If f defines an α-quasihomogeneous isolated singularity, then $\Delta(t)$ is determined by:*

$$\operatorname{Div}(f) = \prod_{i=1}^{n+1} (u_i^{-1} \Lambda_{v_i} - (1)).$$

Example 2.4 Brieskorn–Pham singularities

This example is a well known illustration of the calculations involved in the proof of Theorem 2.3 (see e.g. [N, p. 23]). The reason for presenting it is that, in the case of a 3-fold canonical singularity X, these calculations lead to a formula for $\rho(X)$.

Let $f = \sum_{i=1}^{n+1} x_i^{a_i}$. For any i, we have $\operatorname{Div}(x_i^{a_i}) = \Lambda_{a_i} - (1)$. This can be seen easily because the Milnor fiber $F_i : (x_i^{a_i} = \delta_i)$ consists of a_i points and the geometric monodromy $m_g \colon F_i \to F_i$ is given by the cyclic permutation of these points. In particular, if $h_i \colon \tilde{H}_0(F_i) \to \tilde{H}_0(F_i)$ is the monodromy operator, then

$$\Delta_i(t) := \det(h_i - \mathrm{id}) = \frac{t^{a_i} - 1}{t - 1}.$$

From the Thom–Sebastiani formula for $\mathrm{Div}(f)$ (see Section 1.2.7) it follows that

$$\mathrm{Div}(f) = \prod_{i=1}^{n+1} (\Lambda_{a_i} - (1))$$

or, equivalently, if $a_I := \prod_{i \in I} a_i$ and $[a_I] := \mathrm{lcm}(a_i,\, i \in I)$

$$\Delta(t) = \prod_{I \subset \{1,\dots,n+1\}} (t^{[a_I]} - 1)^{(-1)^{|I|} a_I / [a_I]},$$

where, by convention, $a_\Phi = [a_\Phi] = 1$. In particular, if $n = 3$ and we assume that $0 \in X : (f = 0)$ is canonical, that is, if we assume $\sum_{i=1}^{4} 1/a_i > 1$, then

$$\rho(X) = \sum_{I \subset \{1,2,3,4\}} \frac{(-1)^{|I|} \cdot a_I}{[a_I]},$$

as in [Sto].

There is a larger class of singularities which behave topologically exactly like the quasihomogeneous ones.

Definition 2.5 *A polynomial f is called* α-semiquasihomogeneous*, for some $\alpha \in N_{\mathbb{Q}}$, if its α-tangent cone defines an isolated singularity.*

It is known that the monodromy operator of a quasihomogeneous singularity has finite order. In particular, one has equality in (1.8). Thus, a direct consequence of Proposition 2.1 and Theorem 2.3 is the following formula for the rank of the divisor class group in the three-dimensional canonical case.

Proposition 2.6 *Let $0 \in X \subset \mathbb{C}^4$ be a canonical α-semiquasihomogeneous singularity. Then the Picard number of X is given by*

$$\rho(X) = \sum_{I \subset \{1,2,3,4\}} \frac{(-1)^{|I|}}{\alpha_I \cdot [v_I]}, \tag{2.1}$$

where the summation is taken over all subsets $I \subset \{1,2,3,4\}$, and $\alpha_I = \prod_{i \in I} \alpha_i$, $[v_I] := \mathrm{lcm}(v_i,\, i \in I)$.

2.4 Canonical singularities

The assumption that a given singularity $0 \in X : (f = 0) \subset \mathbb{C}^4$ is canonical gives certain restrictions on the invariants of X; one of these comes from Saito's result (1.10)

$$\mathrm{Sp}(f) \subset (1, 3),$$

which implies that the only spectral number associated to the eigenvalue 1 is 2. In particular, this gives useful information about the divisor class number of X which, in this case, coincides with the rank of the kernel of the intersection form μ_0.

Proposition 2.7 *Let $0 \in X : (f = 0) \subset \mathbb{C}^4$ be an isolated canonical singularity and F its Milnor fiber. Then*

(1) *The only nontrivial Hodge number of F associated to the eigenvalue* 1 *is $h_1^{2,2}$; in particular*

$$\rho(X) = h_1^{2,2}(F).$$

(2) *The Jordan matrix of the monodromy operator associated to the eigenvalue* 1 *is diagonal; in particular*

$$\rho(X) = b_3(F)_1.$$

Proof We have $n_r = \sum_q h_1^{4-r,q}$. Since $n_r = 0$ unless $r = 2$, this implies that $h_1^{p,q} = 0$ unless $p = 2$ and, by the symmetry of the Hodge numbers (1.9), $h_1^{p,q} = 0$ unless $(p, q) = (2, 2)$. In particular $n_2 = h_1^{2,2}$.

From Steenbrink's formula (1.11), it follows that

$$\mu_0 = n_2 = h_1^{2,2},$$

which proves (1), using again the fact that the singularity is canonical.

On $H^3(F, \mathbb{C})_1$, $W_\bullet$ is the weight filtration of N with central index 4. It follows from (1) that $\mathrm{Gr}_l^W H^3(F, \mathbb{C})_1 = 0$ unless $l = 4$, which shows that $W_\bullet$ is trivial on $H^3(F, \mathbb{C})_1$. In particular, (1.8) implies that

$$\rho(X) = b_3(F)_1.$$

$\square$

2.5 Nondegenerate singularities

It is possible to calculate explicitly the spectrum for large classes of isolated singularities. In the semiquasihomogeneous case there is the following answer, due to Steenbrink.

Theorem 2.8 ([S2]) *Suppose that $0 \in X : (f = 0) \subset \mathbb{C}^{n+1}$ is an isolated α-semiquasihomogeneous singularity, and let B be a set of monomials whose residue classes form a basis of the Milnor algebra of f. Then*

$$\mathrm{Sp}(f) = \sum_{m \in B} \alpha(m + \mathbf{1}). \tag{2.2}$$

A similar description holds for nondegenerate singularities, for a suitable choice of a basis for the Milnor algebra $M(f)$.

To see this, suppose that f is nondegenerate and *convenient*. The last assumption means that the Newton diagram of f intersects every coordinate axis; this is not restrictive since f is finitely determined and thus, if we change f by adding monomials of the form x_i^k for sufficiently large k, we obtain an equivalent singularity.

Let Ω^p denote the space of germs at $0 \in \mathbb{C}^{n+1}$ of holomorphic p-forms. In [VK], a filtration $\mathcal{N}^\bullet$ on Ω^{n+1} is introduced as follows.

Let $\Gamma_i \prec \Gamma(f)$ be the facets (top dimensional faces) of the Newton diagram and suppose that each Γ_i is defined by a weighting $\alpha^{(i)} \in N_{\mathbb{Q}}$. The *Newton degree* of f, denoted by α, is defined on monomials $m \in M$ by $\alpha(m) = \min_i \alpha^{(i)}(m)$, and on power series $g \in \mathcal{O}_{n+1}$ by $\alpha(g) = \min_{m \in g} \alpha(m)$. This definition can be extended to forms $\omega = g(\mathbf{x})\mathrm{d}\mathbf{x} \in \Omega^{n+1}$ by $\alpha(\omega) = \alpha(g(\mathbf{x})\mathbf{x})$. The *Newton filtration* $\mathcal{N}^\bullet$ is then the decreasing filtration on Ω^{n+1} associated to α, i.e.,

$$\mathcal{N}^r = \left\{\omega \in \Omega^{n+1} : \alpha(\omega) \geq r\right\}.$$

Let $\overline{\mathcal{N}}^\bullet$ and $\overline{\alpha}$ denote the induced filtration and degree on the quotient space $\Omega_f := \Omega^{n+1}/\mathrm{d}f \wedge \Omega^n$. Denote also by $\mathcal{N}^{>r}$ the set consisting of those forms ω of degree strictly bigger than r; same notation for $\overline{\mathcal{N}}^{>r}$.

Let $\widetilde{\mathrm{Sp}}(f)$ be the spectrum of the Newton filtration $\overline{\mathcal{N}}^\bullet$ on Ω_f, i.e.,

$$\widetilde{\mathrm{Sp}}(f) = \sum n_r(r), \text{ where } n_r = \dim \overline{\mathcal{N}}^r / \overline{\mathcal{N}}^{>r}.$$

M. Saito proved in [Sa1] a conjecture of Steenbrink [S1] relating the spectrum of a singularity $\mathrm{Sp}(f)$ to $\widetilde{\mathrm{Sp}}(f)$. In [VK] there is an alternative proof of this. The result is the following:

Theorem 2.9 ([Sa1],[VK]) *Under the above assumptions,*

$$\mathrm{Sp}(f) = \widetilde{\mathrm{Sp}}(f). \tag{2.3}$$

[AGVI, 12.7] defines and constructs a *regular basis* for the Milnor algebra of f. We make an analogous definition, except that we work in

$$\Omega_f = \Omega^{n+1}/J_f\Omega^{n+1} = M(f) \otimes_{\mathcal{O}_{n+1}} \Omega^{n+1}.$$

Definition 2.10 *A set of elements $B \subset \mathcal{O}_{n+1}$, whose residue classes form a basis for the Milnor algebra $M(f)$, is* regular *for the filtration $\mathcal{N}^\bullet$ on Ω^{n+1} if, for any r, the elements of the set*

$$\{\omega \in \Omega^{n+1} : \omega = g(\mathbf{x})\mathrm{d}\mathbf{x}, g \in B, \alpha(\omega) = r\}$$

are independent modulo $(\mathrm{d}f \wedge \Omega^n) + \mathcal{N}^{>r}$.

A proof identical to that in [AGVI, 12.7] shows that there exists a monomial basis for $M(f)$, that is regular for the filtration $\mathcal{N}^\bullet$ on Ω^{n+1}.

Lemma 2.11 *Let f and B be as above, $m \in B$ a regular element, and $\omega = \mathbf{x}^m\mathrm{d}\mathbf{x}$. Then*

$$\alpha(\omega) = \bar{\alpha}(\overline{\omega}).$$

Proof For any form ω, we have

$$\begin{aligned}\bar{\alpha}(\overline{\omega}) &= \max_{\eta\in \mathrm{d}f\wedge\Omega^n} \alpha(\omega+\eta) \\ &\geq \max_{\eta\in \mathrm{d}f\wedge\Omega^n} \min\{\alpha(\omega), \alpha(\eta)\} \\ &= \alpha(\omega),\end{aligned}$$

since it is possible to find elements $\eta \in \mathrm{d}f \wedge \Omega^n$ of arbitrary large degree (in fact, all $(n+1)$-forms of sufficiently large degree belong to $\mathrm{d}f \wedge \Omega^n$, by the finite-determinancy property).

If we had strict inequality above, $\overline{\alpha}(\overline{\omega}) > \alpha(\omega)$, then there would exist $\eta \in \mathrm{d}f\wedge\Omega^n$ such that $\alpha(\omega+\eta) > \alpha(\omega)$. But this means that ω belongs to the ideal $(\mathrm{d}f \wedge \Omega^n) + \mathcal{N}^{>\alpha(\omega)}$, which is impossible by the regularity assumption. $\square$

Suppose B is a regular set. By definition, this means that the elements $\overline{\omega}_m$ with $m \in B$, of degree $\overline{\alpha}(\overline{\omega}_m) = r$, form a basis of the quotient vector space $\overline{\mathcal{N}}^r / \overline{\mathcal{N}}^{>r}$. Thus, from the above lemma, one can write

$$\widetilde{\mathrm{Sp}}(f) = \sum_{m \in B} \alpha(m + \mathbf{1}),$$

for B a regular set, which gives an alternative way of writing formula (2.3), similar to Steenbrink's formula (2.2).

The divisor class number of a quasihomogeneous singularity was calculated by Flenner [Fl, Theorem 7.5].

Theorem 2.12 ([Fl]) *Let $0 \in X : (f = 0) \subset \mathbb{C}^4$ be an isolated semi-quasihomogeneous singularity of type α, and B any set of monomials whose residue classes give a basis for $M(f)$. Then*

$$\rho(X) \leq \#\{m \in B : \alpha(m + \mathbf{1}) = 2\},$$

with equality if $0 \in X$ is canonical.

Proof It follows from Proposition 2.1, Proposition 2.7 and Theorem 2.8. □

This theorem can be extended to nondegenerate singularities to the following:

Theorem 2.13 *Let $0 \in X : (f = 0) \subset \mathbb{C}^4$ be an isolated nondegenerate singularity of Newton degree α, and B a regular set of monomials whose residue classes give a basis for $M(f)$. Then*

$$\rho(X) \leq \#\{m \in B : \alpha(m + \mathbf{1}) = 2\}, \tag{2.4}$$

with equality holding if $0 \in X$ is canonical.

Proof Apply Proposition 2.1, Proposition 2.7 and Theorem 2.9. □

The following two examples are illustrations of the result above.

Example 2.14 Let $g(x, y, z) = x^p + y^q + z^r + xyz$ with $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$ be a singularity of type $T_{p,q,r}$, and let

$$f(x, y, z, t) = g(x, y, z) + t^n \quad \text{with} \quad n \geq 2.$$

Then f defines a isolated canonical nondegenerate singularity. Let

$$\Gamma_1 = [(1,1,1),(0,q,0),(0,0,r)],$$
$$\Gamma_2 = [(p,0,0),(1,1,1),(0,0,r)],$$
$$\Gamma_3 = [(p,0,0),(0,q,0),(1,1,1)],$$

be the facets of $\Gamma(g)$, where, for a set $A \subset M_{\mathbb{R}}$, we denote by $[A]$ the convex hull of A; also let

$$\alpha^{(1)} = \left(1 - \frac{q+r}{qr}, \frac{1}{q}, \frac{1}{r}\right),$$
$$\alpha^{(2)} = \left(\frac{1}{p}, 1 - \frac{p+r}{pr}, \frac{1}{r}\right),$$
$$\alpha^{(3)} = \left(\frac{1}{p}, \frac{1}{q}, 1 - \frac{p+q}{pq}\right)$$

be the quasihomogeneity types of g. The set

$$B = \left\{1, xyz, x^i, y^j, z^k;\ 1 \le i \le p-1,\ 1 \le j \le q-1,\ 1 \le k \le r-1\right\}$$

is regular. In order to see this, let

$$\left\{\omega \in \Omega^4 : \omega = \mathbf{x}^m \mathrm{d}\mathbf{x},\ m \in B,\ \alpha(m+\mathbf{1}) = s\right\}$$

be a set of forms of a given degree s. The element xyz is clearly regular. For $s \neq 1$ this set can contain at most one element ω of the form $x^i\mathrm{d}\mathbf{x}$, since different elements of this form have different degrees; the same is true for y^j and z^k. Suppose we have a relation

$$c_1 x^i \mathrm{d}\mathbf{x} + c_2 y^j \mathrm{d}\mathbf{x} + c_3 z^k \mathrm{d}\mathbf{x} = \eta + \omega'$$

with $\eta \in \mathrm{d}f \wedge \Omega^3$ and $\omega' \in \mathcal{N}^{>s}$. Then, since $J_g = (px^{p-1} + yz, qy^{q-1} + xz, rz^{r-1} + xy)$, the relation above with, say, $c_1 \neq 0$ implies that $i = p-1$ and $\omega' = yz\mathrm{d}\mathbf{x} +$ (other terms); therefore

$$\alpha(\omega') \le \alpha(yz\mathrm{d}\mathbf{x}) = 1 + \frac{1}{q} + \frac{1}{r}.$$

On the other hand

$$s = \alpha(x^{p-1}\mathrm{d}\mathbf{x}) = \frac{p-1}{p} + 1 > 1 + \frac{1}{q} + \frac{1}{r},$$

and this gives a contradiction. It follows that

$$\mathrm{Sp}(g) = (1) + (2) + \sum_{i=1}^{p-1} \left(\frac{i}{p}\right) + \sum_{j=1}^{q-1} \left(\frac{j}{q}\right) + \sum_{k=1}^{r-1} \left(\frac{k}{r}\right).$$

Since $\mathrm{Sp}(t^n) = \sum_{l=1}^{n-1} \left(\frac{l}{n}\right)$, and the Thom–Sebastiani type formula for $\mathrm{Sp}(f)$ (see Section 1.2.7) gives $\mathrm{Sp}(f) = \mathrm{Sp}(g)\,\mathrm{Sp}(t^n)$, it follows that the Picard number of $X : (f = 0)$ is

$$\rho(X) = \mathrm{hcf}(p, n) + \mathrm{hcf}(q, n) + \mathrm{hcf}(r, n) - 3.$$

A blowup of a factorial singularity need not be locally factorial, as the next example shows (see also Example 4.17).

