Chapter 4

Equivariant and Orbifold Riemann–Roch

4.1 Introduction

4.1.1 Notation

Fix the following notation throughout this chapter. Let G be a finite group acting on a smooth projective variety Z. For a nontrivial element $g \in G$ of order r say, let $H := \langle g \rangle$ be the subgroup generated by g. Fix ε a primitive rth root of unity. Let $\pi \colon Z \to X = Z/H$ be the quotient morphism. Then X is normal and π is a finite morphism.

For $h \in H$ let $Z^h = \left\{ z \in Z \;\middle|\; hz = z \right\}$ and define Z^H by the following formula $Z^H = \bigcup_{h \in H \smallsetminus 1} Z^h$. The action is *coprime* if $Z^H = Z^h$ for any $h \neq 1 \in H$.

If D is a prime divisor and D_1 any $\mathbb{Q}$-Cartier divisor we write $\lambda D \in \operatorname{Supp} D_1$ to mean $D \in \operatorname{Supp} D_1$ but $D \notin \operatorname{Supp}(D_1 - \lambda D)$. We also write $P \in \operatorname{Supp} D_1$ to mean $P \in D$ and $D \in \operatorname{Supp} D_1$.

Let $\mathcal{F}$ be a coherent sheaf on a variety X then denote by $\mathcal{F}^*$ the dual $\mathcal{H}om(\mathcal{F}, \mathcal{O}_X)$ of the sheaf $\mathcal{F}$.

4.1.2 Background to the equivariant Riemann–Roch theorem

Let x be an irreducible component of Z^g and $\mathcal{N}_x = \mathcal{N}_{x\big|Z}$ the normal sheaf of x in Z which is a locally free sheaf of rank $\operatorname{codim}(x, Z)$ on the trivial G-variety x. By Proposition 1.18 this decomposes into eigensheaves. We denote this decomposition by

$$\mathcal{N}_x = \bigoplus_i \mathcal{N}_{xi} \otimes \chi_i,$$

where χ_i is the character of $\langle g \rangle$ given by $\chi_i(g) = \varepsilon^i$. This gives a decomposition of $\mathcal{N}^g$, the normal sheaf of $Z^g \in Z$, as follows, $\mathcal{N}^g = \bigoplus_i \mathcal{N}^g_i \otimes \chi_i$.

For a locally free G-sheaf $\mathcal{F}$ on a trivial G-space X write $\lambda_{-1}(\mathcal{F})$ for the element $\bigoplus (-1)^i \Lambda^i(\mathcal{F})$ in $K^G(X)$ and write $\operatorname{td}(X)$ for the Todd class of X.

Let G be a finite group acting on a smooth projective variety Z and $\mathcal{F}$ a G-sheaf. The vector space $\mathrm{H}^i(Z, \mathcal{F})$ is a representation of G so the character of any element g acting on the virtual representation $\bigoplus_{i \geq 0} (-1)^i \mathrm{H}^i(Z, \mathcal{F})$ can be defined. Equivariant Riemann–Roch gives an expression for this in terms of the irreducible components of the fixed locus.

Theorem 4.1 (Equivariant Riemann–Roch [AS68]) *Let Z be a non-singular projective variety, G a finite group acting on Z and $\mathcal{F}$ a locally free G-sheaf. Then*

$$\operatorname{Tr}\left(g \colon \bigoplus_{p \geq 0} (-1)^p \mathrm{H}^p(Z, \mathcal{F}) \right) = \int_{Z^g} \frac{\operatorname{ch}(\mathcal{F}_{|Z^g})(g) \cdot \operatorname{td}(Z^g)}{\operatorname{ch} \lambda_{-1}\left((\mathcal{N}^g)^*\right)(g)}. \tag{4.1}$$

In this chapter we give an explicit proof of this theorem when Z is a curve or a surface and $\mathcal{F}$ is an invertible G-sheaf. We then use these results to generate a Riemann–Roch theorem for an orbifold of dimension n having only isolated cyclic quotient singularities.

4.1.3 Technical preliminaries on higher direct image sheaves

We begin by recalling some properties of higher direct image sheaves. See [Har77, III.8] for details. The results in this section are not new but some proofs are included here.

Definition 4.2 Let $f\colon Y \to X$ be a continuous map of topological spaces, $i \geq 0$ and $\mathcal{F}$ a sheaf of Abelian groups on Y. The *ith higher direct image* of $\mathcal{F}$ is the sheaf, $\mathcal{R}^i f_*(\mathcal{F})$, associated to the presheaf

$$V \mapsto \mathrm{H}^i\left(f^{-1}V, \mathcal{F}(f_{-1}V)\right)$$

on X. Notice that $\mathcal{R}^0 f_*(\mathcal{F}) = f_*\mathcal{F}$.

Lemma 4.3 *If $f\colon Y \to X$ is an affine morphism and $\mathcal{F}$ a coherent $\mathcal{O}_Y$-module then $\mathcal{R}^i f_*(\mathcal{F}) = 0$ for all $i > 0$.*

Proof. If $V \subset X$ is affine then $f^{-1}(V)$ is affine and so $\mathrm{H}^i\left(f^{-1}V, \mathcal{F}|_{f^{-1}V}\right) = 0$. Thus $\mathcal{R}^i f_*(\mathcal{F}) = 0$ for all $i > 0$. □

The Leray spectral sequence relates the cohomology of the sheaf $\mathcal{F}$ with the cohomology of the sheaves $\mathcal{R}^i f_*(\mathcal{F})$.

Theorem 4.4 (Leray spectral sequence [God73]) *Let $f\colon Y \to X$ be a morphism between ringed spaces and $\mathcal{F}$ an $\mathcal{O}_Y$-module. Then there exists a* *spectral sequence with*

$$E_\infty \implies \mathrm{H}^*(Y,\mathcal{F}) \text{ where } E_2^{p,q} = \mathrm{H}^p(X,\mathcal{R}^q f_* \mathcal{F}).$$

Taking alternating sums in the Leray spectral sequence shows that

$$\chi(Y,\mathcal{F}) = \sum_q (-1)^q \chi(X,\mathcal{R}^q f_* \mathcal{F}). \tag{4.2}$$

It follows immediately from Theorem 4.4 that if $\mathcal{R}^i f_*(\mathcal{F}) = 0$ for all $i > 0$ then $\mathrm{H}^i(Y,\mathcal{F}) \cong \mathrm{H}^i(X, f_*\mathcal{F})$ and so $\chi(Y,\mathcal{F}) = \chi(X, f_*\mathcal{F})$. In particular this is true if f is an finite morphism by Lemma 4.3.