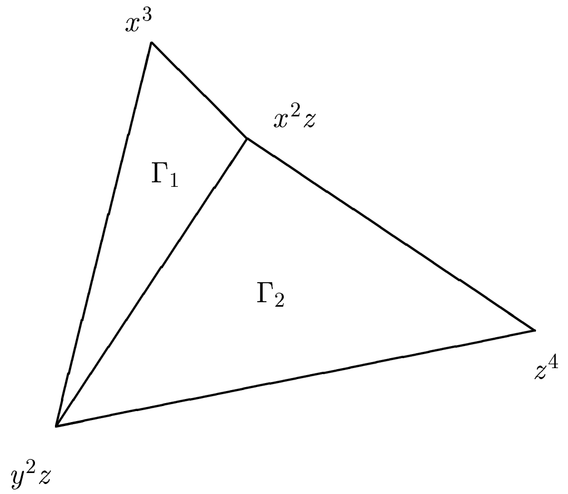
Example 2.15 Let $f(x, y, z, t) = g(x, y, z) + t^n$, where $g(x, y, z) = x^3 + x^2z + y^2z + z^4$ and $n \geq 2$. The polynomial f defines an isolated nondegenerate singularity, which is canonical since $g = 0$ is an elliptic Gorenstein surface singularity (see [R4], [SB1] and [Ste]). Let

$$\begin{aligned} \Gamma_1 &= [(3, 0, 0), (2, 0, 1), (0, 2, 1)], \\ \Gamma_2 &= [(2, 0, 1), (0, 2, 1), (0, 0, 4)] \end{aligned}$$

be the 2-dimensional faces of $\Gamma(g)$ and

$$\alpha^{(1)} = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right), \qquad \alpha^{(2)} = \left(\frac{3}{8}, \frac{3}{8}, \frac{1}{4}\right)$$

the corresponding quasihomogeneity types.



The set of monomials

$$B = \left\{1, x, y, x^2, xy, z^k\right\}_{1 \le k \le 4}$$

induces a regular basis on Ω_g. This implies that the spectrum of g is

$$\mathrm{Sp}(g) = (1) + \left(\frac{5}{4}\right) + 2\left(\frac{4}{3}\right) + \left(\frac{3}{2}\right) + 2\left(\frac{5}{3}\right) + \left(\frac{7}{4}\right) + (2).$$

From Section 1.2.7 and Theorem 2.13 it follows that

$$\rho(X) = \begin{cases} 0 & \text{if } n \equiv \pm 1 \bmod 6 \\ 1 & \text{if } n \equiv \pm 2 \bmod 12 \\ 3 & \text{if } n \equiv \pm 4 \bmod 12 \\ 4 & \text{if } n \equiv \pm 3 \bmod 12 \\ 5 & \text{if } n \equiv 6 \bmod 12 \\ 7 & \text{if } n \equiv 0 \bmod 12. \end{cases}$$

In particular, for $n = 5$, the singularity $(X, 0)$ is (analytically) factorial.

Suppose $n = 5$, and let $X_1 = \mathrm{Bl}_0 X$ be the blowup of X at its singular point. The blown up variety X_1 has only one singular point. Its equation near this point is $x^3 + x^2z + y^2z + tz^4 + t^2 = 0$ or, by a change of variable,

$$X_1 : x^3 + x^2z + y^2z + z^8 + t^2 = 0.$$

The singularity $(0,0,0,0) \in X_1$ is a cD_4 point. Denote $h = x^3 + x^2z + y^2z + z^8$; the corresponding Newton diagram $\Gamma(h)$ has again 2 facets and the quasihomogeneity types are now

$$\beta^{(1)} = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right), \qquad \beta^{(2)} = \left(\frac{7}{16}, \frac{7}{16}, \frac{1}{8}\right).$$

A regular set for Ω_h is

$$B' = \left\{1, x, y, x^2, xy, z^k\right\}_{1 \le k \le 8}$$

and the spectral numbers of h are given by

$$\mathrm{Sp}(h) = (1) + 2\left(\frac{4}{3}\right) + 2\left(\frac{5}{3}\right) + \sum_{i=1}^{7}\left(\frac{i+8}{8}\right) + (2).$$

It follows that $\rho(X_1) = 1$.

In [V1] it is obtained a formula relating the characteristic polynomial of an isolated nondegenerate singularity

$$0 \in X : (f = 0) \subset \mathbb{C}^{n+1}$$

to its Newton diagram $\Gamma(f)$. For any face $\Gamma \prec \Gamma(f)$, denote by $d(\Gamma)$ its dimension and by $V(\Gamma)$ its $(d(\Gamma))$-dimensional volume. Varchenko's formula and Proposition 2.1 give the following:

Proposition 2.16 *Let $0 \in X : (f = 0) \subset \mathbb{C}^4$ be an isolated canonical nondegenerate singularity. Then*

$$\rho(X) = 1 + \sum_{\Gamma} (-1)^{d(\Gamma)+1} d(\Gamma)! V(\Gamma),$$

where the summation is taken over all faces Γ of $\Gamma(f)$ which are contained in a $(d(\Gamma) + 1)$-dimensional coordinate plane.

For instance, if $xy = g(z, t)$ as in Section 2.2, with g nondegenerate and convenient, then

$$\rho(X) = \operatorname{length} \Gamma(g) - 1.$$

2.6 The general case

The idea of this section is that, in principle, it is possible to calculate the rank of the divisor class group $\rho(X)$ for *any* isolated canonical singularity

$$0 \in X : (f = 0) \subset \mathbb{C}^4.$$

The reason is that $\rho(X)$ is related to the characteristic polynomial $\Delta(t)$ by

$$\rho(X) = b_3(F)_1$$

(Proposition 2.7) and $\Delta(t)$ can be calculated from the embedded resolution data of X by A'Campo's formula.

Let $0 \in X : (f = 0) \subset \mathbb{C}^{n+1}$ be an isolated singularity, $\varphi \colon \widetilde{\mathbb{C}}^{n+1} \to \mathbb{C}^{n+1}$ an embedded resolution of the singularity of X. This means that a desingularisation of X is achieved by making a transformation of the ambient space $\mathbb{C}^{n+1}$ with the property that the strict transform $\widetilde{X}$ of X and the exceptional locus $E = \varphi^{-1}(0)$ have only normal crossings. In particular, in suitable local coordinates around any intersection point of k prime divisors from the inverse image of X, $\varphi^{-1}(X) = \varphi^*(X)_{\mathrm{red}}$, the function $f \circ \varphi$ can be written as

$$z_1^{m_1} \cdots z_k^{m_k}.$$

Such a resolution exists by [Hi1]. Let

$$E_m = \{z \in E : f \circ \varphi = z_1^m \text{ in suitable local coordinates near } z\}.$$

The following theorem, due to A'Campo, gives the characteristic polynomial of f in terms of the multiplicities of the exceptional divisors and the Euler characteristics $\chi(E_m)$ of the interior loci above.

Theorem 2.17 ([AC]) *The characteristic polynomial of the monodromy operator h_f is given by*

$$\Delta(t) = \left(\frac{1}{t-1} \prod_{m \geq 1} (t^m - 1)^{\chi(E_m)} \right)^{(-1)^n}$$

This result, together with Proposition 2.7, gives a formula for $\rho(X)$ in the canonical 3-dimensional case.

Proposition 2.18 *Let $0 \in X : (f = 0) \subset \mathbb{C}^4$ be an isolated singularity. Then, with the above notation,*

$$\rho(X) \leq 1 - \sum_{m \geq 1} \chi(E_m),$$

with equality if the singularity is canonical.

Although embedded resolutions of 3-fold singularities are difficult to control, in simple cases this result is useful. For instance, suppose that

$$f(x, y, z, t) = g(x, y) + h(z, t).$$

If $0 \in C : (g = 0) \subset \mathbb{C}^2$ is a curve singularity and $\varphi \colon \widetilde{\mathbb{C}}^2 \to \mathbb{C}^2$ an embedded resolution of g, then A'Campo's formula becomes

$$\Delta_g(t) = (t - 1) \prod (t^{m_i} - 1)^{\delta_i - 2}, \tag{2.5}$$

where the product is taken over all irreducible components E_i of the exceptional set E, $m_i = m(E_i)$ are their multiplicities and δ_i is the number of irreducible components of $\varphi^{-1}(C)$ intersecting E_i, for any i (see e.g. [N, p. 21]).

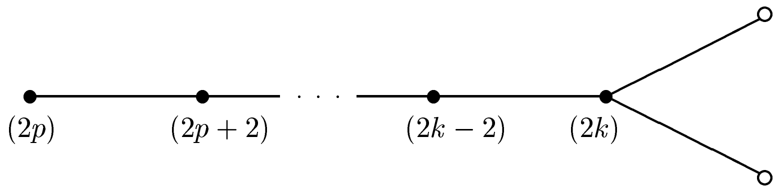
Using the expression (2.5) for both g and h gives, via Thom–Sebastiani, a formula for $\Delta_f(t)$.

Example 2.19 This is an example about the calculation of $\rho(X)$ for degenerate singularities. Let

$$g(z,t) = (z^p + t^p)^2 + t^n,\ p \geq 2,\ n \geq 2p+1,$$
$$f(x,y,z,t) = x^2 + y^3 + g(z,t).$$

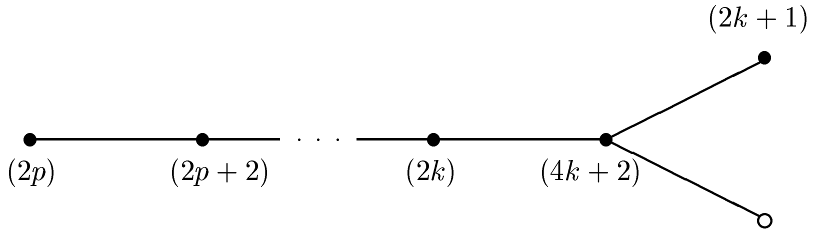
The polynomials g and f define isolated singularities, degenerate with respect to their Newton polyhedrons. This is because the singular locus defined by the hypersurface $g_{\Gamma(g)} = 0$ consists of p lines passing through the origin, each with multiplicity 2.

First we want to compute $\Delta_g(t)$ from the resolution graph of $0 \in C : (g=0) \subset \mathbb{C}^2$. Blowing up $0 \in \mathbb{C}^2$ we obtain an exceptional divisor $E_1 \cong \mathbb{P}^1$ of multiplicity $2p$. The proper transform of C has p singular points of type A_{n-2p-1}, lying on E_1. The following is one of the p identical branches, having their left vertex in common, of the resolution graph of C in the case $n = 2k$:



Vertices of the graph $\bullet$ correspond to the exceptional curves; the irreducible components of the proper transform of C are denoted by $\circ$. An edge is associated to a pair of vertices with nonempty intersection. The numbers in brackets are the multiplicities of the corresponding exceptional divisors.

There is a similar picture for $n = 2k+1$:



From A'Campo's formula (2.5) it follows that

$$\begin{aligned}\mathrm{Div}(g) &= (1) + (p-2)\Lambda_{2p} + p\Lambda_{2k}, \text{ if } n = 2k\\ \mathrm{Div}(g) &= (1) + (p-2)\Lambda_{2p} + p\Lambda_{4k+2} - p\Lambda_{2k+1}, \text{ if } n = 2k+1.\end{aligned}$$

Using Thom–Sebastiani's formula (see Section 1.2.7) we obtain:

$$\begin{aligned}\mathrm{Div}(f) =& p(3,k)\Lambda_{2[3,k]} + (p-2)(3,p)\Lambda_{2[3,p]} - p\Lambda_{2k}\\ &-(p-2)\Lambda_{2p} + \Lambda_6 - \Lambda_3 - \Lambda_2 + (1), \text{ if } n = 2k,\\ \mathrm{Div}(f) =& p(3,2k+1)\Lambda_{[3,2k+1]} + (p-2)(3,p)\Lambda_{2[3,p]} - p\Lambda_{2k+1}\\ &-(p-2)\Lambda_{2p} + \Lambda_6 - \Lambda_3 - \Lambda_2 + (1), \text{ if } n = 2k+1.\end{aligned}$$

In particular, $b_3(F_f)_1 = p((3,n)-1) + (p-2)((3,p)-1)$ in both cases.

The singularity $0 \in X$ is canonical if and only if $p \leq 5$. This is because, taking $\alpha = (3p, 2p, 3, 3)$, the condition $\alpha(\mathbf{1}) \geq \alpha(f) + 1$ implies $p \leq 5$. On the other hand, if $p = 2$, the singularity $0 \in X$ is cDV, hence canonical (see [R1, Theorem 2.6]), while for $3 \leq p \leq 5$

$$x^2 + y^3 + z^{2p}$$

define elliptic Gorenstein singularities (see [R4]), and thus $0 \in X$ is canonical by [SB1] and [Ste]. Therefore

$$\rho(X) \leq p((3,n)-1) + (p-2)((3,p)-1),$$

with equality for $p \leq 5$.

Chapter 3

Crepant divisors

3.1 The number $c(X)$

Let $0 \in X : (f = 0) \subset \mathbb{C}^{n+1}$ be a canonical hypersurface singularity. Given a morphism $\varphi\colon Y \to X$ from a partial resolution Y with at most terminal singularities, we can write

$$K_Y = \varphi^* K_X + \sum a_i E_i,$$

where E_i are the φ-exceptional prime divisors and $a_i \geq 0$. If $a_i = 0$ the divisor E_i is called *crepant.*

The discrepancy a_i depends only on the valuation of the divisor E_i, since, given any partial resolution $\varphi'\colon Y' \to X$, and φ'-exceptional prime divisor E_i' birational to E_i, the varieties Y and Y' are locally isomorphic near the generic points of E_i and E_i'. A crepant valuation v_{E_i} has centre a divisor E_i' on any partial resolution Y' of X with at most terminal singularities. The number of crepant valuations

$$c(X) := \#\{i : a_i = 0\},$$

measuring how far is X from being terminal, is finite [R1, Lemma (2.3)] and independent of Y.

Under the assumption that the hypersurface X is nondegenerate with respect to its Newton polyhedron, we develop in this chapter the calculations in toric geometry enabling us to detect the birationally defined divisors E_i in terms of weightings and the Newton polyhedron of f; in particular, we derive a formula for their number, $c(X)$. The arguments work for a n-dimensional hypersurface X, but the main application is to the 3-fold case, since, for 3-dimensional singularities, $c(X)$ is the second invariant we need to compute the Betti numbers of a minimal model.

3.2 Toric resolutions

We begin with an example to illustrate the method of calculating the number of crepant divisors $c(X)$ from a toric resolution of X. For notation, see Section 1.2.4.

Example 3.1 Let $f = x^3+y^3+z^3+t^n$ with $n \geq 2$. The polynomial f defines an isolated canonical nondegenerate singularity X, which is non-terminal if $n \geq 4$.

The blowup $\varphi_1 \colon X_1 \to X$ of X at 0 is a crepant morphism, and X_1 is given on one affine piece by

$$X_1 : x^3 + y^3 + z^3 + t^{n-3} = 0.$$

After repeating the blowup, at the new singular point, $[n/3]$ times, where $[n/3]$ denotes the integer part of $n/3$, we obtain either a nonsingular variety, if $n \equiv 0$ or $1 \bmod 3$, or a variety which can be desingularised by a discrepant morphism, if $n \equiv 2 \bmod 3$. Therefore

$$c(X) = [n/3].$$

The example above is very special; in order to compute $c(X)$ for more general singularities, we need to use toric geometry.

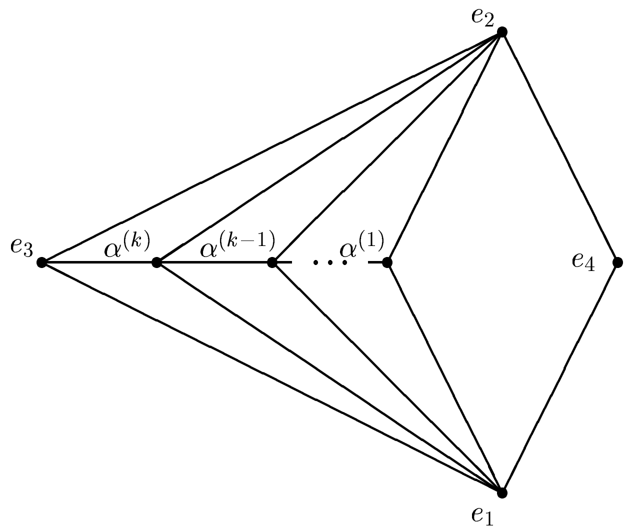
We treat the cases $n \equiv 0$, 1 and $2 \bmod 3$ separately.