It could be hoped that if $f\colon Y \to X$ is a resolution of rational singularities on X and $\mathcal{M} = f^*\mathcal{L}/(\text{torsion})$ then $\chi(Y,\mathcal{M}) = \chi(X, f_*\mathcal{M})$. However this is only true when $\dim Y \leq 2$ as we will see in Lemma 4.5. In higher dimensional cases we will also have terms $\chi(X,\mathcal{R}^i f_*(\mathcal{M}))$, where $\mathcal{R}^i f_*(\mathcal{M})$ is a sheaf supported only over the singular locus of X. This will be proved in the following lemma.

Lemma 4.5 *Let X be a projective variety having at worst rational singularities and $f\colon Y \to X$ be a resolution of X. Let $\mathcal{L}$ be a divisorial sheaf on X. Then $\mathcal{M} = (f^*\mathcal{L})/(\text{torsion})$ is a rank one torsion free sheaf on Y and*

$$\chi(Y,\mathcal{M}) = \chi(X, f_*\mathcal{M}) + \sum_{i=1}^{n-2} (-1)^i \chi(X,\mathcal{R}^i f_* \mathcal{M}).$$

If $i > 0$ and X has only isolated singularities then

$$\chi(X,\mathcal{R}^i f_* \mathcal{M}) = h^0(X,\mathcal{R}^i f_* \mathcal{M}).$$

Proof. For any coherent sheaf $\mathcal{L}$ on a projective variety there exists a surjective morphism

$$\mathcal{O}_X^N(-n) \to \mathcal{L} \to 0$$

for some $n, N \in \mathbb{Z}$. See [Har77, II.5 Cor 5.18]. Since $f^*\mathcal{O}_X = \mathcal{O}_Y$ this gives a short exact sequence

$$0 \to \mathcal{K} \to \mathcal{O}_Y^N \otimes f^*\mathcal{O}_X(-n) \to \mathcal{M} \to 0$$

where $\mathcal{M} = (f^*\mathcal{L})/(\text{torsion})$ as in the definition and $\mathcal{K}$ is the defined as the kernel of the surjective morphism $\mathcal{O}_Y^N \otimes f^*\mathcal{O}_X(-n) \to \mathcal{M}$. Applying f_* gives the following long exact sequence.

$$\begin{array}{ccccccccc} 0 \longrightarrow & f_*\mathcal{K} & \longrightarrow & f_*(\mathcal{O}_Y^N \otimes f^*(\mathcal{O}_X(-n)) & \longrightarrow & f_*\mathcal{M} & \longrightarrow \\ \to & \mathcal{R}^1 f_*(\mathcal{K}) & \to & \mathcal{R}^1 f_*(\mathcal{O}_Y^N \otimes f^*(\mathcal{O}_X(-n)) & \to & \mathcal{R}^1 f_*(\mathcal{M}) & \to & \cdots \end{array}$$

By the Projection Formula [Har77, III Ex 8.3] $\mathcal{R}^i f_*(\mathcal{O}_Y^N \otimes f^*(\mathcal{O}_X(-n)) = \mathcal{R}^i f_*(\mathcal{O}_Y^N) \otimes \mathcal{O}_X(-n)$ and if $i > 0$ then this is zero by definition of rational singularities. Thus $\mathcal{R}^i f_*(\mathcal{M}) \cong \mathcal{R}^{i+1} f_*(\mathcal{K})$. Since $\mathcal{K}$ is supported on the fibres of f, the sheaves $\mathcal{R}^i f_*(\mathcal{M}) = 0$ for $i \geq n-1$.

Now $f_*\mathcal{M} = \mathcal{L}$ since $\mathcal{M}$ is generated by the sections of $\mathcal{L}$. Thus by Equation 4.2

$$\chi(Y, \mathcal{M}) = \sum_{i=0}^{n-2} (-1)^i \chi(X, \mathcal{R}^i f_*(\mathcal{M})) = \chi(X, \mathcal{L}) + \sum_{i=1}^{n-2} (-1)^i \chi(X, \mathcal{R}^i f_*(\mathcal{M})).$$

For $i > 0$ the sheaves $\mathcal{R}^i f_*(\mathcal{M})$ are supported only over the singularities of X. If the singularities are all isolated then the sheaves $\mathcal{R}^i f_*(\mathcal{M})$ are supported only over points so all its higher cohomology groups vanish. □

Lemma 4.6 *In notation of Lemma 4.5 if X is 2 dimensional then $\mathcal{M}$ is invertible and $\mathcal{R}^i f_*(\mathcal{M}) = 0$ for all $i > 0$ so*

$$\chi(X, \mathcal{L}) = \chi(Y, \mathcal{M}).$$

PROOF. The fact that $\mathcal{R}^i f_*(\mathcal{M}) = 0$ follows immediately from Lemma 4.5. Since $\mathcal{M}$ is torsion free there exists a short exact sequence

$$0 \to \mathcal{M} \to \mathcal{M}^{**} \to \mathcal{Q} \to 0$$

where $\mathcal{Q}$ is defined to be the cokernel of the embedding $\mathcal{M} \to \mathcal{M}^{**}$. Applying f_* gives the long exact sequence

$$\begin{aligned} 0 \longrightarrow f_*\mathcal{M} \longrightarrow f_*(\mathcal{M}^{**}) \longrightarrow f_*\mathcal{Q} \longrightarrow \\ \longrightarrow \mathcal{R}^1 f_*(\mathcal{M}) = 0. \end{aligned}$$

As in the proof of Lemma 4.5 $f_*\mathcal{M} = \mathcal{L}$. The sheaf $f_*(\mathcal{M}^{**})$ is torsion free, contains the reflexive sheaf $\mathcal{L}$ and $\mathrm{Supp}(f_*(\mathcal{M}^{**})/\mathcal{L})$ is contained in the singular locus of X. Therefore by Proposition 1.9 $f_*(\mathcal{M}^{**}) = \mathcal{L}$. This implies that $f_*\mathcal{Q} = 0$ and so $\mathcal{Q} = 0$. Hence $\mathcal{M} \cong \mathcal{M}^{**}$ and so $\mathcal{M}$ is reflexive. A reflexive rank 1 sheaf on a smooth variety is invertible so $\mathcal{M}$ is an invertible sheaf. □

In particular the next proposition implies that if $\pi\colon Z \to X$ is the quotient morphism and $\mathcal{F}$ a locally free sheaf then $\pi_*\mathcal{F}$ is locally free away from the singularities.

Proposition 4.7 (II.5 Cor 2 [Mum85]) *Let $f\colon Z \to X$ be a proper morphism of Noetherian schemes, X integral and $\mathcal{F}$ a coherent sheaf on Z which flat over X. If for some $i \geq 0$ the function $x \mapsto h^i(Z_x, \mathcal{F}_x)$ is constant then the sheaf $\mathcal{R}^i f_*(\mathcal{F})$ is locally free of rank $h^i(Z_x, \mathcal{F}_x)$.*

4.1.4 Roots of unity

It will also be useful to recall the following properties of roots of unity. For any nontrivial rth root of unity, ε, we have

$$1+\varepsilon+\varepsilon^2+\cdots+\varepsilon^{r-1} = 0 \tag{4.3}$$

$$\frac{-\varepsilon-2\varepsilon^2-\cdots-(r-1)\varepsilon^{r-1}}{r} = \frac{1}{1-\varepsilon}. \tag{4.4}$$

4.2 Results on G-sheaves

The proof of Theorem 4.1 also uses the following results.

Lemma 4.8 *Let Z be a smooth projective variety, G a finite cyclic group acting on Z such that only the trivial element of G acts trivially on Z and $\mathcal{F}$ an invertible G-sheaf on Z. Let $\pi\colon Z\to X=Z/G$ be the quotient morphism. Then*

$$\pi_*\mathcal{F}=\bigoplus_i \mathcal{F}_i$$

and each $\mathcal{F}_i$ is a divisorial sheaf which is invertible away from the singularities of X.

Proof. Let $\mathcal{F}_i$ be the torsion free sheaf $\mathcal{H}om(V_i,\pi_*\mathcal{F})^G$. We show that this is invertible away from the singularities. Since X is normal and the singular locus has codimension ≥ 2 this means that the $\mathcal{F}_i$ are divisorial by Proposition 1.13 and therefore $\mathcal{F}_i=\mathcal{O}(D_i)$ for some $\mathbb{Q}$-Cartier divisor D_i.

Let X^0 be the nonsingular locus of X and $Z^0=\pi^{-1}(X^0)$. Then the restriction of the morphism π to Z^0, is proper and $\mathcal{F}_{|Z^0}$ is flat over X^0. The function $x\mapsto h^0(Z_x,\mathcal{F}_x)$ is equal to $|G|$ for all $x\in X^0$. Therefore by Proposition 4.7 the sheaf $\pi_*(\mathcal{F}_{|X^0})$ is locally free of rank $|G|$. Since X^0 is a trivial G-space $\pi_*\left(\mathcal{F}_{|X^0}\right)$ decomposes into eigensheaves $\mathcal{F}_i^0$ which are locally free by Proposition 1.18 and $\mathcal{F}_i^0\cong\mathcal{F}_{i|X^0}$. □

Lemma 4.9 *Let G be a finite group acting on a smooth projective variety Z such that only the trivial element of G acts trivially on Z and $\mathcal{F}$ an invertible G-sheaf. Then $\mathcal{F}$ is isomorphic to a subsheaf of $k(Z)$ as G-sheaves.*

Proof. Let $\mathcal{F}$ be an invertible G-sheaf. As in [Har77, II Prop 6.13] we know that $\mathcal{F} \otimes k(Z)$ is 1 dimensional vector space over $k(Z)$. The compatibility conditions of λ_g imply that the map $\mathcal{F} \otimes k(Z) \to g^*\mathcal{F} \otimes k(Z)$ is quasilinear. Then by [Sha94, Appendix 3 Prop 1] we may assume that $\mathcal{F} \otimes k(Z)$ is based by an invariant element of $\mathcal{F} \otimes k(Z)$ over $k(Z)$. Thus $\mathcal{F} \subset \mathcal{F} \otimes k(Z) \cong k(Z)$ as G-sheaves and so $\mathcal{F}$ is isomorphic to a subsheaf of $k(Z)$ as G-sheaves. $\square$

Lemma 4.10 ("Hilbert's Theorem 90") *Let G be a cyclic group, $\mathcal{F}$ an invertible G-subsheaf of $k(Z)$ and assume that the map $\lambda_g \colon \mathcal{F} \to g^*\mathcal{F}$ is induced by the action of g on $k(Z)$. Then $\mathcal{F} \cong \mathcal{O}(D)$ where D is G-invariant.*

Proof. Compare with [Lan93, proof of VI Thm 6.1]. Let G be a cyclic group of order r and g a generator of G. Let $\mathcal{F}$ be an invertible G-sheaf and f a generator of $\mathcal{F}_P$ where P is a fixed point. Now $\mathcal{F}_P = (g^*\mathcal{F})_P$ and since λ_g is an isomorphism $\lambda_g(f)$ is also a generator of the stalk $\mathcal{F}_P$. Therefore $\beta = \lambda_g(f)/f \in \mathcal{O}_P^*$.

We want to find $\gamma \in \mathcal{O}_P^*$ such that $\lambda_g(\gamma)/\gamma = \varepsilon^{-i}\beta$ and then f/γ is a generator of $\mathcal{F}_P$ and $\lambda_g(f/\gamma) = \varepsilon^i f/\gamma$.

By Artin's Theorem on characters [Lan93, VI Thm 4.1] the map Θ on $k(Z)$ given by

$$\Theta(\xi) = \mathrm{id}^*(\xi) + \beta^{-1}g^*(\xi) + \cdots + \beta^{-1}g^*\left(\beta^{-1}\right)\cdots(g^{r-2})^*\left(\beta^{-1}\right)g^{r-1*}(\xi)$$

for $\xi \in k(Z)$ is not identically zero. So there exists $\eta \in k(Z)$ such that $\gamma = \Theta(\eta) \neq 0$ and then $\lambda_g(\gamma)/\gamma = \beta$.