(i) $n = 3k$

The fan $\Delta(f)$ associated to f (see Section 1.2.4) consists of the cones σ_i generated by e_j for $j \neq i$ and $\alpha^{(k)} := (k, k, k, 1)$, together with all their faces. Recall that a desingularisation of X is obtained by an arbitrary nonsingular subdivision of $\Delta(f)$. Since e_1, e_2, e_3 and $\alpha^{(k)}$ are basic, the cone σ_4 is nonsingular. The remaining cones can be resolved by taking a triangulation w.r.t. the weightings

$$\alpha^{(l)} = (l, l, l, 1) \quad \text{for} \quad 1 \leq l \leq k-1$$

as in the following picture:



Notice that $\alpha^{(l)}$ for $1 \leq l \leq k$ satisfy the equality

$$\alpha^{(l)}(\mathbf{1}) = \alpha^{(l)}(f) + 1. \tag{3.1}$$

Weightings satisfying an equation of this form will play an important role in the sequel (see Definition 3.2). Their corresponding exceptional divisors turn out to be crepant; in particular, a minimal model of X is nonsingular in this case, and

$$c(X) = k.$$

(ii) $n = 3k + 1$

The fan $\Delta(f)$ consists of the cones $\sigma_i = (e_j, \alpha)_{j \neq i}$, where

$$\alpha = (n, n, n, 3),$$

together with all their faces. Taking a triangulation w.r.t. the weightings

$$\alpha^{(l)} = (l, l, l, 1), \quad 1 \leq l \leq k,$$

resolves σ_i for $1 \leq i \leq 3$. In the case of σ_4 the weightings are

$$\alpha^{(k+1)} = (k+1, k+1, k+1, 1) \quad \text{and} \quad \beta = (2k+1, 2k+1, 2k+1, 2).$$

Only $\alpha^{(l)} = (l, l, l, 1)$ for $1 \leq l \leq k$ satisfy the equation (3.1) above. Therefore

$$c(X) = k.$$

(iii) $n = 3k+2$

In this case $\alpha = (n,n,n,3)$. The cones σ_i, for $1 \le i \le 3$, are subdivided w.r.t.

$$\alpha^{(l)} = (l,l,l,1), \quad 1 \le l \le k,$$

and σ_4 w.r.t. $\alpha^{(k+1)} = (k+1,k+1,k+1,1)$. The number of crepant divisors is again k.

3.3 Weighted blowups

A nondegenerate hypersurface singularity $0 \in X : (f = 0) \subset \mathbb{C}^{n+1}$ can be desingularised by a toric resolution of $\mathbb{C}^{n+1}$, that is, by a resolution of $\mathbb{C}^{n+1}$ corresponding to a subdivision of the positive quadrant $\sigma \subset N_{\mathbb{R}}$ (see Section 1.2.4).

It is possible to resolve X by successive barycentric subdivisions (defined below) of σ; a weighted blowup is a 1-step process, giving a prime divisor isomorphic to a weighted projective space. Weighted blowups can be described in several different ways, and this section reviews these descriptions.

3.3.1 Toric description

Let $\alpha \in \sigma \cap N$ be a primitive weighting. The *barycentric subdivision of σ* w.r.t. α is the fan $\Delta(\alpha)$ consisting of the cones

$$\sigma_i(\alpha) = \sum_{j \neq i} \mathbb{R}_{\geq 0} e_j + \mathbb{R}_{\geq 0} \alpha$$

for $1 \le i \le n+1$, together with all their faces. Define the *α-blowup* of $\mathbb{C}^{n+1}$ as the proper birational morphism

$$\varphi_\alpha \colon \mathbb{C}^{n+1}(\alpha) \to \mathbb{C}^{n+1}$$

from the toric variety $\mathbb{C}^{n+1}(\alpha)$ associated to the fan $\Delta(\alpha)$ to $\mathbb{C}^{n+1}$. This is obtained by gluing together the birational morphisms

$$\varphi_{\alpha,i} \colon \mathbb{C}_i^{n+1}(\alpha) = \operatorname{Spec} \mathbb{C}[\sigma_i(\alpha)^\vee \cap M] \to \mathbb{C}^{n+1},$$

for $1 \le i \le n+1$. Let $E_\alpha := \varphi_\alpha^{-1}(0)$ be the exceptional divisor. The restriction of φ_α to the proper transform $X(\alpha)$ of X

$$\varphi_\alpha| \colon X(\alpha) \to X$$

is called the *α-blowup of X*. Denote this morphism, for simplicity, still by φ_α, and let $E(\alpha) = E_\alpha \cap X(\alpha)$ be its exceptional set. The n-dimensional torus $\mathbb{T}^n$ corresponding to the lattice $\alpha^\perp \cap M$ is

$$\mathcal{O}_\alpha = \operatorname{Spec} \mathbb{C}[\alpha^\perp \cap M] \cong \mathbb{T}^n$$

and E_α is its closure (see [Fu, Section 3.1]).

In [R3, (4.8)] a neighborhood U_α of the generic point of E_α is described as follows. Let $U_\alpha = \operatorname{Spec} \mathbb{C}[\alpha^\vee \cap M] \cong \mathbb{T}^n \times \mathbb{C}$. The surjection

$$\mathbb{C}[\alpha^\vee \cap M] \twoheadrightarrow \mathbb{C}[\alpha^\perp \cap M]$$

$$x^m \longmapsto \begin{cases} x^m & \text{if } m \in \mathbb{C}[\alpha^\perp \cap M] \\ 0 & \text{otherwise} \end{cases}$$

corresponds to an embedding $\mathcal{O}_\alpha \hookrightarrow U_\alpha$. Choose a monomial $m_0 \in M$ such that $\alpha(m_0) = 1$ and denote $u = x^{m_0}$. Then $\mathcal{O}_\alpha = E_\alpha \cap U_\alpha$ is given in U_α by the equation $u = 0$. This description has the advantage that one can work on an open piece of the α-blowup rather than on one of its cyclic covers (see Section 3.3.3).

3.3.2 Global description

A weighted blowup is just a generalisation of the usual blowup of the origin of $\mathbb{C}^{n+1}$, and we can see it in a different way.

Let $\mathbf{x} = (x_i)_{1\le i\le n+1}$ be local coordinates on $\mathbb{C}^{n+1}$ around 0. Consider another copy of $\mathbb{C}^{n+1}$, with coordinates $\mathbf{z} = (z_i)_{1\le i\le n+1}$, and the map

$$\begin{aligned} \psi_\alpha &: \mathbb{C}^{n+1}_{\mathbf{z}} \to \mathbb{C}^{n+1}_{\mathbf{x}} \\ z_i &\mapsto z_i^{\alpha_i} = x_i. \end{aligned}$$

Let $\psi\colon \operatorname{Bl}_0 \mathbb{C}^{n+1}_{\mathbf{z}} \to \mathbb{C}^{n+1}_{\mathbf{z}}$ be the blowup of the origin and $\xi = (\xi_j)_{1\le j\le n+1}$ homogeneous coordinates on $\mathbb{P}^n$. Denote by $G = \mu_{\alpha_1} \times \cdots \times \mu_{\alpha_{n+1}}$ the direct product of the groups of α_ith roots of unity and consider the action of G on $\operatorname{Bl}_0 \mathbb{C}^{n+1}_{\mathbf{z}}$ given by

$$(\varepsilon, (\mathbf{z};\xi)) \mapsto (\varepsilon_1 z_1, \dots, \varepsilon_{n+1} z_{n+1}; \varepsilon_1 \xi_1, \dots, \varepsilon_{n+1}\xi_{n+1}),$$

for $\varepsilon = (\varepsilon_1, \dots, \varepsilon_{n+1}) \in G$. Then the α-blowup of $\mathbb{C}^{n+1}_{\mathbf{x}}$ is the quotient

$$\mathbb{C}^{n+1}(\alpha) = \operatorname{Bl}_0 \mathbb{C}^{n+1}_{\mathbf{z}}/G.$$

3.3.3 Local description

The local version of this description is as follows. Let U_i be the affine piece of $\mathrm{Bl}_0\, \mathbb{C}^{n+1}_{\mathbf{z}}$ given by $(\xi_i \neq 0)$. This is a copy of $\mathbb{C}^{n+1}$ with coordinates z_i and $\rho_j := \xi_j/\xi_i$ for $j \neq i$. The induced action of G on U_i is

$$\begin{aligned} z_i &\mapsto \varepsilon_i z_i \\ \rho_j &\mapsto \varepsilon_j \varepsilon_i^{-1} \rho_j \end{aligned}$$

and the corresponding quotient $U_i/G = \operatorname{Spec} \mathbb{C}[z_i, \rho_j]^G_{j\neq i}$.

Denote $\zeta_j = \rho_j^{\alpha_j}$ for $j \neq i$, and let $\widetilde{\mathbb{C}^{n+1}_i}(\alpha)$ be a copy of $\mathbb{C}^{n+1}$ with coordinates z_i and ζ_j.

We can realise U_i/G alternatively as the quotient $\widetilde{\mathbb{C}^{n+1}_i}(\alpha)/\mu_{\alpha_i}$, where μ_{α_i} acts on $\widetilde{\mathbb{C}^{n+1}_i}(\alpha)$ by

$$\begin{aligned} z_i &\mapsto \varepsilon_i z_i \\ \zeta_j &\mapsto \varepsilon_i^{-\alpha_j} \zeta_j. \end{aligned}$$

This is immediate, since the rings of invariants of G on U_i and μ_{α_i} on $\widetilde{\mathbb{C}^{n+1}_i}(\alpha)$ are the same:

$$\begin{aligned} \mathbb{C}[z_i, \rho_j]^G_{j\neq i} &= \mathbb{C}\left[z_i^{a_i} \prod_{j\neq i} \rho_j^{a_j},\ a_i \equiv \sum_{j\neq i} a_j \bmod \alpha_i \text{ and } a_j \equiv 0 \bmod \alpha_j,\ j \neq i \right] \\ &= \mathbb{C}\left[z_i^{b_i} \prod_{j\neq i} \zeta_j^{b_j},\ b_i \equiv \sum_{j\neq i} \alpha_j b_j \bmod \alpha_i \right] \\ &= \mathbb{C}[z_i, \zeta_j]^{\mu_{\alpha_i}}_{j\neq i}. \end{aligned}$$

Furthermore, both algebras are isomorphic to $\mathbb{C}[\sigma_i^\vee(\alpha) \cap M]$, therefore define the ith affine piece of the blowup. For instance

$$\mathbb{C}[z_i, \zeta_j]^{\mu_{\alpha_i}}_{j\neq i} \cong \mathbb{C}[\sigma_i^\vee(\alpha) \cap M],$$

via the correspondence

$$z_i^{b_i} \prod_{j\neq i} \zeta_j^{b_j} \mapsto x_i^{(b_i - \sum_{j\neq i} b_j \alpha_j)/\alpha_i} \prod_{j\neq i} x_j^{b_j}.$$

The right hand side makes sense since $(b_i - \sum_{j\neq i} b_j \alpha_j)/\alpha_i$ is an integer; moreover, this monomial belongs to $\mathbb{C}[\sigma_i^\vee(\alpha) \cap M]$ since, by definition,

$$\sigma_i^\vee(\alpha) \cap M = \{m \in M : \alpha(m) \geq 0 \text{ and } m_j \geq 0 \text{ for all } j \neq i\}$$

and $b_j \geq 0$, $\alpha((b_i - \sum b_j \alpha_j)/\alpha_j, b_j) = b_i \geq 0$.

Since $x_i = z_i^{\alpha_i}$ and $x_j = z_j^{\alpha_j} = z_i^{\alpha_j} \rho_j^{\alpha_j} = z_i^{\alpha_j} \zeta_j$, the morphism

$$\widetilde{\varphi_{\alpha,i}} \colon \widetilde{\mathbb{C}_i^{n+1}(\alpha)} \to \mathbb{C}^{n+1}$$

from the cyclic cover of $\mathbb{C}_i^{n+1}(\alpha)$ to $\mathbb{C}^{n+1}$, descending to $\varphi_{\alpha,i}$, is given by

$$\begin{aligned} z_i &\mapsto z_i^{\alpha_i} \\ \zeta_j &\mapsto z_i^{\alpha_j} \zeta_j. \end{aligned}$$

3.3.4 Algebraic description

For terminology and notation throughout this section see [Ha, II.7].

The weighted projective space $\mathbb{P}(\alpha)$, for $\alpha \in \sigma \cap N$, can be seen either geometrically, as $\mathbb{P}(\alpha) = (\mathbb{C}^{n+1} \setminus 0)/\mathbb{C}^*$, where $\mathbb{C}^*$ acts by

$$\lambda \cdot (x_1, \ldots, x_{n+1}) = (\lambda^{\alpha_1} x_1, \ldots, \lambda^{\alpha_{n+1}} x_{n+1}),$$

or algebraically, as $\mathbb{P}(\alpha) = \operatorname{Proj} A$, where the algebra $A = \mathbb{C}[x_1, \ldots, x_{n+1}]$ is graded by $\deg(x_i) = \alpha_i$.

Similarly, weighted blowups can be realised in an algebraic way, as well. Consider the filtration of ideals $\mathcal{I}_\bullet(\sigma)$ on $A = \mathbb{C}[\sigma^\vee \cap M]$ given by

$$\mathcal{I}_k(\sigma) = \{g \in A : \alpha(g) \geq k\},$$

where the α-degree of g is defined by $\alpha(g) = \min_{m \in g} \alpha(m)$. Then $\mathbb{C}^{n+1}(\alpha) = \operatorname{Proj} \mathcal{A}(\sigma)$, where $\mathcal{A}(\sigma) := \bigoplus_{k \geq 0} \mathcal{I}_k(\sigma)$ is the blowup algebra of $\mathcal{I}_\bullet(\sigma)$ in A. The corresponding graded ring

$$gr_{\mathcal{I}_\bullet(\sigma)} A := \bigoplus_{k \geq 0} \mathcal{I}_k(\sigma)/\mathcal{I}_{k+1}(\sigma)$$

defines the exceptional set by

$$\begin{aligned} E_\alpha &= \operatorname{Proj} gr_{\mathcal{I}_\bullet(\sigma)} A \\ &\cong \mathbb{P}(\alpha). \end{aligned}$$

Furthermore, for any face $\tau \prec \sigma$ and any open set

$$U_\tau = \operatorname{Spec} \mathbb{C}[\tau^\vee \cap M],$$

we have $\varphi_\alpha^{-1}(U_\tau) = \operatorname{Proj} \mathcal{A}(\tau)$, where $\mathcal{I}_\bullet(\tau)$ and $\mathcal{A}(\tau)$ are defined similarly.

We can see that this definition of a weighted blowup is consistent with the previous one by looking at an open covering. The open affine sets $D_+(g)$, for $g \in \mathcal{I}_k(\sigma) \setminus \mathcal{I}_{k+1}(\sigma)$, covering $\operatorname{Proj} \mathcal{A}(\sigma)$, are

$$\begin{aligned} D_+(g) &= \mathcal{A}(\sigma)_{(g)} \\ &= \operatorname{Spec} \mathbb{C}[h/g^a : a \geq 0,\ h \in \mathcal{I}_{ka}(\sigma)]. \end{aligned}$$

In particular, the standard open subsets of $\operatorname{Proj} \mathcal{A}(\sigma)$

$$D_+(x_i) = \operatorname{Spec} \mathbb{C}[\sigma_i(\alpha)^\vee \cap M]$$

coincide with $\mathbb{C}_i^{n+1}(\alpha)$.

If there exists an i such that $\alpha_i = 1$, then a minimal set of generators for $\sigma_i(\alpha)^\vee \cap M$ is $\{e_i^*, e_j^* - \alpha_j e_i^*\}_{j \neq i}$, and thus

$$D_+(x_i) = \operatorname{Spec} \mathbb{C}[\zeta_j, x_i]_{j \neq i} \cong \mathbb{C}^{n+1},$$

where $\zeta_j = x_j / x_i^{\alpha_i}$ for all $1 \leq j \leq n+1$ with $j \neq i$.

In general, $D_+(x_i) = \operatorname{Spec} \mathbb{C}[\zeta_j, z_i]_{j \neq i}^{\mu_{\alpha_i}}$, as we have noticed in Section 3.3.3. We can avoid working on the cyclic cover by choosing a different open set. Let $m_0 \in M$ be a monomial with $\alpha(m_0) = 1$, as in Section 3.3.1, and $u = x^{m_0}$. Suppose that p of its components, say the first p, are nonnegative and let $\tau = \sum_{i=1}^{p} \mathbb{R}_{\geq 0} e_i$. Then

$$\begin{aligned} D_+(u) &= \operatorname{Spec} \mathcal{A}(\tau)_{(u)} \\ &= \operatorname{Spec} \mathbb{C}\left[u, \frac{x_i}{u^{\alpha_i}} \text{ for } 1 \leq i \leq n+1\right]. \end{aligned}$$

If p_0 denotes the number of vanishing components of m_0, then $D_+(u)$ is an open set of $\mathbb{C}^{n+1}(\alpha)$ containing U_α (see Section 3.3.1), equal to U_α if $p_0 = 0$, and equal to one of the standard open sets $D_+(x_i)$ if $p_0 = n$. With the notation $y_i = x_i / u^{\alpha_i}$, the morphism $\varphi_\alpha| \colon D_+(u) \to \mathbb{C}^{n+1}$ is given then by

$$x_i = u^{\alpha_i} y_i \text{ for } 1 \leq i \leq n+1.$$

3.4 Crepant weightings

Let $0 \in X : (f = 0) \subset \mathbb{C}^{n+1}$ be an isolated canonical hypersurface singularity and assume, from now on, that f defines a nondegenerate singularity (see Section 1.2.4).