Now $\Theta(\eta)$ may have a zero or pole at P. Define a map Θ_i by

$$\Theta_i(\xi) = \mathrm{id}^*(\xi) + \varepsilon^i\beta^{-1}g^*(\xi) + \cdots + \varepsilon^{-i}\beta^{-1}g^*\left(\beta^{-1}\right)\cdots(g^{r-2})^*\left(\beta^{-1}\right)(g^{r-1})^*(\xi)$$

for $\xi \in k(Z)$. Then $\lambda_g(\Theta_i(\eta))/\Theta_i(\eta) = \varepsilon^{-i}\beta$. If $\Theta_i(\eta)$ has a zero at P for all i then η has a zero at P. If η has a zero at P then $\eta + n$ does not for any $n \in \mathbb{Z}$. At least one of $\Theta_i(\eta + n)$ or $\Theta_i(n)$ is nonzero and for some i this has no zero at P. Set γ equal to $\Theta_i(\eta + n)$ or $\Theta_i(n)$. Then γ does not have a zero at P and $\lambda_g(\gamma)/\gamma = \varepsilon^{-i}\beta$. Similarly we can find γ such that γ has no pole at P.

Consider an open set $U \subset Z$ on which G acts freely. By [Mum85, 2.7 Prop 2] $\mathcal{F}_{|U} \cong \pi^*\left((\pi_*\mathcal{F})^G\right)_{|U}$. Thus for any $P \in Z \smallsetminus Z^g$ the stalks $\mathcal{F}_P$ and $\left(\pi^*(\pi_*\mathcal{F})^G\right)_P$ are isomorphic as G-modules over $\mathcal{O}_P$. Therefore we can choose a generator of $\mathcal{F}_P$ which is G-invariant.

We have shown that for all $P \in Z$ we can find a generator of $\mathcal{F}_P$ such that $\lambda_P(f) = \varepsilon^i f$. Thus F is a G-invariant divisor. □

4.3 Proof of Theorem 4.1 on a smooth projective curve

Let $Z = C$ be a smooth projective curve. Theorem 4.1 can be restated as follows.

Theorem 4.11 *Let G be a finite group, C a smooth curve on which G acts and $\mathcal{F}$ an invertible G-sheaf on C. Assume that only the trivial element of G acts trivially on C. Then for all nontrivial $g \in G$*

$$\mathrm{Tr}\left(g\colon \bigoplus_{i\geq 0}(-1)^i\,\mathrm{H}^i(C,\mathcal{F})\right) = \sum_{P\in C^g}\frac{\mathrm{Tr}\left(g\colon \mathcal{F}_{|P}\right)}{1-\mathrm{d}g_P} \qquad (4.5)$$

where $\mathrm{d}g_P \colon T^*_P C \to T^*_P C$ *is the induced map on the cotangent space and* $\mathcal{F}_{|P} \in K_G(P) = R(G)$.

The trace of the identity acting on a virtual representation is its dimension. Thus, by Hirzebruch Riemann–Roch for curves,

$$\begin{aligned} \operatorname{Tr}\left(1\colon \bigoplus_{i\geq 0}(-1)^i \,\mathrm{H}^i(C,\mathcal{F})\right) &= \chi(C,\mathcal{F}) \\ &= \deg\mathcal{F} + 1 - \mathrm{p_a}(C). \end{aligned} \tag{4.6}$$

Recall the notation, $g \in G$ is of order r, $H = \langle g\rangle$ and $\pi\colon C \to X = C/H$ is the quotient morphism.

Let $\pi_*\mathcal{F} = \bigoplus_{i=0}^{r-1}\mathcal{F}_i$ be the decomposition of $\pi_*\mathcal{F}$ into eigensheaves. Now X is smooth and H is cyclic so each of the sheaves $\mathcal{F}_i$ is invertible by Lemma 4.8. If $\mathcal{F}$ is the trivial G-sheaf then denote the decomposition of $\pi_*\mathcal{O}_C$ into eigensheaves by $\pi_*\mathcal{O}_C = \bigoplus_i \mathcal{L}_i$ and write $\mathcal{L}_i = \mathcal{O}_X(D_i)$. Since the sheaves $\mathcal{L}_i$ are invertible $\mathcal{L}_i \otimes \mathcal{L}_j \cong \mathcal{O}(D_i + D_j)$.

Using the previous results

$$\begin{aligned} &\operatorname{Tr}\left(g\colon \bigoplus_{i\geq 0}(-1)^i\,\mathrm{H}^i(C,\mathcal{F})\right) \\ &\quad= \operatorname{Tr}\left(g\colon \bigoplus_{i\geq 0}(-1)^i\,\mathrm{H}^i(X,\pi_*\mathcal{F})\right) && \text{by Lemma 4.3} \\ &\quad= \sum_{k=0}^{r-1}\operatorname{Tr}\left(g\colon \bigoplus_{i\geq 0}(-1)^i H^i(X,\mathcal{F}_k)\right) && \text{by Lemma 4.8} \\ &\quad= \sum_{k=0}^{r-1}\varepsilon^k\big(\deg\mathcal{F}_k + 1 - \mathrm{p_a}(X)\big) && \text{by Equation 4.6} \\ &\quad= \sum_{k=0}^{r-1}\varepsilon^k\deg\mathcal{F}_k && \text{by Equation 4.3.} \end{aligned}$$

So it remains to calculate the degrees of the invertible sheaves $\mathcal{F}_k$.

4.3.1 Proof of Theorem 4.1 for the trivial G-sheaf when the action is coprime.

Example 4.12 Let the action of $G = \mathbb{Z}_r$ on $C = \mathbb{P}^1$ be generated by $g = \frac{1}{r}(0,1)$ and let $\mathcal{F}$ be the trivial G-sheaf. The vector space $\mathrm{H}^0(C, \mathcal{O}_C) \cong \mathbb{C}$ and g acts trivially on it so

$$\mathrm{Tr}\left(g\colon \bigoplus_{i\geq 0}(-1)^i\,\mathrm{H}^i(C,\mathcal{O}_C)\right) = 1.$$

The fixed locus of g acting on C is $C^g = \{P_0 = (1:0), P_1 = (0:1)\}$. At P_0 the local coordinate is x_1/x_0 and so the induced action on the cotangent space is $\mathrm{d}g_P\mathrm{d}\,(x_1/x_0) = \varepsilon\mathrm{d}\,(x_1/x_0)$. At P_1 the coordinate is x_0/x_1 and so the induced action on the cotangent space is $\mathrm{d}g_P\mathrm{d}\,(x_0/x_1) = \varepsilon^{-1}\mathrm{d}\,(x_0/x_1)$. Thus

$$\sum_{P\in C^g}\frac{\mathrm{Tr}(g\colon \mathcal{O}_{|P})}{1-\mathrm{d}g_P} = \frac{1}{1-\varepsilon} + \frac{1}{1-\varepsilon^{-1}} = 1.$$

The ith eigensheaf $\mathcal{L}_i$ of $\pi_*(\mathcal{O}_Z)$ is not a subsheaf of $k(X)$ but the sheaf $\mathcal{L}_i^{\otimes r}$ is. Therefore its Cartier divisor can be recovered along with its embedding into $k(X)$ [Har77, II Prop 6.13]. In what follows we show how to construct the divisor corresponding to rD_i.

Let P be any point in C, $Q = \pi(P)$ and u_i a local generator of $\mathcal{L}_{iQ}$. If P is not fixed by g, then $u_i^r \in \mathcal{O}_Q$ is a unit. Therefore $\mathcal{L}_{iQ}^{\otimes r}$ is generated by 1 as an $\mathcal{O}_Q$-module and so $Q \notin \mathrm{Supp}\, rD_i$.

If P is fixed by g then the quotient morphism $\pi\colon C \to X$ is ramified with order r at P. If t is a local coordinate of C at P then $s = t^r$ is a local coordinate of X at Q. The element g induces an action on the cotangent space of C at P which is given by $\mathrm{d}g_P(\mathrm{d}t) = \varepsilon^a\mathrm{d}t$ for some a coprime to r.

Let u_i be a local generator of $\mathcal{L}_{iQ}$ then $u_i^r \in \mathcal{O}_Q$ is no longer a unit so $u_i^r = (\text{unit})s^\lambda$. Since $\mathrm{d}g\mathrm{d}t = \varepsilon^a \mathrm{d}t$ it follows that $u_i = t^{b_i}$ where $b_i a \equiv i \pmod r$. Therefore u_i^r has a zero of order b_i at Q and $-b_i Q \in \operatorname{Supp} rD_i$.

Continuing this process over all $P \in C$ we obtain

$$\mathcal{L}^{\otimes r} = \mathcal{O}_X \left(- \sum_Q b_{i,Q} Q \right)$$

where $0 \leq b_{i,Q} \leq r-1$ is defined as follows. If Q is not a branch point then $b_{i,Q} = 0$. If Q is a branch point and the induced action on the cotangent space to the point $P = \pi^{-1}(Q)$ is $\mathrm{d}g_P(\mathrm{d}t) = \varepsilon^{a_P}\mathrm{d}t$ then $b_{i,Q}$ is defined such that $b_{i,Q} a_P \equiv i \pmod r$. Hence

$$\deg(D_i) = -\sum_Q \frac{b_{i,Q}}{r}.$$

The divisor D_i is Cartier so $\deg D_i$ is an integer and the following corollary is immediate.

Corollary 4.13 *The order of g divides $\sum_Q b_{i,Q}$.*

This corollary restricts the way a group may act on a curve. For example if $\mathbb{Z}_r$ acts on a curve then any $g \in \mathbb{Z}_r$ cannot fix only one point. If g fixes exactly two points then g acts by ε^a at one point and ε^{-a} at the other.

Example 4.14 Continuing with the above example. Let $Q_0 = \pi(P_0)$. Now x_1/x_0 is a local coordinate of P_0, $g(x_1/x_0) = \varepsilon x_1/x_0$ and $(x_1/x_0)^r$ is a local coordinate at Q_0. If u_i is the generator of $\mathcal{L}_{iQ_0}$ then u_i^r is invariant so $u_i^r = (x_1/x_0)^{r\lambda}$ and since $u_i = (x_1/x_0)^\lambda$ it follows that $\lambda = i$. Similarly if $Q_1 = \pi(P_1)$ and u_i is the generator of $\mathcal{L}_{iQ_1}$ then $u_i = (x_1/x_0)^{r-i}$. Thus

$$\deg \mathcal{L}_i = -\frac{i}{r} - \frac{r-i}{r} = -1$$

and

$$\sum_i \varepsilon^i \deg \mathcal{L}_i = -\varepsilon - \varepsilon^2 - \cdots - \varepsilon^{r-1} = 1.$$

So far Equation 4.5 has been shown to equal

$$\operatorname{Tr}\left(g\colon \bigoplus_{i\geq 0}(-1)^i \operatorname{H}^i(C,\mathcal{O}_C)\right) = \sum_{k=0}^{r-1} \varepsilon^k \deg \mathcal{L}_k = -\sum_{k=0}^{r-1}\sum_Q \frac{b_{k,Q}}{r}\varepsilon^k \tag{4.7}$$

where Q runs through all the branch points of π. Now fix a branch point Q, the contribution this point makes to $\sum_i \varepsilon^i \deg \mathcal{L}_i$ is

$$\begin{aligned} -\sum_{k=0}^{r-1}\frac{b_{k,Q}}{r}\varepsilon^k = -\sum_{i=1}^{r-1}\frac{i\varepsilon^{ia_P}}{r} \\ = \frac{-\varepsilon^{a_p} - 2\varepsilon^{2a_P} - \cdots - (r-1)\varepsilon^{(r-1)a_P}}{r} = \frac{1}{1-\varepsilon^{a_P}} \end{aligned}$$

by Equation 4.4. Substituting this into Equation 4.7 gives

$$\operatorname{Tr}\left(g\colon \bigoplus_{i\geq 0}(-1)^i \operatorname{H}^i(C,\mathcal{O}_C)\right) = \sum_Q \frac{1}{1-\mathrm{d}g_P} = \sum_{P\in C^g}\frac{1}{1-\mathrm{d}g_P}.$$