For a finite set of weightings $\Lambda \subset \sigma \cap N$, let Δ be a subdivision of the positive quadrant σ in $N_{\mathbb{R}}$ such that $\mathrm{Sk}^1(\Delta) \setminus \mathrm{Sk}^1(\sigma) = \Lambda$. Let also

$$\varphi_\Lambda \colon \mathbb{C}^{n+1}(\Lambda) \to \mathbb{C}^{n+1}$$

be the morphism from the corresponding toric variety, and $X(\Lambda)$ the proper transform of X.

As a part of the proof of Theorem 1.4, it is shown in [R3, Section 4.8] that, for any primitive weighting $\alpha \in \sigma \cap N$, there is a relation

$$K_{\mathbb{C}^{n+1}(\alpha)} + X(\alpha) = \varphi_\alpha^*(K_{\mathbb{C}^{n+1}} + X) + (\alpha(\mathbf{1}) - \alpha(f) - 1)E_\alpha, \tag{3.2}$$

where $\mathbf{1} = (1, \dots, 1) \in M$. Iterating this for the finite set Λ, it follows that

$$K_{\mathbb{C}^{n+1}(\Lambda)} + X(\Lambda) = \varphi_\Lambda^*(K_{\mathbb{C}^{n+1}} + X) + \sum_{\alpha \in \Lambda} (\alpha(\mathbf{1}) - \alpha(f) - 1)E_{\alpha,\Lambda},$$

where $E_{\alpha,\Lambda}$ denotes the exceptional divisor corresponding to α on $\mathbb{C}^{n+1}(\Lambda)$.

In particular, if Δ is a nonsingular subdivision of σ, the adjunction formula implies that

$$K_{X(\Lambda)} = \varphi_\Lambda^*(K_X) + \sum_{\alpha \in \Lambda} (\alpha(\mathbf{1}) - \alpha(f) - 1)(E_{\alpha,\Lambda} \cap X(\Lambda))$$

(see also [I]). Therefore, the irreducible components of $E_{\alpha,\Lambda} \cap X(\Lambda)$ are crepant if and only if $\alpha(\mathbf{1}) = \alpha(f) + 1$. This justifies the following:

Definition 3.2 *A primitive weighting $\alpha \in \sigma \cap N$, not belonging to any proper face of σ (see the remark below), is called* crepant *if it satisfies the equation*

$$\alpha(\mathbf{1}) = \alpha(f) + 1. \tag{3.3}$$

Denote by $W(f)$ the set of crepant weightings.

Remark 3.3 The reason for restricting to interior weightings $\alpha \in \sigma^\circ \cap N$ in the above definition is that, since it is always possible to assume that f is convenient, the only solution α of the equation (3.3) with, say, $\alpha_i = 0$ is $\alpha = e_j^*$, for some j.

Remark 3.4 A weighting $\alpha \in \sigma \cap N$ satisfying (3.3) is automatically primitive.

It is a common phenomenon that a weighted blowup of a given singularity is not normal. Consider, for instance, the singularity given by

$$f = x^3 + y^3 + z^3 + t^n,\ n \geq 3,$$

and $\alpha = (k, k, 1, 1)$ with $1 \leq k \leq [n/3]$ (compare Example 3.1). One affine piece of the α-blowup of X is given by

$$x = t^k x_1, \quad y = t^k y_1 \quad \text{and} \quad z = t z_1$$

and thus

$$X(\alpha) : t^{3(k-1)}(x_1^3 + y_1^3) + z_1^3 + t^{n-3} = 0$$

is not normal, unless $k = 1$. Taking its normalisation by replacing z_1 with z_1/t^{k-1} leads to the new weighting $\beta = (k, k, k, 1)$, which is crepant.

This example can be formalised into the following:

Proposition 3.5 *Let $0 \in X \subset \mathbb{C}^{n+1}$ be a canonical hypersurface singularity and α a crepant weighting. Then the α-blowup of X also has canonical singularities.*

Proof We only need to check that $X(\alpha)$ is normal. This is a direct consequence of the adjunction and subadjunction formulas [R5].

(1) Adjunction [R5, 2.12].

From the relation (3.2) it follows, since α is crepant, that the sheaf $\omega_{\mathbb{C}^{n+1}(\alpha)}(X(\alpha))$ of rational sections of $\omega_{\mathbb{C}^{n+1}(\alpha)}$ with poles along $X(\alpha)$ satisfies

$$\begin{aligned} \omega_{\mathbb{C}^{n+1}(\alpha)}(X(\alpha)) &= \varphi_\alpha^* \omega_{\mathbb{C}^{n+1}}(X) \\ &= \mathcal{O}_{\mathbb{C}^{n+1}(\alpha)}. \end{aligned} \tag{3.4}$$

Part of the long exact sequence of sheaves $\mathcal{E}xt^i\left(\,\cdot\,, \omega_{\mathbb{C}^{n+1}(\alpha)}\right)$ on $\mathbb{C}^{n+1}(\alpha)$ associated to the standard exact sequence

$$0 \to \mathcal{I}_{X(\alpha)} \to \mathcal{O}_{\mathbb{C}^{n+1}(\alpha)} \to \mathcal{O}_{X(\alpha)} \to 0$$

is the following:

$$\begin{aligned} &\omega_{\mathbb{C}^{n+1}(\alpha)} \to \mathcal{H}om\left(\mathcal{I}_{X(\alpha)}, \omega_{\mathbb{C}^{n+1}(\alpha)}\right) \to \\ &\mathcal{E}xt^1\left(\mathcal{O}_{X(\alpha)}, \omega_{\mathbb{C}^{n+1}(\alpha)}\right) \to \mathcal{E}xt^1\left(\mathcal{O}_{\mathbb{C}^{n+1}(\alpha)}, \omega_{\mathbb{C}^{n+1}(\alpha)}\right). \end{aligned}$$

We have the equalities

$$\begin{aligned}
\mathcal{E}xt^1\left(\mathcal{O}_{\mathbb{C}^{n+1}(\alpha)}, \omega_{\mathbb{C}^{n+1}(\alpha)}\right) &= \mathcal{O}_{\mathbb{C}^{n+1}(\alpha)} \\
\mathcal{H}om\left(\mathcal{I}_{X(\alpha)}, \omega_{\mathbb{C}^{n+1}(\alpha)}\right) &= \omega_{\mathbb{C}^{n+1}(\alpha)}(X(\alpha)) \\
\mathcal{E}xt^1\left(\mathcal{O}_{X(\alpha)}, \omega_{\mathbb{C}^{n+1}(\alpha)}\right) &= \omega_{X(\alpha)},
\end{aligned}$$

where the last follows from [R5, Theorem 2.12], once we notice that $\mathbb{C}^{n+1}(\alpha)$ is a toric variety, hence Cohen-Macaulay. From the sequence above and (3.4) it follows that

$$\omega_{X(\alpha)} = \mathcal{O}_{X(\alpha)}. \tag{3.5}$$

(2) Subadjunction [R5, 2.3].

Suppose $X(\alpha)$ is not normal and take its normalisation

$$\pi\colon \widetilde{X(\alpha)} \to X(\alpha).$$

Let $C \subset \widetilde{X(\alpha)}$ be defined by $\mathcal{I}_C = \widetilde{\mathcal{C}}$, where

$$\begin{aligned}
\mathcal{C} &= \mathrm{Ann}\left(\pi_*\mathcal{O}_{\widetilde{X(\alpha)}}/\mathcal{O}_{X(\alpha)}\right) \\
\widetilde{\mathcal{C}} &= \mathcal{C}\mathcal{O}_{\widetilde{X(\alpha)}}
\end{aligned}$$

as in [R5, 2.1]. Then [R5, Proposition 2.3]

$$\pi^*\omega_{X(\alpha)} = \omega_{\widetilde{X(\alpha)}}(C). \tag{3.6}$$

Relations (3.5) and (3.6) with $C \neq 0$ contradict the assumption that $0 \in X$ is canonical. $\square$

3.5 Crepant valuations

In order to calculate the number $c(X)$ it is more convenient to use a birational invariant language, namely that of geometric discrete valuations v of the function field $k(X)$. To any prime divisor E on a partial resolution of X, associate the corresponding valuation

$$v_E\colon k(X)^* \to \mathbb{Z}.$$

By Section 3.4, for any crepant valuation v, there exist a crepant weighting α and a prime divisor $E \in E(\alpha)$ such that $v = v_E$. This gives a relation between crepant valuations and exceptional prime divisors on weighted blowups with crepant weightings. In order to make this correspondence 1-to-1, we introduce the following:

Definition 3.6 *Let α be a crepant weighting. A prime divisor E on the α-blowup of X is called* essential for its valuation *if $v_E(x_i) = \alpha_i$. Denote by $E(f)$ the set of divisors which are essential for their valuations.*

Let $f = f_\alpha + f_{>\alpha}$ be the α-homogeneous decomposition of f. Denote $f^{(j)} = \sum a_m x^m$, where the summation is taken over all monomials $m \in f$ with $m_j = 0$; similar notation for f_α and $f_{>\alpha}$.

Lemma 3.7 *Let $0 \in X : (f = 0)$ be a canonical singularity, $\alpha \in W(f)$. Suppose that $x_j | f_\alpha$ for some j with $1 \le j \le n+1$. Then*

$$\gcd(\alpha_1, \dots, \alpha_{j-1}, \alpha_{j+1}, \dots, \alpha_{n+1}) = 1.$$

Proof Let $d = \gcd(\alpha_1, \dots, \alpha_{j-1}, \alpha_{j+1}, \dots, \alpha_{n+1})$ and denote

$$l = \min_{m \in f_\alpha} m_j.$$

Define a new weighting $\beta \in \sigma^\circ \cap N$ by $\beta_j = \lceil \alpha_j / d \rceil$, and $\beta_i = \alpha_i / d$ for $i \ne j$, where, for r a rational number, $\lceil r \rceil$ is the smallest integer $\ge r$.

Since $0 \in X$ is canonical, we have $\beta(\mathbf{1}) \ge \beta(f) + 1$ (see Theorem 1.4). Therefore, using the expression of β in terms of α, at least one of the following conditions must hold:

$$\alpha(\mathbf{1}) + d\beta_j - \alpha_j \ge \min_{m \in f_\alpha} (\alpha(f) + d\beta_j m_j - \alpha_j m_j) + d; \qquad (a)$$

$$\alpha(\mathbf{1}) + d\beta_j - \alpha_j \ge \min_{m \in f_{>\alpha}} (\alpha(m) + d\beta_j m_j - \alpha_j m_j) + d. \qquad (b)$$

(a) is equivalent to $d\beta_j - \alpha_j + 1 \ge l(d\beta_j - \alpha_j) + d$, which is possible if and only if $d = 1$.

(b) is equivalent to the existence of a monomial $m \in f_{>\alpha}$ such that

$$\alpha(f) + d\beta_j - \alpha_j + 1 \ge \alpha(m) + (d\beta_j - \alpha_j) m_j + d,$$

which implies that $d\beta_j - \alpha_j \ge (d\beta_j - \alpha_j) m_j + d$. Therefore $m_j = 0$, which gives $d\beta_j - \alpha_j \ge d$. This contradicts the definition of β_j. $\square$

Lemma 3.8 *Under the same assumptions as above, denote also*

$$l' = \min_{m \in f^{(j)}_{>\alpha}} (\alpha(m) - \alpha(f)).$$

Then either l or l' is equal to 1.

Proof Define $\beta \in \sigma^\circ \cap N$ by $\beta_j = \alpha_j + 1$ and $\beta_i = \alpha_i$ for $i \neq j$. As before, since $\beta(\mathbf{1}) \geq \beta(f) + 1$, at least one of the following holds:

(i) $\alpha(\mathbf{1}) \geq \min_{m \in f_\alpha}(\alpha(m) + m_j)$.

In this case, $\alpha(\mathbf{1}) \geq \alpha(f) + l$, hence $l = 1$.

(ii) $\alpha(\mathbf{1}) \geq \min_{m \in f_{>\alpha}}(\alpha(m) + m_j)$.

Since $\min_{m \in f_{>\alpha} - f^{(j)}_{>\alpha}}(\alpha(m) + m_j) \geq \alpha(f) + 2$ and α is crepant, the condition *(ii)* is equivalent to

$$\alpha(\mathbf{1}) \geq \min_{m \in f^{(j)}_{>\alpha}} \alpha(m).$$

Therefore, since α is crepant, it follows that

$$\alpha(f) + 1 \geq \min_{m \in f^{(j)}_{>\alpha}} \alpha(m),$$

and this is equivalent to $l' = 1$.

□

Lemma 3.9 *Same assumptions and notation as before. Then the new weighting $\beta \in \sigma \cap N$ defined by*

$$\begin{aligned} &(i)\ \beta_i = l\alpha_i \text{ and } \beta_j = l\alpha_j + 1 \text{ if } l' = 1; \\ &(ii)\ \beta_i = \alpha_i \text{ and } \beta_j = \alpha_j + l' \text{ if } l = 1. \end{aligned}$$

is crepant.

Proof (i) Using the fact that α is crepant it follows that

$$\beta(\mathbf{1}) = l\alpha(\mathbf{1}) + 1 = l(\alpha(f) + 1) + 1.$$

In order to express $\beta(f)$ in terms of $\alpha(f)$, notice that any monomial $m \in f_{>\alpha}$ satisfies $l\alpha(m) + m_j \geq l(\alpha(f) + 1)$. Therefore

$$\begin{aligned}\beta(f) &= \min_{m \in f}(l\alpha(m) + m_j) = \min_{m \in f_{>\alpha}}(l\alpha(m) + m_j) \\ &= l\alpha(f) + \min_{m \in f_{>\alpha}} m_j = l\alpha(f) + l.\end{aligned}$$

(ii) Similarly,

$$\beta(\mathbf{1}) = \alpha(\mathbf{1}) + l' = \alpha(f) + 1 + l'.$$

In this case $\alpha(m) + l'm_j \geq \alpha(f) + 1 + l'$, for any monomial $m \in f_{>\alpha}$ with $m_j \neq 0$. Since $l = 1$, we also have

$$\begin{aligned}\min_{m \in f_\alpha}(\alpha(m) + l'm_j) &= \alpha(f) + l' \min_{m \in f_\alpha} m_j \\ &= \alpha(f) + l', \quad \text{and} \\ \min_{m \in f^{(j)}_{>\alpha}}(\alpha(m) + l'm_j) &= \alpha(f) + l'.\end{aligned}$$

Therefore it follows that β is a crepant weighting, since

$$\beta(f) = \min_{m \in f}(\alpha(m) + l'm_j) = \alpha(f) + l'.$$

□

3.6 The set $E(f)$

Recall that, given an isolated canonical nondegenerate singularity $0 \in X : (f = 0) \subset \mathbb{C}^{n+1}$, we are interested in the exceptional divisors $E(\alpha)$ on weighted blowups $X(\alpha)$ of X with α crepant weightings, that is, weightings satisfying

$$\alpha(\mathbf{1}) = \alpha(f) + 1.$$

In this section we prove that $E(f)$ (see Definition 3.6) is a set of representatives of the crepant valuations. We start with an illustration of the general situation.

Example 3.10 The polynomial $f = x^3 + x^2z + y^2z + z^4 + t^n$ defines an isolated canonical nondegenerate singularity $0 \in X : (f = 0) \subset \mathbb{C}^4$ (see also Examples 2.15 and 4.17). A calculation shows that all the weightings in the set

$$\left\{(k, k, k-i, 1) : i \geq 0 \text{ and } 3i \leq k \leq \left[\frac{n+i}{3}\right]\right\}$$

are crepant. The following are some of the exceptional divisors on weighted blowups with weightings in this set:

- $E(\alpha) \cong (x^3 + x^2z + y^2z = 0) \subset \mathbb{P}^3$; this is the case for $\alpha = (k, k, k, 1)$ with $3k < n$. One affine piece of the α-blowup of X is given by

  $$x = t^k x_1 \quad y = t^k y_1 \quad z = t^k z_1,$$

  and thus

  $$X(\alpha) : x_1^3 + x_1^2 z_1 + y_1^2 z_1 + t^k z_1 + t^{n-3k} = 0.$$

  Since t is a uniformising parameter of the ring of regular function on $X(\alpha)$ at the generic point of $E(\alpha)$, it follows that $E(\alpha) \in E(f)$.