□

4.3.2 Proof of Theorem 4.1 for any invertible G-sheaf

Example 4.15 Let C and G be as above. Let $\mathcal{F} = \mathcal{O}_C(\mu P_0)$ and assume $\mu > 0$ so that $\mathcal{F}_{P_1}$ is generated by 1 and $\mathcal{F}_{P_0}$ is generated by $(x_0/x_1)^\mu$. Let the action $\mathcal{F} \to g^*\mathcal{F}$ be induced by the action of g on $k(C)$. Then isomorphism $\mathcal{F}_{|P_1} \to \mathcal{F}_{|P_1}$ is multiplication by 1 and the isomorphism $\mathcal{F}_{|P_0} \to \mathcal{F}_{|P_0}$ is multiplication by ε^μ. The rational functions $1, x_0/x_1, \ldots, (x_0/x_1)^\mu$ generate the global sections of the sheaf $\mathcal{F}$ and so

$$\operatorname{Tr}\left(g\colon \bigoplus_{i\geq 0}(-1)^i \operatorname{H}^i(C,\mathcal{F})\right) = 1 + \varepsilon^{-1} + \cdots + \varepsilon^{-\mu}.$$

At the fixed points $\mathrm{Tr}\left(g\colon \mathcal{F}_{|P_0}\right) = \varepsilon^{-\mu}$ and $\mathrm{Tr}\left(g\colon \mathcal{F}_{|P_1}\right) = 1$ so

$$\sum_{P\in C^g} \frac{\mathrm{Tr}\left(g\colon \mathcal{F}_{|P}\right)}{1-\mathrm{d}g_P} = \frac{\varepsilon^{-\mu}}{1-\varepsilon} + \frac{1}{1-\varepsilon^{-1}} = 1 + \varepsilon^{-1} + \cdots + \varepsilon^{-\mu}.$$

Now let $\mathcal{F} = \mathcal{O}(F)$ be an invertible G-sheaf and so $\mathcal{F}$ is also a H-sheaf. By Lemma 4.10 we may assume F is H-invariant so $F = \sum_S \lambda_S S + \sum_R \mu_R R$ where the S are isolated fixed points and the R are free H-orbits. Let $\mathcal{F}_i$ be the ith eigensheaf of $\pi_*(\mathcal{F})$. As in the case when $\mathcal{F}$ was the trivial G-sheaf the theorem is proved by computing the degree of the sheaves $\mathcal{F}_i$. Let f be the generator of $\mathcal{F}_P$.

If $P \notin \operatorname{Supp} F$ then $f = 1$ and so if f_i is a generator of $\mathcal{F}_{iQ}$ then $f_i = u_i$. If $\lambda P \in \operatorname{Supp} F$ and $P \notin C^g$ then $f_i = u_i s^{-\lambda}$ where s is the local coordinate at $Q = \pi(P)$. If $P \in C^g$ then $f_{ia} = u_{ia} s^{-\lfloor \frac{i+\lambda}{r} \rfloor}$. If $\mathcal{L}_i = \mathcal{O}_X\left(-\sum_Q \frac{b_{i,Q}}{r} Q\right)$ then

$$\mathcal{F}_i = \mathcal{O}_X\left(-\sum_Q \frac{b_{i,Q}}{r} Q + \sum_S \left\lfloor \frac{i+\lambda_S}{r} \right\rfloor \pi(S) + \sum_R \mu_R \pi(R)\right).$$

Now $\pi(R)$ makes the same contribution to each of the sheaves $\mathcal{F}_i$ so makes no contribution to $\sum \varepsilon^i \deg \mathcal{F}_i$ by Equation 4.3. If P is fixed by g but $P \notin \operatorname{Supp} F$ then P makes the same contribution $\sum \varepsilon^i \deg \mathcal{F}_i$ as it made to $\sum \varepsilon^i \deg \mathcal{L}_i$ and since $\mathrm{Tr}(g\colon \mathcal{F}_{|P}) = \mathrm{Tr}(g\colon \mathcal{O}_{|P})$ the theorem holds.

If $P \in C^g$ and $\lambda P \in \operatorname{Supp} F$ then P contributes

$$\sum_{i=0}^{r-1} \left(\left\lfloor \frac{i+\lambda}{r} \right\rfloor - \frac{i}{r} \right) \varepsilon^{ia}$$

to $\sum \varepsilon^i \deg \mathcal{F}_i$.

Now there exists $0 \le \alpha < r$ such that $\alpha \equiv -\lambda \pmod r$. Thus

$$\begin{aligned}
&\sum_{i=0}^{r-1}\left(\left\lfloor\frac{i+\lambda}{r}\right\rfloor - \frac{i}{r}\right)\varepsilon^{ia} \\
&\quad = \left(\frac{\lambda+\alpha}{r} - \frac{\alpha}{r}\right)\varepsilon^{\alpha a} + \cdots + \left(\frac{\lambda+\alpha}{r} - \frac{\alpha+r-1}{r}\right)\varepsilon^{(\alpha+r-1)a} \\
&\quad = \frac{\lambda}{r}\varepsilon^{\alpha a} + \frac{\lambda-1}{r}\varepsilon^{(\alpha+1)a} + \cdots + \frac{\lambda-(r-1)}{r}\varepsilon^{(\alpha+r-1)a} \\
&\quad = \frac{-\varepsilon^{\alpha a}}{r}\left(\varepsilon^{a} + 2\varepsilon^{2a} + \cdots + (r-1)\varepsilon^{(r-1)a}\right) = \frac{\mathrm{Tr}(g\colon \mathcal{F}_P)}{1-\mathrm{d}g_P}
\end{aligned}$$

□

4.3.3 Proof of Theorem 4.1 when the action is not coprime

In this case we show that the points in $C^H \smallsetminus C^g$ will contribute to $\deg \mathcal{F}_i$ but these terms will cancel in the summation $\sum_i \varepsilon^i \deg \mathcal{F}_i$. Assume for simplicity of notation that $\mathcal{F} = \mathcal{O}_C$ but the proof of the general case is the same.