- $E(\alpha) \cong (x^3 + x^2z + y^2z + t^{3k} = 0) \subset \mathbb{P}(\alpha)$, where $\alpha = (k, k, k, 1)$. The same calculation as above shows that $E(\alpha) \in E(f)$.

- $E(\alpha) \cong (x^2z + y^2z = 0) \subset \mathbb{P}(k, k, k-i, 1)$; this is the case for $\alpha = (k, k, k-i, 1)$ with $3k \leq n$ and $4i < k$. The blowup $X(\alpha)$ is now given by

  $$x = t^k x_1 \quad y = t^k y_1 \quad z = t^{k-i} z_1.$$

  The divisor $E(\alpha)$ has 3 irreducible components, and only 2 of them belong to $E(f)$. We can see this by calculating the valuation v_E along the 3rd component, given on

  $$X(\alpha) : t^i x_1^3 + x_1^2 z_1 + y_1^2 z_1 + t^{k-3i} z_1 + t^{n-3k+i} = 0$$

  by $t = z_1 = 0$. In the local ring $\mathcal{O}_{X(\alpha),E}$ the element t is still a uniformising parameter; the difference from the previous cases is that z_1 is no longer a unit in this ring. A calculation gives

  $$\begin{array}{lll} v_E(t) = 1 & v_E(z_1) = i & v_E(x) = k \\ v_E(y) = k & v_E(z) = k. & \end{array}$$

  Taking $\beta := (v_E(x), v_E(y), v_E(z), v_E(t))$, we obtain one of the weightings considered before.

Theorem 3.11 *Let $0 \in X : (f = 0) \subset \mathbb{C}^{n+1}$ be an isolated canonical nondegenerate singularity, α a crepant weighting, and E an irreducible component of $E(\alpha) \subset \mathbb{P}(\alpha)$. Then*

(1) *The divisor E is essential for its valuation if and only if $E \subset \mathbb{P}(\alpha)$ is not one of the coordinate hyperplanes.*

(2) *Any crepant valuation is the valuation along a divisor in $E(f)$.*

Proof The proof has two parts:

(A) The first part is based on explicit calculations of valuations along crepant divisors and proves (1); it also defines, for $E \notin E(f)$, a new weighting β, by $\beta_i = v_E(x_i)$.

(B) In the second part we prove that $E(\beta)$ has precisely one component E' which is essential for its valuation, and that $X(\alpha)$ and $X(\beta)$ are locally isomorphic over X at the generic points of E and E' .

(A) Valuations along crepant divisors

Choose an arbitrary Laurent monomial $m_0 \in M$ satisfying $\alpha(m_0) = 1$ and let $u = x^{m_0}$ as in Section 3.3.4. Denote $y_i = u^{-\alpha_i} x_i$ and

$$U'_\alpha = \operatorname{Spec} \mathbb{C}[u, y_i \text{ for } 1 \le i \le n+1] .$$

Notice that the y_i are related by $y^{m_0}=1$.

The α-blowup φ_α is then given on U'_α by $x_i = u^{\alpha_i} y_i$. Let $U := X(\alpha) \cap U'_\alpha$ be the proper transform of X on U'_α. Its equation is

$$\sum_{m \in f} u^{\alpha(m)-\alpha(f)} y^m = 0. \tag{3.7}$$

On U the exceptional divisor $E(\alpha)$ is given by $u = f_\alpha(y) = 0$. Let g be the irreducible polynomial such that the component E has the expression

$$E : u = g(y) = 0 \text{ on the open set } U.$$

Denote by $A = \mathcal{O}_{X(\alpha),E}$ the localisation of the ring of regular function on $X(\alpha)$ at the generic point of E. Denote also by $v = v_E$ the corresponding valuation.

If $g(y) \neq y_i$ for any i, it follows from the nondegeneracy assumption that u is an uniformising parameter and all the y_i are units in A. Therefore $v(x_i) = \alpha_i$, i.e., E is essential.

Suppose that $g(y) = y_j$ for some j. Then $v(y_j) \geq 1$ and E is no longer essential. This proves (1).

Let now $l = \min_{m \in f_\alpha} m_j$ and choose a monomial m_0 as above with the additional property that its jth component is zero. Such a monomial exists by Lemma 3.7. In the local ring A the relation (3.7) becomes

$$\begin{aligned} \sum_{m \in f_{>\alpha}} u^{\alpha(m)-\alpha(f)} y^m &= - \sum_{m \in f_\alpha} y^m \\ &= y_j^l(\text{unit}). \end{aligned} \tag{3.8}$$

We have to distinguish two cases:

(a) $l \geq 2$

In this case it follows from Lemma 3.8 that

$$l' := \min_{m \in f^{(j)}_{>\alpha}} (\alpha(m) - \alpha(f)) = 1$$

and (3.8) reads $u = y_j^l(\text{unit})$. This implies that $v(y_j) = 1$ and $v(u) = l$; thus

$$v(x_j) = l\alpha_j + 1 \text{ and } v(x_i) = l\alpha_i \text{ if } i \neq j.$$

The new weighting β defined by

$$\beta_j = l\alpha_j + 1 \text{ and } \beta_i = l\alpha_i \text{ if } i \neq j$$

is crepant by Lemma 3.9.

(b) $l = 1$

In this case (3.8) becomes

$$\sum_{m \in f^{(j)}_{>\alpha}} u^{\alpha(m)-\alpha(f)} y^m + \sum_{m \in f_{>\alpha} - f^{(j)}_{>\alpha}} u^{\alpha(m)-\alpha(f)} y^m = y_j(\text{unit})$$

and this reduces to $u^{l'} = y_j(\text{unit})$. Thus u is a local parameter, $v(u) = 1$, $v(y_j) = l'$ and

$$v(x_j) = \alpha_j + l', \quad v(x_i) = \alpha_i, \text{ if } i \neq j.$$

As in the previous case, define a new weighting β by

$$\beta_j = \alpha_j + l', \quad \beta_i = \alpha_i, \text{ if } i \neq j.$$

Again β is crepant by Lemma 3.9.

(B) The local isomorphism

Let the monomial $m_0' \in M$ with $\beta(m_0') = 1$ be defined by

$$m_0' = e_j^* - \alpha_j m_0, \quad \text{in case } (a),$$
$$m_0' = m_0, \quad \text{in case } (b).$$

Define $u' = x^{m_0'}$ and $z_i = (u')^{-\beta_i} x_i$, and let

$$U_\beta' = \operatorname{Spec} \mathbb{C}\left[u', z_i \text{ for } 1 \leq i \leq n+1\right]$$

and $U' := X(\beta) \cap U_\beta'$. Look at the birational map $\varphi_{\beta\alpha}$ over X between U' and U

$$\begin{tikzcd}
U' \arrow[rr, dashed, "\varphi_{\beta\alpha}"] \arrow[dr, "\varphi_\beta"'] & & U \arrow[dl, "\varphi_\alpha"] \\
& X &
\end{tikzcd}$$

This map, restricted to the locus $U' \setminus \{\prod_{i=1}^{n+1} z_i = 0\}$, turns out to be a morphism, as the following calculation shows. We have $x_i = u^{\alpha_i} y_i = (u')^{\beta_i} z_i$ and, taking the product over all i, it follows that $u^{\alpha(m_0)} = (u')^{\beta(m_0)} z^{m_0} y^{-m_0}$. Since $\alpha(m_0) = 1$, $y^{m_0} = 1$ and $\beta(m_0) = l$, this implies that the map $\varphi_{\beta\alpha}$ is given by

$$y_j = u', \ u = (u')^l z^{m_0}, \ y_i = z_i z^{-\alpha_i m_0}, \quad \text{in case } (a),$$
$$y_j = (u')^{l'} z_j, \ u = u', \ y_i = z_i, \quad \text{in case } (b),$$

and therefore is a morphism.

Take a monomial $m \in f$; we have

$$m \in f_\beta \quad \text{iff} \quad \begin{cases} \text{either} & m \in f_\alpha \text{ and } m_j = l, \\ \text{or} & m \in f_{>\alpha}^{(j)} \text{ and } \alpha(m) = \alpha(f) + 1. \end{cases}$$

This shows, in particular, that $\dim \Gamma_\beta \geq 1$; let E' be any essential component of $E(\beta)$. The morphism $\varphi_{\beta\alpha}$ gives an injective morphism

$$\varphi_{\beta\alpha}^* \colon A \to B,$$

where $B := \mathcal{O}_{X(\beta),E'}$ is the localisation of the ring of regular function on $X(\beta)$ at the generic point of E'. Therefore B dominates A for any choice of E', which gives the desired isomorphism. This completes the proof. $\square$

3.7 A formula for $c(X)$

Let $0 \in X : (f = 0) \subset \mathbb{C}^{n+1}$ be an isolated nondegenerate singularity, $\alpha \in \sigma^\circ \cap N$ a primitive weighting, and

$$f_\alpha = \prod_{i=1}^{n+1} x_i^{l_i} \prod_{j=1}^{c(\alpha)} g_j$$

the decomposition of f_α into irreducible components. From Theorem 3.11 it follows that

$$c(X) = \sum_{\alpha \in W(f)} c(\alpha).$$

This formula becomes more precise once we notice that

$$c(\alpha) = \begin{cases} \operatorname{length} \Gamma_\alpha & \text{if } \dim \Gamma_\alpha = 1 \\ 1 & \text{if } \dim \Gamma_\alpha \geq 2. \end{cases}$$

For the case $\dim \Gamma_\alpha = 1$ see e.g. [O, Lemma 4.8.], while if $\dim \Gamma_\alpha \geq 2$ it is enough to view $(f_\alpha = 0)$ as a hypersurface in the torus T_{Γ_α}, and then apply for instance the toric Lefschetz property proved in [DKh]. Thus we have proved the following:

Corollary 3.12 *Let $0 \in X : (f = 0) \subset \mathbb{C}^{n+1}$ be an isolated canonical nondegenerate singularity. Then the number of crepant divisors $c(X)$ is given by*

$$c(X) = \sum_{\substack{\alpha \in W(f) \\ \dim \Gamma_\alpha = 1}} \operatorname{length} \Gamma_\alpha + \# \{\alpha \in W(f) : \dim \Gamma_\alpha \geq 2.\}$$

Chapter 4

The cohomology of a minimal model

4.1 Aims of this chapter

Let X be an algebraic 3-fold with canonical singularities. By a theorem of Reid [R2], there exists a minimal model Y' of X with terminal algebraically $\mathbb{Q}$-factorial singularities.

A given 3-fold X may have many different minimal models. Nevertheless, they are closely related, and many of their invariants are independent of the choice. This is the content of the next result, due to Kollár.

Theorem 4.1 ([K1],[K2]) *Let Y' be a 3-fold minimal model with terminal algebraically $\mathbb{Q}$-factorial singularities. The following objects are independent of the choice of Y':*

(1) *The intersection cohomology groups $IH^k(Y', \mathbb{C})$, together with their Hodge structures.*

(2) *The integer cohomology groups $H^k(Y', \mathbb{Z})$.*

(3) *The collection of analytic singularities of Y'.*

(4) *The divisor class group $\operatorname{Cl} Y'$ and the Picard group $\operatorname{Pic} Y'$.*

If Y is a minimal model of X with terminal analytically $\mathbb{Q}$-factorial singularities, which exists by [R2], then $IH^k(Y', \mathbb{Q}) = H^k(Y, \mathbb{Q})$, as remarked in Section 1.1; denote by $b_k(Y) = \dim H^k(Y, \mathbb{Q})$ the Betti numbers of Y.

The goal of this chapter is two-fold.

Firstly, given a germ (X, x) of an isolated 3-dimensional canonical singularity and a minimal model Y of X, assumed throughout this chapter to have only terminal analytically $\mathbb{Q}$-factorial singularities, to determine $b_2(Y)$ and $b_4(Y)$. The answer we obtain becomes effective together with the results from the previous chapters.

Secondly, and this occupies most of this chapter, to find methods to calculate the remaining interesting Betti number, namely $b_3(Y)$. For this, an explicit analysis of how the topology changes under a birational morphism is necessary.

4.2 A basic exact sequence

Let $\psi\colon Z' \to Z$ be a proper birational morphism between algebraic varieties or analytic spaces, such that for $C \subset Z$ and $E = \psi^{-1}(C)$ we have $Z' \setminus E \cong Z \setminus C$. There is then a diagram

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & H^k(Z', E; \mathbb{Z}) & \longrightarrow & H^k(Z', \mathbb{Z}) & \longrightarrow & H^k(E, \mathbb{Z}) & \longrightarrow & \\
 & & \cong\uparrow & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & H^k(Z, C; \mathbb{Z}) & \longrightarrow & H^k(Z, \mathbb{Z}) & \longrightarrow & H^k(C, \mathbb{Z}) & \longrightarrow &
\end{array}$$

$$\begin{array}{cccc}
\longrightarrow & H^{k+1}(Z', E; \mathbb{Z}) & \longrightarrow & \cdots \\
 & \cong\uparrow & & \\
\longrightarrow & H^{k+1}(Z, C; \mathbb{Z}) & \longrightarrow & \cdots
\end{array}$$

where the rows are the long exact sequences of the corresponding pairs, and the vertical arrows at the ends are isomorphisms. This leads to the following long exact sequence in reduced cohomology

$$\begin{aligned}\cdots \to H^k(Z, \mathbb{Z}) \to H^k(Z', \mathbb{Z}) \oplus H^k(C, \mathbb{Z}) \to H^k(E, \mathbb{Z}) \to \\ H^{k+1}(Z, \mathbb{Z}) \to \cdots\end{aligned} \tag{4.1}$$

If Z and Z' are algebraic varieties, their cohomology groups carry mixed Hodge structures and the sequence (4.1) becomes a sequence of mixed Hodge structures [De, 8.3.9 and 8.3.10].

This sequence is the basic tool in studying how the cohomology groups change under a morphism ψ as above. For instance, in the smooth case, it is known that the exact sequence (4.1) breaks into short exact sequences

$$0 \to H^k(Z, \mathbb{Z}) \to H^k(Z', \mathbb{Z}) \oplus H^k(C, \mathbb{Z}) \to H^k(E, \mathbb{Z}) \to 0$$

(see e.g. [GH, p. 605]). The same holds for algebraic 3-folds with isolated canonical analytically $\mathbb{Q}$-factorial singularities (cf. Proposition 4.3), but is false in general (cf. the examples in Section 4.8).

4.3 A "model case"

The following result from [K1] relates topological and geometric properties of a threefold with canonical singularities.

Proposition 4.2 *Let X be a threefold with at most isolated canonical singularities. Then X is a rational homology manifold if and only if its singular points are analytically $\mathbb{Q}$-factorial.*

As a consequence of the previous result and of the fact that (4.1) is a sequence of mixed Hodge structures if Z and Z' are algebraic varieties, we obtain the following:

Proposition 4.3 *Let $\psi\colon Z' \to Z$ be as in Section 4.2. Suppose that Z and Z' are algebraic threefolds and Z has at most isolated canonical analytically $\mathbb{Q}$-factorial singularities. Then the exact sequence (4.1) breaks into short exact sequences*

$$0 \to H^k(Z,\mathbb{Q}) \to H^k(Z',\mathbb{Q}) \oplus H^k(C,\mathbb{Q}) \to H^k(E,\mathbb{Q}) \to 0.$$

Proof Indeed, Z is a $\mathbb{Q}$-homology manifold by Proposition 4.2 and therefore $W_{k-1}H^k(Z,\mathbb{Q}) = 0$ (i.e., $H^k(Z,\mathbb{Q})$ has weights at least k) for any k ([De, Théorème 8.2.4]).

Since E is projective, we also have $W_kH^k(E,\mathbb{Q}) = H^k(E,\mathbb{Q})$ that is, $H^k(E,\mathbb{Q})$ has weights at most k, for any k, again by [De, Théorème 8.2.4]. This shows that the morphisms

$$H^k(E,\mathbb{Q}) \to H^{k+1}(Z,\mathbb{Q}),$$

which are morphisms of mixed Hodge structures, are all zero, hence the conclusion. □

Let (X,x) be a germ of 3-dimensional canonical singularity, $c = c(X)$ the number of crepant divisors of X. Suppose that we have a chain of crepant morphisms

$$Y = X_c \xrightarrow{\varphi_c} X_{c-1} \longrightarrow \cdots \longrightarrow X_1 \xrightarrow{\varphi_1} X_0 \xrightarrow{\varphi_0} X$$

from a minimal model Y of X, where φ_0 is an analytic $\mathbb{Q}$-factorialisation of (X, x), assumed to be an isomorphism if (X, x) is already $\mathbb{Q}$-factorial, and φ_i for $i \geq 1$ are divisorial contractions, contracting the divisor E_i corresponding to the crepant valuation v_i.

The next result is a "model theorem" for the 3rd Betti number of Y; under strong hypotheses, we obtain a formula expressing $b_3(Y)$ as the sum of the 3rd Betti numbers of arbitrary nonsingular representatives of the crepant valuations. Although these assumptions are too strong, there are examples satisfying them (see Section 4.7); moreover, there are certain obstructions to extend the result below to more general singularities (see the discussion at the beginning of Section 4.9).

Theorem 4.4 *With the above assumptions and notation, suppose that the partial resolutions X_i have isolated locally analytically $\mathbb{Q}$-factorial singularities, and the φ_i-exceptional divisors E_i are normal for $i = 1, \dots, c$. Then*

$$b_3(Y) = \sum_{i=1}^{c} b_3(\widetilde{E}_i), \tag{4.2}$$

where $\widetilde{E}_i$ is any nonsingular representative of the valuation v_i.

Proof Apply Proposition 4.3 to each morphism φ_i for $1 \leq i \leq c$, and notice that $b_3(X_0) = 0$. $\square$

Remark 4.5 The above argument shows that $b_3(Y) = \sum_{i=1}^{c} b_3(E_i)$. The assumption about the normality of E_i is only made to emphasise the fact that we are looking for a similar birational invariant statement in the general case (see Conjecture 4.23 at the end of this chapter).

4.4 Some lemmas

Lemma 4.6 *Let $(Z, E) \to (X, x)$ be a birational morphism. Take a common resolution of Z and X, $\varphi\colon Z' \to Z$, $\psi\colon Z' \to X$ and suppose that $R^1\varphi_*\mathcal{O}_{Z'} = R^1\psi_*\mathcal{O}_{Z'} = 0$. Then $H^1(Z, \mathbb{Z}) = 0$.*

Proof The exponential exact sequence $0 \to \mathbb{Z}_{Z'} \to \mathcal{O}_{Z'} \to \mathcal{O}^*_{Z'} \to 0$ gives

$$0 \to R^1\varphi_*\mathbb{Z}_{Z'} \to R^1\varphi_*\mathcal{O}_{Z'} \to \cdots$$

and this yields $R^1\varphi_*\mathbb{Z}_{Z'} = 0$. From the Leray spectral sequence of the morphism φ we obtain

$$0 \to H^1(Z,\mathbb{Z}) \to H^1(Z',\mathbb{Z}) \to H^0(Z, R^1\varphi_*\mathbb{Z}_{Z'}) = 0,$$

hence the isomorphism $H^1(Z,\mathbb{Z}) \cong H^1(Z',\mathbb{Z})$.

The above argument applied to the morphism ψ shows that

$$H^1(Z',\mathbb{Z}) \cong H^1(X,\mathbb{Z}),$$

and, since $H^1(X,\mathbb{Z}) = 0$, this proves the lemma. $\square$

Remark 4.7 In [SB1] it is shown that if (X,x) is an isolated canonical 3-fold singularity and $Y \to X$ a crepant morphism from Y with terminal singularities, then Y is, in fact, simply connected.

From now on fix a 3-dimensional canonical singularity (X,x); we take X to be a representative of the germ which is Stein and topologically contractible.

Lemma 4.8 *Let Z be a partial crepant resolutions of X. Then* $\operatorname{Pic} Z \cong H^2(Z,\mathbb{Z})$.

Proof Take a common resolution Z' of X and Z as in the following diagram:

$$\begin{tikzcd} Z' \arrow[rr, "\varphi"] \arrow[dr, "\psi"'] & & Z \arrow[dl] \\ & X & \end{tikzcd}$$

Since Z has canonical singularities, it follows from the Leray spectral sequence associated to φ that, for each $i \geq 0$,

$$\begin{aligned} H^i(Z',\mathcal{O}_{Z'}) &= H^i(Z,\varphi_*\mathcal{O}_{Z'}) \\ &= H^i(Z,\mathcal{O}_Z). \end{aligned}$$

The same applies to ψ, giving, for each $i > 0$,

$$\begin{aligned} H^i(Z',\mathcal{O}_{Z'}) &= H^i(X,\psi_*\mathcal{O}_{Z'}) \\ &= H^i(X,\mathcal{O}_X) \\ &= 0. \end{aligned}$$

The conclusion now follows from the exponential sequence

$$H^1(Z, \mathcal{O}_Z) \to \operatorname{Pic} Z \to H^2(Z, \mathbb{Z}) \to H^2(Z, \mathcal{O}_Z).$$

$\square$

Remark 4.9 If X is algebraic, then the same proof shows that

$$\operatorname{Pic}^{\mathrm{alg}} Z \cong H^2(Z, \mathbb{Z}),$$

where $\operatorname{Pic}^{\mathrm{alg}} Z$ is the algebraic Picard group of Z.

Lemma 4.10 *Let $\varphi\colon Z' \to Z$ be a morphism between partial crepant resolutions of X. Then the natural maps*

$$\varphi^2 \colon H^2(Z, \mathbb{Z}) \to H^2(Z', \mathbb{Z})$$
$$\varphi^4 \colon H^4(Z, \mathbb{Z}) \to H^4(Z', \mathbb{Z})$$

are injective.

Proof By Lemma 4.8, the first injection is equivalent to $\varphi^*\colon \operatorname{Pic} Z \hookrightarrow \operatorname{Pic} Z'$. But this is clearly injective since

$$\varphi_*(\varphi^* \mathcal{L}) = \mathcal{L} \quad \text{for any} \quad \mathcal{L} \in \operatorname{Pic} Z.$$

For the second injection, it is enough to notice that, if E denotes the exceptional set of the morphism $Z \to X$, and E_i its prime divisor components, then

$$\begin{aligned} H^4(Z, \mathbb{Z}) &\cong H^4(E, \mathbb{Z}) \\ &\cong \bigoplus H^4(E_i, \mathbb{Z}), \end{aligned}$$

where the last identification follows from Lefschetz duality [Sp, p. 297].

The same holds for Z' and, if E_i' denotes the proper transform of E_i on Z', then

$$H^4(E_i, \mathbb{Z}) \cong H^4(E_i', \mathbb{Z}).$$

$\square$

Let $\psi\colon Z' \to Z$ be a morphism such that $Z' \setminus E \cong Z \setminus C$ as in Section 4.2. Suppose that Z and Z' are partial crepant resolutions of X. Then, by Lemma 4.10, the exact sequence (4.1) becomes

$$\begin{aligned} 0 &\to H^2(Z,\mathbb{Z}) \to H^2(Z',\mathbb{Z}) \oplus H^2(C,\mathbb{Z}) \to H^2(E,\mathbb{Z}) \\ &\to H^3(Z,\mathbb{Z}) \to H^3(Z',\mathbb{Z}) \to H^3(E,\mathbb{Z}) \to 0. \end{aligned}$$

In particular, we have

$$b_3(Z') = b_3(Z) + b_3(E) + b_2(Z') - b_2(Z) + b_2(C) - b_2(E). \tag{4.3}$$

In some examples, this information is enough to calculate the 3rd Betti number of a minimal model (see Sections 4.7 and 4.8).

4.5 The 2nd and 4th Betti numbers of a minimal model

There are several differences between the algebraic and analytic category related to divisor class groups. For algebraic varieties, there is a surjection

$$\operatorname{Cl} Z \to \operatorname{Cl}(Z \setminus W) \to 0$$

given by restricting divisors on Z to $Z \setminus W$, for any proper closed subset W of Z (see e.g. [Ha, p. 133]). If (Z, W) is a pair consisting of an analytic space and its compact subset, this is no longer true, the reason being the difference between topologies.

The following result from [BS] gives an instance when this difference between the algebraic and the analytic case does not occur.

Proposition 4.11 *Let $\psi\colon Z' \to Z$ be a proper birational morphism between algebraic varieties (resp. analytic spaces), and $C \subset Z$ an algebraic (resp. analytic) set of codimension at least 2 such that $Z' \setminus \psi^{-1}(C) \cong Z \setminus C$. Let K be the subgroup of* $\operatorname{WDiv} Z'$ *with support contained in $\psi^{-1}(C)$. There is then a diagram with exact rows*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & \operatorname{WDiv} Z' & \xrightarrow{\psi_*} & \operatorname{WDiv} Z & \longrightarrow & 0 \\ & & \cong\downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K & \longrightarrow & \operatorname{Cl} Z' & \xrightarrow{\psi_*} & \operatorname{Cl} Z & \longrightarrow & 0 \end{array}$$

Moreover, if Z is algebraic or compact analytic, then K is a finitely generated free group.

In Chapter 2 and Chapter 3 we obtained formulas for the divisor class number $\rho(X)$ and the number of crepant divisors $c(X)$ for a large class of isolated canonical singularities $0 \in X : (f = 0) \subset \mathbb{C}^4$. Together with the following result, these formulas often calculate the 2nd and 4th Betti numbers of a minimal model of X.

Theorem 4.12 *Let (X, x) be 3-fold isolated canonical singularity and let $\varphi\colon Y \to X$ be a minimal model. Then the Betti numbers of Y other than $b_3(Y)$ are given by:*

(1) $b_0(Y) = 1$;

(2) $b_1(Y) = b_5(Y) = b_6(Y) = 0$;

(3) $b_4(Y) = c(X)$;

(4) $b_2(Y) = \rho(X) + c(X)$.

Proof The exceptional set $E = \varphi^{-1}(x)$ is connected, since X is irreducible. The retraction $X \searrow x$ coming from the cone structure of X can be lifted via φ to a retraction $Y \searrow E$. This, together with Lemma 4.6, gives (1) and (2).

If E_i are the prime divisor components of E for $1 \le i \le c(X)$, then

$$\begin{aligned} H^4(Y, \mathbb{Z}) &\cong H^4(E, \mathbb{Z}) \\ &\cong \bigoplus_{i=1}^{c(X)} H^4(E_i, \mathbb{Z}), \end{aligned}$$

which shows (3).

Finally, to prove (4), look at the short exact sequence

$$0 \to K \to \operatorname{Cl} Y \to \operatorname{Cl} X \to 0,$$

of (analytic) divisor class groups from Proposition 4.11. The result follows once we show that the kernel K is equal to $\bigoplus_{i=1}^{c(X)} \mathbb{Z}[E_i]$, that is, we have an exact sequence

$$0 \to \bigoplus_{i=1}^{c(X)} \mathbb{Z}[E_i] \to \operatorname{Cl} Y \to \operatorname{Cl} X \to 0.$$

To see this, it suffices to prove that the components E_i are linearly independent in $\operatorname{Cl} Y$. Since (X, x) is an isolated singularity, we can assume that X

and Y are algebraic. Take then a general hyperplane section H on Y; the curves $C_i = E_i \cap H$ are exceptional on the nonsingular surface H, hence the matrix

$$(C_i \cdot C_j)$$

is negative definite. □

Remark 4.13 The above argument about the components E_i being linearly independent in $\operatorname{Cl} Y$ also shows that, given any crepant partial resolution $\psi\colon Z \to X$, the prime divisor components E_i' of $\psi^{-1}(x)$ are linearly independent in $\operatorname{Cl} Z$ as well.

To see this, notice that, by Proposition 4.11, $\operatorname{Cl} Z$ does not change under a $\mathbb{Q}$-factorialisation; thus we can assume that Z is $\mathbb{Q}$-factorial. Regard now an exceptional divisor D on Z as an element of $\operatorname{Pic} Z$. Then, if $\varphi\colon Y \to Z$ is a minimal model of Z and D' denotes the proper transform of D on Y, the equality

$$\varphi^*(D) = D' + \sum a_i E_i,$$

where the E_i are now φ-exceptional, reduces the problem to Y, where it is already solved.

4.6 The cohomology of weighted surfaces

For the remainder of this chapter we are interested in calculating the 3rd Betti number of a minimal model of X.

This section collects together for future use (see the examples in Sections 4.7 and 4.8) known facts about the topology of weighted projective surfaces

$$S : (h = 0) \subset \mathbb{P}(\alpha),$$

where α_i are positive integers and $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$. We are mainly interested in the third cohomology group of S.

Let $G = \mu_{\alpha_1} \times \mu_{\alpha_2} \times \mu_{\alpha_3} \times \mu_{\alpha_4}$. The ramified covering $\pi\colon \mathbb{P}^3 \to \mathbb{P}(\alpha)/G$, given by $\pi(x, y, z, t) = (x^{\alpha_1}, y^{\alpha_2}, z^{\alpha_3}, t^{\alpha_4})$ can provide informations about the topology of S in terms of the topology of $\pi^{-1}(S)$ via the relation

$$H^*(S, \mathbb{Q}) = H^*(\pi^{-1}(S), \mathbb{Q})^G \tag{4.4}$$

(see e.g. [Di, Appendix B]).

By the weak Lefschetz Theorem (see e.g. [Di, Appendix B]), $H^1(S, \mathbb{Q}) = 0$. For the higher groups we discuss some special cases that will be useful for the examples in the following sections:

- $h = h(x, y, z, t)$

If S is quasismooth, i.e., if the vertex of the affine cone over S is an isolated singularity, then S is a V-manifold, hence a $\mathbb{Q}$-homology manifold (see e.g. [Di, Appendix B]), therefore $H^k(S, \mathbb{Q}) = H^k(\mathbb{P}(\alpha), \mathbb{Q})$ for $k \neq 2$ by Lefschetz and Poincaré duality. The dimension of the interesting cohomology group $H^2(S)$ is given by

$$b_2(S) = \#\{m \in B : \alpha(m + \mathbf{1}) = 2\alpha(f)\} + 1,$$

where B is a set of monomials whose residue classes in $M(f)$ form a basis (see [S2], [Do, 4.3.2 and 4.3.3] and Theorem 2.12).

- $h = h(x, y, z)$

In this case S is a weighted projective cone over the curve

$$C : (h(x, y, z) = 0) \subset \mathbb{P}(\alpha_1, \alpha_2, \alpha_3).$$

The topology of the usual projective cones is known. Their cohomology groups are given by

$$H^k(S, \mathbb{Z}) \cong H^{k-2}(C, \mathbb{Z}) \quad \text{for } k \geq 2$$

(see e.g. [Di, Chapter 5, (4.18) and (4.19)]). The same holds in the weighted case if we work with $\mathbb{Q}$-coefficients. This is immediate, since (4.4) implies

$$H^k(S, \mathbb{Q}) \cong H^k(S', \mathbb{Q})^G \cong H^{k-2}(C', \mathbb{Q})^G \cong H^{k-2}(C, \mathbb{Q}),$$

where $S' := \pi^{-1}(S)$ and $C' := \pi^{-1}(C)$. In particular, $b_3(S) = b_1(C)$ and, if C is smooth, then $b_1(C) = g(C)$ can be calculated in terms of the weights α_i by the following formula from [Do, 3.5.2]:

$$g(C) = \text{coefficient of } t^{\alpha(h) - \alpha(\mathbf{1})} \text{ in } \frac{1 - t^{\alpha(h)}}{\prod(1 - t^{\alpha_i})}. \tag{4.5}$$

- $h = h(x, y)$

The third Betti numbers of these surfaces are zero, since $h = h(x, y)$ can be written as a product of homogeneous linear factors in x^γ and y^δ, for some γ and δ.

We end this section by discussing how the cohomology of a surface S changes under a proper modification $S' \to S$. Suppose that $S' \setminus C' \cong S \setminus C$. Then the exact sequence (4.1) gives a surjection

$$H^3(S, \mathbb{Z}) \to H^3(S', \mathbb{Z}) \to 0,$$

which shows that, in general, $b_3(S)$ drops under a proper modification. It is known that the third Betti number is a birational invariant in the class of projective surfaces with only isolated singularities [BK, Corollaire 3.D.2]. In particular this implies that, having a morphism $S' \to S$ as above, with S normal,

$$b_3(S') = b_3(S).$$

It is also shown in [BK] that, for a normal surface S, one has

$$b_3(S) = 0 \text{ iff } S \text{ is rational.}$$

4.7 Examples with normal exceptional divisors

In all the examples in this section and in the next one, the approach to calculate the 3rd cohomology of a minimal model is "bottom-up". We start with X and then make explicit blowups leading to Y; at each step $Z' \to Z$, we use formula (4.3), Proposition 4.11 and the material in Section 4.6 to relate the cohomologies of Z and Z'. Finally we use the knowledge of the 2nd and the 4th Betti numbers of Y from Theorem 4.12 and the formulas from Chapters 2 and 3. All the examples we consider are nondegenerate, and, after calculating $b_3(Y)$, we also give a toric interpretation of the result.