Take a point $P \in C^H \smallsetminus C^g$. Then $\mathrm{Stab}_H P$ is generated by some element $g^a \in H$ and let a be the least such a. Now $g^a t_P = \varepsilon^{\lambda a} t_P$ and assume for simplicity that $\lambda = 1$. Let $a' = r/a$ and consider the short exact sequence

$$0 \to \mathbb{Z}_{a'} \to \mathbb{Z}_r \to \mathbb{Z}_a \to 1.$$

Then consider the chain of quotients

$$C \to C_1 = C/\mathbb{Z}_{a'} \to C_1/\mathbb{Z}_a = X.$$

Let $\pi_1\colon C \to C_1$ and $\pi_1(P) = Q_1$ then $\pi_{1*}\mathcal{O}_C = \bigoplus_{i=0}^{a'-1} \mathcal{L}'_i$. The morphism $\pi_2\colon C_1 \to X$ is unramified at Q_1 and $\pi_{2*}\mathcal{L}'_i = \bigoplus_{j=0}^{a-1} \mathcal{L}_{i+ja'}$. Then as in the previous subsection, for fixed i, the sheaf $\mathcal{L}_{i+ja'}^{\otimes r}$ has the same degree b_i at

$Q = \pi_2(Q_1)$ for all j. Thus this point contributes

$$\sum_{i=0}^{a'-1}\sum_{j=0}^{a-1} b_i \varepsilon^{i+ja'} = \sum_{i=0}^{a-1} b_i \left(\varepsilon^i + \varepsilon^{i+a'} + \cdots + \varepsilon^{i+a'(a-1)}\right) = 0$$

to $\operatorname{Tr}\left(g\colon \bigoplus_{i\geq 0}(-1)^i \operatorname{H}^i(C,\mathcal{F})\right)$. The same argument holds for any invertible G-sheaf $\mathcal{F}$ on C.

Hence for any nontrivial $g \in G$

$$\operatorname{Tr}\left(g\colon \bigoplus_{i\geq 0}(-1)^i \operatorname{H}^i(C,\mathcal{F})\right) = \sum_{P\in C^g} \frac{\operatorname{Tr}\left(g\colon \mathcal{F}_{|P}\right)}{(1-\mathrm{d}g_P)}.$$

$\square$

4.4 Proof of Theorem 4.1 on a smooth projective surface.

Let $Z = S$ a smooth projective surface then Theorem 4.1 can be rewritten in the following way.

Theorem 4.16 *Let G be a finite group acting on S and $\mathcal{F}$ an invertible G-sheaf on S. Assume that only the trivial element of G acts trivially on S. Then for any nontrivial $g \in G$*

$$\operatorname{Tr}\left(g\colon \bigoplus_{p\geq 0}(-1)^p \operatorname{H}^p(S,\mathcal{F})\right) = \sum_{P\in S^g} \frac{\operatorname{Tr}\left(g\colon \mathcal{F}_{|P}\right)}{\det(I-\mathrm{d}g_P)} + \sum_{C\subset S^g} \operatorname{Tr}\left(g\colon \mathcal{F}_{|C}\right)\left(\frac{1-\mathrm{p_a}(C)+\mathrm{c}_1(\mathcal{F}_{|C})}{\det(I-\mathrm{d}g_C)} - \frac{\mathrm{d}g_C C^2}{\det(I-\mathrm{d}g_C)^2}\right). \quad (4.8)$$

The first sum runs through the isolated fixed points of g and the second sum runs though prime divisors C fixed by g. The genus of the curve C is denoted by $\mathrm{p_a}(C)$ and $\mathrm{d}g_C\colon N^*_{C/S} \to N^*_{C/S}$ is the induced action of g on the conormal space to $C \subset S$.

As in Theorem 4.11 if $g = 1$ then by Hirzebruch Riemann–Roch

$$\operatorname{Tr}\left(1\colon \bigoplus_{p\geq 0}(-1)^p \operatorname{H}^p(S,\mathcal{F})\right) = \chi(S,\mathcal{F}) = \frac{c_1(\mathcal{F})\,(c_1(\mathcal{F}) - K_S)}{2} + 1 + \operatorname{p_a}(S).$$

To prove this theorem first show that there exists a formula of the form $\operatorname{Tr}\left(g\colon \bigoplus_{p\geq 0}(-1)^p \operatorname{H}^p(S,\mathcal{O}_S)\right) = \sum_{F\subset S^g} \operatorname{RR}(F)$ and then compute the terms $\operatorname{RR}(F)$ for each irreducible $F \subset S^g$.

4.4.1 Notation

Let $S^g = \{C_1,\dots,C_n,P_1,\dots,P_m\}$, where C_i are divisors and P_i are isolated points. The C_i are disjoint smooth curves and $P_j \notin C_i$ for all i,j. Let $X = S/H$ and $\pi\colon S \to X$ be the quotient morphism. Let $B_i = \pi(C_i)$ and $Q_i = \pi(P_i)$.

The sheaf $\pi_*\mathcal{F}$ on X decomposes as a direct sum of eigensheaves $\bigoplus_{i=0}^{r-1}\mathcal{F}_i$. Each of the $\mathcal{F}_i$ are divisorial sheaves on X which are invertible away from the singularities of X by Lemma 4.8. Again the decomposition of $\pi_*\mathcal{O}_S$ into eigensheaves is denoted by $\bigoplus_{i=0}^{r-1}\mathcal{L}_i$ when $\mathcal{O}_S$ is the trivial G-sheaf. Denote the $\mathbb{Q}$-Cartier divisor corresponding to $\mathcal{L}_i$ by D_i.

Let $\mathcal{O}(D_1)$ and $\mathcal{O}(D_2)$ be reflexive sheaves on X and let X^0 be the nonsingular locus of X. It may be that $\mathcal{O}(D_1)\otimes\mathcal{O}(D_2) \neq \mathcal{O}(D_1+D_2)$ but $(\mathcal{O}(D_1)\otimes\mathcal{O}(D_2))^{**} = \mathcal{O}(D_1+D_2)$. Since $\operatorname{codim}(X \smallsetminus X^0, X) = 2$ the sheaf $(\mathcal{O}(D_1)\otimes\mathcal{O}(D_2))^{**}$ is the unique reflexive sheaf on X such that

$$(\mathcal{O}(D_1)\otimes\mathcal{O}(D_2))^{**}_{|X^0} = \mathcal{O}(D_1)_{|X^0}\otimes\mathcal{O}(D_2)_{|X^0}.$$

See [KM98] for details. We denote by $\mathcal{L}^{[r]}$ the reflexive sheaf $(\mathcal{L}^{\otimes r})^{**}$.