The singularities $(X, 0)$ in this section are quasihomogeneous; their various partial resolutions have only isolated quasihomogeneous singularities, as well. Therefore, in these cases, there is no subtlety related to the difference between algebraic and analytic divisor classes. For $(X, 0)$ factorial, these examples are also instances where the strong assumptions of Theorem 4.4 are satisfied.

Example 4.14 Let $0 \in X : (f = 0) \subset \mathbb{C}^4$, where $f = x^2 + y^3 + z^6 + t^n$ with $n \geq 6$, and Y a minimal model of X (see also Example 3.1). We want to show that

$$b_3(Y) = \begin{cases} 2([n/6]-1) & \text{if } n \equiv 0 \bmod 6 \\ 2[n/6] & \text{otherwise.} \end{cases} \tag{4.6}$$

The number of crepant divisors and the Picard number of X are given by $c(X) = [n/6]$ and

$$\rho(X) = \begin{cases} 0 & \text{if } n \equiv \pm 1 \bmod 6 \\ 2 & \text{if } n \equiv \pm 2 \bmod 6 \\ 4 & \text{if } n \equiv 3 \bmod 6 \\ 8 & \text{if } n \equiv 0 \bmod 6. \end{cases}$$

Let $\alpha = (3,2,1,1)$ and $X_1 = X(\alpha)$ the α-blowup of X. This is a crepant morphism and, on one affine piece

$$X_1 : x^2 + y^3 + z^6 + t^{n-6} = 0.$$

Blowing up again, we obtain a chain of crepant morphisms

$$X_c \to X_{c-1} \to \cdots \to X_1 \to X,$$

where $X_{i+1} = X_i(\alpha)$, and X_c has at most terminal singularities, $c = c(X)$. The exceptional divisor of the morphism $X_{i+1} \to X_i$ is given by

$$\begin{cases} x^2 + y^3 + z^6 + t^6 = 0 \subset \mathbb{P}(3,2,1,1) & \text{if } i = c-1 \text{ and } n \equiv 0 \bmod 6 \\ x^2 + y^3 + z^6 = 0 \subset \mathbb{P}(3,2,1,1) & \text{otherwise.} \end{cases}$$

Therefore (see Section 4.6)

$$b_2(E_{i+1}) = \begin{cases} 9 & \text{if } i = c-1 \text{ and } n \equiv 0 \bmod 6 \\ 1 & \text{otherwise,} \end{cases} \tag{4.7}$$

$$b_3(E_{i+1}) = \begin{cases} 0 & \text{if } i = c-1 \text{ and } n \equiv 0 \bmod 6 \\ 2 & \text{otherwise.} \end{cases} \tag{4.8}$$

Let $\varphi \colon X_i \to X$ be the composite morphism. Its exceptional divisor E' is a chain of elliptic ruled surfaces, ending with a rational surface for $i = c$ and $n \equiv 0 \bmod 6$, and with a cone over an elliptic curve otherwise. If E_i' denote its irreducible components, $1 \le i \le c(X) - c(X_i)$, the short exact sequence

$$0 \to \bigoplus \mathbb{Z}[E_i'] \to \operatorname{Cl} X_i \to \operatorname{Cl} X \to 0$$

coming from Proposition 4.11 and Remark 4.13 shows that $\operatorname{rank}\operatorname{Cl}X_i = \rho(X) + c(X) - c(X_i)$. Apart from the case $i = c$ and $n \equiv 0 \bmod 6$, we have $\sigma(X_i) = \rho(X)$. Thus

$$\begin{aligned} b_2(X_i) &= \operatorname{rank}\operatorname{Pic}X_i \\ &= c(X) - c(X_i). \end{aligned} \tag{4.9}$$

In the remaining case $n \equiv 0 \bmod 6$ the variety X_c is $\mathbb{Q}$-factorial (which is the same as locally $\mathbb{Q}$-factorial by the observation before this example), and

$$\begin{aligned} b_2(X_c) &= \operatorname{rank}\operatorname{Pic}X_c \\ &= \operatorname{rank}\operatorname{Cl}X_c \\ &= c(X) + \rho(X). \end{aligned} \tag{4.10}$$

From (4.3), (4.7), (4.9), (4.10) and (4.8) it follows that

$$\begin{aligned} b_3(X_c) &= \sum_{i=1}^{c} b_3(E_i) \\ &= \begin{cases} 2(c-1) & \text{if } i = c-1 \text{ and } n \equiv 0 \bmod 6 \\ 2c & \text{otherwise.} \end{cases} \end{aligned}$$

A minimal model is obtained from X_c by a $\mathbb{Q}$-factorialisation. For $n \equiv 0$, 1 or $5 \bmod 6$ X_c is already $\mathbb{Q}$-factorial, while in the remaining cases it is not. To prove the claim, we are left with showing that, in this example, b_3 does not change under a $\mathbb{Q}$-factorialisation. But this is clear since, if C denotes the exceptional curve, $b_2(Y) = \rho(X) + c(X)$, $b_2(X_c) = c(X)$, $b_2(C) = \rho(X)$ and, again by (4.3),

$$b_3(Y) = b_3(X_c) + b_2(Y) - b_2(X_c) - b_2(C).$$

Formula (4.6) can also be interpreted in toric terms. The set of crepant weightings is

$$W(f) = (3k, 2k, k, 1),\ 1 \leq k \leq [n/6]$$

and the corresponding divisors are isomorphic to $(x^2 + y^3 + z^6 + t^6 = 0) \subset \mathbb{P}(3,2,1,1)$ if $n = 6k$, and $(x^2 + y^3 + z^6 = 0) \subset \mathbb{P}(3,2,1,1)$ otherwise (see Section 3.6).

Example 4.15 The singularity defined by $f = x^3 + y^3 + z^3 + t^n = 0$ for $n \geq 3$ can be treated in a similar way. A minimal model Y can be obtained in this case by a sequence of usual blowups, followed, for $n \equiv 2 \bmod 3$, by a $\mathbb{Q}$-factorialisation. Calculations as in the previous example show that

$$b_3(Y) = \begin{cases} 2([n/3]-1) & \text{if } n \equiv 0 \bmod 3 \\ 2[n/3] & \text{otherwise.} \end{cases}$$

The other Betti numbers of Y are $b_2(Y) = \rho(X) + c(X)$ and $b_4(Y) = c(X)$, where

$$\begin{aligned} c(X) &= [n/3], \\ \rho(X) &= \begin{cases} 6 & \text{if } n \equiv 0 \bmod 3 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

In particular the Euler number of Y is given by

$$e(Y) = \begin{cases} 9 & \text{if } n \equiv 0 \bmod 3 \\ 1 & \text{otherwise.} \end{cases}$$

Example 4.16 ([G]) Let $0 \in X : (f=0) \subset \mathbb{C}^4$ be the singularity defined by $f = x^3 + y^4 + z^4 + t^4$, and Y a minimal model of X. Then

$$b_3(Y) = 12.$$

To see this, let $X_1 = \operatorname{Bl}_0 X \to X$ be the blowup of X. This is a crepant morphism and the blown up variety X_1 is singular along the quartic curve of A_2-singularities

$$C : (y^4 + z^4 + t^4 = 0) \subset \mathbb{P}^2$$

of genus $g(C) = 3$; the corresponding exceptional divisor $E_1 : (x^3 = 0) \subset \mathbb{P}^3$ is a triple plane. For instance, on one affine piece, X_1 is given by

$$x^3 + t(y^4 + z^4 + 1) = 0.$$

Blowing up X_1 along its singular curve C, leads to a nonsingular variety Y. Again the morphism $Y = \operatorname{Bl}_C X_1 \to X_1$ is crepant and, since the term $t(y^4 + z^4 + 1)$ splits, its exceptional locus E is a union of two $\mathbb{P}^1$-bundles, E_2 and E_3, over C. The singularity $0 \in X$ is factorial thus, by Theorem 4.12, $b_2(Y) = 3$. Since $b_2(X_1) = 1$, $b_2(C) = 1$ and $b_2(E) = 3$, it follows by (4.3) that

$$\begin{aligned} b_3(Y) &= b_3(X_1) + b_3(E) \\ &= 4g(C) \\ &= 12. \end{aligned}$$

The toric interpretation of this formula is as follows. In this case

$$W(f) = \{(1,1,1,1), (2,1,1,1), (3,2,2,2), (4,3,3,3)\},$$

and the essential divisors are the plane $E_1' : (x^3 + y^4 + z^4 + t^4 = 0) \subset \mathbb{P}(4,3,3,3)$, together with the two projective cones $E_2' : (y^4 + z^4 + t^4 = 0) \subset \mathbb{P}(2,1,1,1)$ and $E_3' : (y^4 + z^4 + t^4 = 0) \subset \mathbb{P}(3,2,2,2)$. Notice that

$$b_3(Y) = \sum b_3(E_i').$$

4.8 Examples with nonnormal exceptional divisors

In the next example the singularity (X, x) is factorial and its blowup X_1 is globally factorial, but not locally factorial near its singular point $p \in X_1$. If Y is a $\mathbb{Q}$-factorialisation of X_1 at p, then the global $\mathbb{Q}$-factoriality of X_1 implies that $b_2(Y) = b_2(X_1)$, while b_3 changes under this morphism precisely by $\rho(p)$.

Example 4.17 We first recall from Example 2.15 that the singularity $0 \in X : (f = 0) \subset \mathbb{C}^4$ given by $x^3 + x^2z + y^2z + z^4 + t^5$ is factorial. Its blowup $\varphi_1 \colon X_1 \to X$ has one singular point p and the divisor class number of X_1 at this point is 1. The exceptional divisor $E = \varphi_1^{-1}(0)$, given by

$$E : (x^3 + x^2z + y^2z = 0) \subset \mathbb{P}^3$$

is a projective cone over the nodal curve $(x^3 + x^2z + y^2z = 0) \subset \mathbb{P}^2$; thus $b_2(X_1) = b_3(X_1) = 1$. Since $p \in X_1$ is a terminal singularity, $\varphi \colon Y \to X_1$, the $\mathbb{Q}$-factorialisation of X_1 at p, is a minimal model of X. Its Betti numbers are $b_2(Y) = b_4(Y) = 1$ and, if $\widetilde{\Sigma} = \varphi^{-1}(p)$, it follows by (4.3) that

$$\begin{aligned} b_3(Y) &= b_3(X_1) + b_2(Y) - b_2(X_1) - b_2(\widetilde{\Sigma}) \\ &= 0. \end{aligned}$$

Note that $b_3(Y)$ does not coincide to $b_3(E)$, but to b_3 of the normalisation of E.

The neighbourhood X_1 of E gives an example of a variety which is globally analytically $\mathbb{Q}$-factorial, but is not locally analytically $\mathbb{Q}$-factorial (see Section 1.2.5).

The next example is based on an idea from [C]. We start with a singularity (X, x) which is factorial iff $\operatorname{hcf}(n, 4) = 1$. Its blowup X_1 has 3 planes which are not $\mathbb{Q}$-Cartier. Under a $\mathbb{Q}$-factorialisation of X_1 at $\operatorname{Sing} X_1$, both the 2nd and the 3rd Betti numbers change, b_2 by $\sigma(X_1)$, and b_3 by $\sum_{p\in\mathrm{SingX_1}} \rho(p) - \sigma(X_1)$.

Example 4.18 The polynomial $f = xyz + x^n + y^n + z^n + t^4$ for $n \geq 4$, defines an isolated canonical singularity at 0. Its local divisor class number and number of crepant divisors are

$$\begin{aligned} \rho(X) &= 3(\mathrm{hcf}(n,4) - 1), \\ c(X) &= 3. \end{aligned}$$

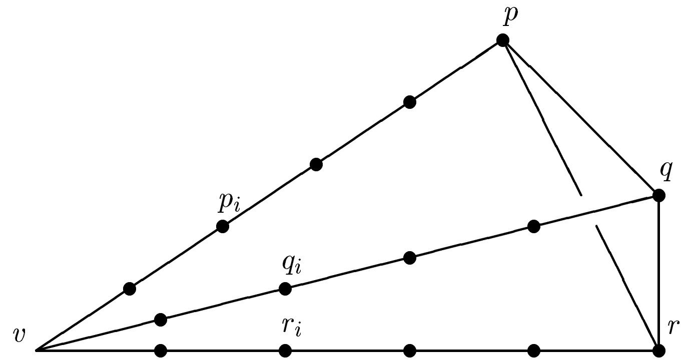
The blowup of X at 0 is a crepant morphism from $X_1 = \mathrm{Bl}_0 X$ with terminal singularities. Its exceptional divisor E is the projective cone over the curve $C : (xyz = 0) \subset \mathbb{P}^2$, therefore

$$\begin{aligned} b_3(X_1) &= b_1(C) = 1, \\ b_4(X_1) &= b_2(C) = 3. \end{aligned}$$

Since the 3 components of E are not $\mathbb{Q}$-Cartier, we also have

$$\begin{aligned} \sigma(X_1) &= \rho(X) + 2 \\ &= 3\,\mathrm{hcf}(n,4) - 1. \end{aligned}$$

For $n = 4$, the singular locus $\Sigma = \operatorname{Sing} X_1$ (in a neighbourhood of E) consists of 12 ordinary double points, p_i, q_i and r_i, as in the following figure:



The 3 components of E pass through $v = (0 : 0 : 0 : 1)$, and the singular points lie on the lines out of v.

Let $\varphi \colon Y \to X_1$ be a $\mathbb{Q}$-factorialisation of X_1 at Σ, obtained, for instance, by blowing up the divisors which are not $\mathbb{Q}$-Cartier in some chosen order.

Then Y is a minimal model of X and $\widetilde{\Sigma} = \varphi^{-1}(\Sigma)$ is a union of 12 lines. Therefore

$$b_2(Y) = 12, \quad b_4(Y) = 3 \quad \text{and} \quad b_3(Y) = 0.$$

For $n \geq 5$ the blown up variety X_1 has 3 singular points on E. These are the vertices $p = (1:0:0:0)$, $q = (0:1:0:0)$ and $r = (0:0:1:0)$ of the triangle C. The (local analytic) divisor class numbers of X_1 at these points are

$$\rho(p) = \rho(q) = \rho(r) = \operatorname{hcf}(n, 4).$$

For instance, p belongs to the first affine piece of X_1, which is given by

$$yz + x^{n-3}(1 + y^n + z^n) + xt^4 = 0,$$

and, near p, this equation is equivalent to $yz + x(x^{n-4} + t^4) = 0$. With the same notation as in the case $n = 4$, it follows that $b_2(\widetilde{\Sigma}) = 3\operatorname{hcf}(n, 4)$, hence

$$b_2(Y) = 3\operatorname{hcf}(n, 4), \quad b_4(Y) = 3 \quad \text{and} \quad b_3(Y) = 0.$$

Notice that $b_3(Y) = b_3(\widetilde{E})$, where $\widetilde{E}$ is the normalisation, or a resolution of E.

Denote by $E_1 : (x = 0)$, $E_2 : (y = 0)$ and $E_3 : (z = 0)$, the irreducible components of E. The essential divisors birational to the planes E_i are

$$\begin{aligned} E_1' &: xyz + x^4 + y^4 + z^4 + t^4 = 0 \subset \mathbb{P}(2,1,1,1) \\ E_2' &: xyz + x^4 + y^4 + z^4 + t^4 = 0 \subset \mathbb{P}(1,2,1,1) \\ E_3' &: xyz + x^4 + y^4 + z^4 + t^4 = 0 \subset \mathbb{P}(1,1,2,1), \end{aligned}$$

if $n = 4$ and

$$\begin{aligned} E_1' &: xyz + t^4 = 0 \subset \mathbb{P}(2,1,1,1) \\ E_2' &: xyz + t^4 = 0 \subset \mathbb{P}(1,2,1,1) \\ E_3' &: xyz + t^4 = 0 \subset \mathbb{P}(1,1,2,1), \end{aligned}$$

for $n \geq 5$. These are all rational normal surfaces, hence their 3rd cohomology groups are 0.

4.9 The 3rd Betti number of a minimal model

It seems clear from the above examples that the obstructions to obtaining a formula like (4.2), giving the third Betti number of a minimal model $b_3(Y)$ purely in terms of the third Betti numbers of the crepant divisors, consist both of problems related to factoriality of the intermediate X_i and the nonnormality of the exceptional divisors E_i (and the change in their $b_3(E_i)$ on resolving).

In this section we give an approximation to formula (4.2) for an arbitrary 3-dimensional isolated canonical singularity (X, x). We also give arguments to justify Conjecture 4.23 that the conclusion of Theorem 4.4 holds in general.

In contrast to the previous examples, the approach in this section is "top-down"; we start from a minimal model Y and use, at each step given by the theorem below, certain refinements of formula (4.3).

Theorem 4.19 ([Ka],[K1]) *Let X be a 3-fold with canonical singularities. Then there exists a chain of crepant morphisms*

$$Y = X_c \xrightarrow{\varphi_c} X_{c-1} \longrightarrow \cdots \longrightarrow X_1 \xrightarrow{\varphi_1} X_0 \xrightarrow{\varphi_0} X \quad (4.11)$$

with the following properties:

(1) *Y is a minimal model of X;*

(2) *X_i is globally analytically $\mathbb{Q}$-factorial for any i;*

(3) *X_i is projective over X for any i;*

(4) *φ_i contracts precisely one crepant divisor E_i for any $i > 0$;*

(5) *φ_0 is small.*

Let (X, x) be a germ of 3-dimensional isolated canonical singularity, and consider a sequence (4.11) satisfying the conclusion of Theorem 4.19.

Recall from Section 1.2.5 that, for a pair (Z, W) consisting of an analytic space and its compact subset, $\sigma(Z, W)$ denotes the rank of the group of Weil divisors modulo Cartier divisors near W. Notice that, even if the X_i are $\mathbb{Q}$-factorial, their smaller open neighbourhoods do not necessarily have this property (cf. Section 1.2.5 and the examples in Section 4.8). In other words, analytic $\mathbb{Q}$-factoriality is not preserved by passing to smaller open sets.

Proposition 4.20 *Let $\psi\colon Z' \to Z$ be one of the above morphisms φ_i, with $i \geq 1$. Suppose it contracts the divisor E to C. Then*

$$b_3(Z') = b_3(Z) + b_3(E) + \sigma(Z', E) - \sigma(Z, C).$$

Proof Formula (4.3) applied to ψ gives

$$b_3(Z') = b_3(Z) + b_3(E) + b_2(Z') - b_2(Z) + b_2(C) - b_2(E).$$

From the exact sequence

$$0 \to \mathbb{Z}[E] \to \operatorname{Cl} Z' \to \operatorname{Cl} Z \to 0$$

it follows that

$$\begin{aligned} b_2(Z') - b_2(Z) &= \operatorname{rank} \operatorname{Pic} Z' - \operatorname{rank} \operatorname{Pic} Z \text{ (by Lemma 4.8)} \\ &= \operatorname{rank} \operatorname{Cl} Z' - \operatorname{rank} \operatorname{Cl} Z \text{ (by } \mathbb{Q}\text{-factoriality)} \\ &= 1. \end{aligned}$$

Look now at the map of germs

$$\psi\colon (Z', E) \to (Z, C)$$

and choose representatives U' and U of (Z', E) and (Z, C) respectively, such that

$$H^k(U', \mathbb{Z}) = H^k(E, \mathbb{Z}) \text{ and } H^k(U, \mathbb{Z}) = H^k(C, \mathbb{Z}) \text{ for any } k.$$

Suppose that U is chosen also with the property that $H^2(U, \mathcal{O}_U) = 0$. This is possible since we are working in a neighbourhood of a curve C, and thus C can be covered by 2 Stein open sets. Then $H^2(U', \mathcal{O}_{U'}) = 0$ as well, therefore part of the commutative diagram with rows coming from the exponential sequences of U and U' is

$$\begin{tikzcd} H^1(U', \mathcal{O}_{U'}) \arrow[r] & \operatorname{Pic} U' \arrow[r] & H^2(U', \mathbb{Z}) \arrow[r] & 0 \\ H^1(U, \mathcal{O}_U) \arrow[r] \arrow[u, "\cong"] & \operatorname{Pic} U \arrow[r] \arrow[u] & H^2(U, \mathbb{Z}) \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

The morphism $\psi^*\colon \operatorname{Pic} U \to \operatorname{Pic} U'$ is injective by the same argument as in the proof of Lemma 4.10. Thus, the exact sequence of analytic class groups

$$0 \to \mathbb{Z}[E] \to \operatorname{Cl} U' \to \operatorname{Cl} U \to 0$$

gives

$$\begin{aligned} b_2(E) - b_2(C) &= \operatorname{rank} \operatorname{Pic} U' - \operatorname{rank} \operatorname{Pic} U \\ &= \sigma(U') - \sigma(U) - 1, \end{aligned}$$

hence the conclusion. □

Let Z be a 3-fold with canonical singularities. By [R1, Corollary 1.14], there exists a finite set Σ of points, called dissident, such that, in a neighbourhood of any point in $Z \setminus \Sigma$,

$$Z \cong (\text{Du Val singularity}) \times \mathbb{C}. \tag{4.12}$$

By [Ka, Theorem 6.1′], there exists an analytic $\mathbb{Q}$-factorialisation of the pair (Z, Σ), that is, there exists a proper birational morphism $\psi\colon Z' \to Z$ satisfying:

(1) the pair $(Z', \psi^{-1}(\Sigma))$ is analytically $\mathbb{Q}$-factorial;

(2) ψ is an isomorphism in codimension one.

Furthermore

$$R^i\psi_*\mathcal{O}_{Z'} = 0 \text{ for } i > 0. \tag{4.13}$$

The last assertion is a consequence of rationality. Take a common resolution

$$\begin{tikzcd} Z'' \arrow[rr, "\psi'"] \arrow[dr, "\psi''"'] & & Z' \arrow[dl, "\psi"] \\ & X & \end{tikzcd}$$

Since Z and Z' have rational singularities

$$R^p\psi''_*\mathcal{O}_{Z''} = R^q\psi'_*\mathcal{O}_{Z''} = 0 \text{ for } p, q > 0.$$

Now (4.13) follows from the Grothendieck spectral sequence

$$E_2^{p,q} = R^p\psi_*(R^q\psi'_*\mathcal{O}_{Z''}) \Rightarrow R^{p+q}\psi''_*\mathcal{O}_{Z''}.$$

Then, by [Ka, Lemma 3.4], the morphism ψ also satisfies:

(1) Z' is analytically $\mathbb{Q}$-factorial at every point of $\psi^{-1}(\Sigma)$;

(2) $\psi^{-1}(\Sigma)$ is a union of curves, each isomorphic to $\mathbb{P}^1$.

As we have noticed before, an analytically $\mathbb{Q}$-factorial variety Z may not be locally analytically $\mathbb{Q}$-factorial. The next result quantifies the difference between $\sigma(Z)$ and the sum $\sum \rho(p)$ of the local divisor class numbers at the points $p \in Z$ which are not analytically $\mathbb{Q}$-factorial, in the case of a 3-fold Z with canonical singularities.

Proposition 4.21 *Let Z be a 3-fold with canonical singularities, Σ the set of dissident points of Z, and $\psi\colon \widetilde{Z} \to Z$ an analytic $\mathbb{Q}$-factorialisation of Z at Σ. Suppose that $H^2(Z, \mathcal{O}_Z) = 0$ and the map $\psi^*\colon H^4(Z, \mathbb{Z}) \to H^4(\widetilde{Z}, \mathbb{Z})$ is injective. Then*

$$b_3(\widetilde{Z}) = b_3(Z) + \sigma(Z) - \sum_{p\in\Sigma} \rho(p).$$

Proof Denote $\widetilde{\Sigma} = \psi^{-1}(\Sigma)$. Since $H^1(\widetilde{\Sigma}, \mathbb{Z}) = 0$ and $\psi^*\colon H^4(Z, \mathbb{Z}) \to H^4(\widetilde{Z}, \mathbb{Z})$ is assumed to be injective, the exact sequence (4.1)

$$\begin{aligned} \cdots \to H^k(Z, \mathbb{Z}) \to H^k(\widetilde{Z}, \mathbb{Z}) \oplus H^k(\Sigma, \mathbb{Z}) \to H^k(\widetilde{\Sigma}, \mathbb{Z}) \to \\ H^{k+1}(Z, \mathbb{Z}) \to \cdots \end{aligned}$$

gives

$$b_3(\widetilde{Z}) = b_3(Z) + b_2(\widetilde{Z}) - b_2(Z) - b_2(\widetilde{\Sigma}).$$

Look at the map of germs

$$\psi\colon (\widetilde{Z}, \widetilde{\Sigma}) \to (Z, \Sigma)$$

and choose representatives $\widetilde{U}$ and U of $(\widetilde{Z}, \widetilde{\Sigma})$ and (Z, C) such that

$$\begin{aligned} H^k(\widetilde{U}, \mathbb{Z}) = H^k(\widetilde{\Sigma}, \mathbb{Z}) \text{ and } H^k(U, \mathbb{Z}) = H^k(\Sigma, \mathbb{Z}) \text{ for any } k, \\ H^k(\widetilde{U}, \mathcal{O}_{\widetilde{U}}) = H^k(U, \mathcal{O}_U) = 0 \text{ for any } k > 0. \end{aligned}$$

It follows that

$$\begin{aligned} b_2(\widetilde{\Sigma}) &= b_2(\widetilde{U}) = \operatorname{rank} \operatorname{Pic} \widetilde{U} \\ &= \operatorname{rank} \operatorname{Cl} \widetilde{U} = \operatorname{rank} \operatorname{Cl} U \\ &= \sum_{p\in\Sigma} \rho(p). \end{aligned}$$

Since $H^2(Z,\mathcal{O}_Z) = H^2(\widetilde{Z},\mathcal{O}_{\widetilde{Z}}) = 0$, and ψ is an isomorphism in codimension 1, we also have

$$\begin{aligned} b_2(\widetilde{Z}) - b_2(Z) &= \operatorname{rank}\operatorname{Pic}\widetilde{Z} - \operatorname{rank}\operatorname{Pic} Z \\ &= \sigma(Z), \end{aligned}$$

hence the conclusion. □

Now we are ready to prove the main result of this section.

Theorem 4.22 *Let (X,x) be a germ of a 3-fold isolated canonical singularity, $c = c(X)$ the number of crepant divisors of X, and write a morphism $\varphi\colon Y \to X$ from a minimal model Y of X as a composite of crepant morphisms $\varphi_i\colon X_i \to X_{i-1}$ as in Theorem 4.19. Let E_i be the φ_i-exceptional divisors and $(\widetilde{X}_i,\widetilde{E}_i)$ a $\mathbb{Q}$-factorialisation of the germ (X_i,E_i) near the set Σ_i of its dissident points. Then*

$$b_3(Y) = \sum_{i=1}^{c} b_3(\widetilde{E}_i). \tag{4.14}$$

Proof Suppose that the morphism φ_i contracts E_i to C_{i-1}. By Proposition 4.20

$$b_3(X_i) = b_3(X_{i-1}) + b_3(E_i) + \sigma(X_i,E_i) - \sigma(X_{i-1},C_{i-1}),$$

for $i \geq 2$. Choose representatives $\widetilde{U}$ and U of $(\widetilde{X}_i,\widetilde{E}_i)$ and (X_i,E_i) such that

$$\begin{aligned} &H^k(\widetilde{U},\mathbb{Z}) = H^k(\widetilde{E}_i,\mathbb{Z}) \text{ and } H^k(U,\mathbb{Z}) = H^k(E_i,\mathbb{Z}) \text{ for any } k, \\ &H^k(\widetilde{U},\mathcal{O}_{\widetilde{U}}) = H^k(U,\mathcal{O}_U) = 0 \text{ for any } k > 1. \end{aligned}$$

By Proposition 4.21

$$b_3(\widetilde{U}) = b_3(U) + \sigma(U) - \sum_{p\in\Sigma_i} \rho(p).$$

The same argument, applied to a neighbourhood of C_{i-1} gives

$$\sigma(X_{i-1},C_{i-1}) = \sum_{p\in\Sigma(C_{i-1})} \rho(p),$$

where $\Sigma(C_{i-1})$ denotes the set of dissident points on that neighbourhood. It follows that

$$b_3(X_i) = b_3(X_{i-1}) + b_3(\widetilde{E}_i) + \sum_{p\in\Sigma_i} \rho(p) - \sum_{p\in\Sigma(C_{i-1})} \rho(p). \tag{4.15}$$

Adding the relations (4.15) up for $i \geq 1$, formula (4.14) follows from the $\mathbb{Q}$-factoriality of X_0 and Y, once we notice that

$$\sum_{i=1}^{c-1} \sum_{p\in\Sigma_i} \rho(p) = \sum_{i=1}^{c-1} \sum_{p\in\Sigma(C_{i-1})} \rho(p).$$

But this is clear, again by the $\mathbb{Q}$-factoriality of X_0. $\square$

4.10 Further remarks

We end this chapter with a conjecture and a possible application.

Conjecture 4.23 *Let (X, x) be a germ of a 3-dimensional isolated canonical singularity, Y a minimal model of X, and E_i any nonsingular projective representative of the crepant valuation v_i of X. Then*

$$b_3(Y) = \sum_{i=1}^{c(X)} b_3(E_i).$$

Remark 4.24 In the notation of Theorem 4.22, the only missing ingredient to proving the conjecture above is the equality between $b_3(\widetilde{E_i})$ and the 3rd Betti number of a nonsingular projective representative of E_i. Since, by (4.12), $\widetilde{X_i}$ is a $\mathbb{Q}$-homology manifold outside its dissident points, the following should be relevant for this property.

Claim 4.25 *With the notation from Theorem 4.22, let $\widetilde{\Sigma}_i$ be the set of dissident points of $\widetilde{X_i}$. Then the map*

$$H^3(\widetilde{X_i}, \mathbb{Q}) \to H^3(\widetilde{X_i} \setminus \widetilde{\Sigma}_i, \mathbb{Q})$$

is injective.

Proof Take a neighbourhood U of $\widetilde{\Sigma_i}$ in $\widetilde{X_i}$ such that $H^k(U,\mathbb{Z})=0$ for $k>0$. The Mayer–Vietoris sequence of the covering $\widetilde{X_i}=(\widetilde{X_i}\setminus\widetilde{\Sigma_i})\cup U$ then gives

$$H^2(U\setminus\widetilde{\Sigma_i},\mathbb{Z})\to H^3(\widetilde{X}_i,\mathbb{Z})\to H^3(\widetilde{X_i}\setminus\widetilde{\Sigma_i},\mathbb{Z}).$$

Let $\Sigma=\operatorname{Sing}\widetilde{X_i}$ be the singular locus of $\widetilde{X_i}$. The local $\mathbb{Q}$-factoriality of $\widetilde{X}_i$ implies that

$$\begin{aligned} H^2(U_p\setminus\Sigma,\mathbb{Q}) &= (\operatorname{Cl}_p\widetilde{X}_i)_{\mathbb{Q}} \\ &= 0, \end{aligned}$$

for any point p and any small enough neighbourhood U_p of p (see [Fl]). Since $\widetilde{X_i}$ has canonical singularities, we have an injection

$$H^2(U_p\setminus p,\mathbb{Z})\to H^2(U_p\setminus\Sigma,\mathbb{Z}),$$

for any neighbourhood U_p of p (see [K2, 2.1.7.4]). Therefore the group $H^2(U_p\setminus p,\mathbb{Q})$ is zero near p. The equality

$$H^2(U\setminus\widetilde{\Sigma_i},\mathbb{Z})=\bigoplus_{p\in\widetilde{\Sigma_i}}H^2(U\setminus p,\mathbb{Z}),$$

with U shrunk if necessary, proves now the claim. □

Remark 4.26 Another way of stating Conjecture 4.23 is that

$$b_3(Y)=2\sum_{i=1}^{c(X)}q(E_i),$$

where $q(E_i)$ is the irregularity of E_i. Since, by [R1, Corollary 2.14], each E_i is rational or ruled, this is further equivalent to

$$b_3(Y)=2\sum g(C_i),$$

where the summation is taken over the surfaces E_i which are ruled over C_i.

Suppose now that $0\in X:(f=0)\subset\mathbb{C}^4$ is an isolated canonical non-degenerate singularity and Y a minimal model of X. Then Conjecture 4.23 implies that

$$b_3(Y)=2\sum_{\substack{\alpha\in W(f)\\ \dim\Gamma_\alpha=2}}\#(\Gamma_\alpha^\circ\cap N).$$

This follows from Theorem 3.11, together with the following fact. Let $E\subset E(\alpha)$ be a divisor which is essential for its valuation. Then

- if $\dim \Gamma_\alpha = 3$, then E is rational;
- if $\dim \Gamma_\alpha = 2$, i.e., E is a weighted projective cone over a curve C, and if $\widetilde{C}$ is a resolution of C, then $g(\widetilde{C}) = \#(\Gamma_\alpha^\circ \cap N)$ by [O, Theorem 6.1];
- if $\dim \Gamma_\alpha = 1$, then E is rational (see [O, Lemma 4.8]).

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