Let $f\colon Y \to X$ be a resolution of X such that f is a log resolution of each pair (X, D_i). Let $\mathcal{M}_i = (f^*\mathcal{L}_i)/(\text{torsion})$. Then by Lemma 4.6 $\mathcal{M}_i$ is an invertible sheaf. Write M_i for the corresponding divisor. Let $A_i = f^*B_i$ and $E_i = \sum_j E_{i,j} = f^{-1}Q_i$ where each $E_{i,j}$ is an irreducible curve. Lemma 4.5 implies that

$$\chi(\mathcal{M}_i) \quad = \quad \chi(\mathcal{L}_i). \tag{4.9}$$

As in the case of curves

$$\begin{aligned}
\operatorname{Tr}\left(g\colon \bigoplus_{p\geq 0}(-1)^p \operatorname{H}^p(S,\mathcal{F})\right) &= \sum_{i=0}^{r-1}\varepsilon^i\chi(\mathcal{L}_i) = \sum_{i=0}^{r-1}\varepsilon^i\chi(\mathcal{M}_i)\\
&= \sum_{i=0}^{r-1}\varepsilon^i\left(\chi(\mathcal{O}_Y)+\frac{1}{2}M_i(M_i-K_Y)\right)\\
&= \sum_{i=0}^{r-1}\frac{\varepsilon^i}{2}M_i(M_i-K_Y).
\end{aligned}$$

Theorem 4.16 is proved in three steps. First assume that the action is coprime and that the invertible sheaf is trivial, then prove the theorem for any invertible G-sheaf and finally prove it for any action.

4.4.2 Proof of Theorem 4.1 for the trivial G-sheaf when the action is coprime

If $S^H = S^g$ then the points Q_i are the only singularities of X.

Example 4.17 Let $\mathbb{Z}_r$ act on $S = \mathbb{P}^2$ by $g = \frac{1}{r}(0,0,1)$. Let $L_2\colon\ (x_2 = 0)$ and $P_2 = (0:0:1)$ then $S^g = \{L_2, P_2\}$. Again let $\mathcal{F}$ be the trivial G-sheaf so

$$\operatorname{Tr}\left(g\colon \bigoplus_{p\geq 0}(-1)^p \operatorname{H}^p(S,\mathcal{O}_S)\right) = 1.$$

The coordinate on the conormal space to L_2 in S is $\mathrm{d}\,(x_2/x_0)$ and so $\mathrm{d}g_{L_2} = \varepsilon$. The coordinates on the cotangent space to P in S are $\mathrm{d}(x_0/x_2)$

and $\mathrm{d}(x_1/x_2)$ so $\mathrm{d}g_P = \begin{pmatrix} \varepsilon^{-1} & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix}$. Therefore

$$\sum_{C\subset S^g} \mathrm{Tr}\left(g\colon \mathcal{O}_{|L_2}\right)\left(\frac{1-\mathrm{p_a}(L_2)+\mathrm{c}_1(\mathcal{O}_{|L_2})}{\det(I-\mathrm{d}g_{L_2})} - \frac{\mathrm{d}g_{L_2}L_2^2}{\det(I-\mathrm{d}g_{L_2})^2}\right)+$$
$$\sum_{P\in S^g}\frac{\mathrm{Tr}\left(g\colon \mathcal{O}_{|P}\right)}{\det(I-\mathrm{d}g_P)} = \frac{1-0}{1-\varepsilon} - \frac{\varepsilon}{(1-\varepsilon)^2} + \frac{1}{(1-\varepsilon^{-1})^2} = 1.$$

The theorem will be proved by constructing the divisor M_i and computing M_i^2 and M_iK_Y. Let P be any point on S which is not fixed by g so $Q = \pi(P)$ is a nonsingular point of X and let $u_{i,Q}$ be the generator of the free $\mathcal{O}_Q$-module $\mathcal{L}_{iQ}$. As in the case of curves $\mathcal{L}_{iQ}^{[r]}$ is generated by 1 as an $\mathcal{O}_Q$-module. In a neighbourhood of $\widetilde{Q} = f^{-1}(Q)$, the resolution Y is isomorphic to X so $u_{i,Q}$ is also a generator for $\mathcal{M}_{i\widetilde{Q}}$.

Next let $P \in C$, one of the fixed divisors, again $Q = \pi(P)$ is a nonsingular point of X and $\mathcal{L}_{iQ}$ is a free $\mathcal{O}_Q$-module generated by $u_{i,Q}$. Now $\mathcal{L}_{iQ}$ is a submodule of $\mathcal{O}_P$ and choose $s,t \in m_P/m_P^2$ such that s is the local equation for C and $g(s,t) = (\varepsilon^a s, t)$. Therefore $u_{i,Q} = (\mathrm{unit})s^{b_i}$ where $b_i a \equiv i \pmod r$. Now $u_{i,Q}^r = s^{rb_i} \in \mathcal{O}_Q$ and s^r is locally the equation of $B = \pi(C)$. Thus $-b_iB \in \operatorname{Supp} rD_i$.

Figure 4.1: Exceptional locus for the blowup of Q

Finally let $P = P_j$ be an isolated fixed point. Then $s,t \in m_P/m_P^2$ can be chosen so that $g(s,t) = (\varepsilon^a s, \varepsilon^b t)$ and, since we are assuming the action is coprime, a and b are coprime to r. Now $Q = \pi(P)$ is a singular point of X. Figure 4.1 shows the exceptional locus of Y in a neighbourhood of $f^{-1}Q$.

Let $\widetilde{Q}$ be any point in $f^{-1}Q$. At $\widetilde{Q}$ the local coordinates are s_k and s_{k+1} where s_k is the local coordinate along the line E_k. Now $s_k = s^{\alpha_k}/t^{\beta_k}$ and $s_{k+1} = t^{\beta_{k+1}}/s^{\alpha_{k+1}}$ where α_i and β_i depend only on the local analytic nature of the singularity Q, that is only on how g acts in a local analytic neighbourhood of P.

Now $\mathcal{M}_{i\widetilde{Q}}$ is a free $\mathcal{O}_{\widetilde{Q}}$-module generated by some $u_i \in \mathcal{L}_{iQ}$. Since $u_i^r \in \mathcal{O}_{\widetilde{Q}}$ it follows that $u_i^r = (\text{unit})s_k^{c_{i,k+1}}s_{k+1}^{c_{i,k}}$ where $c_{i,k+1}$ and $c_{i,k}$ are defined such that

$$ac_{i,k+1}\left(\alpha_k - \alpha_{k+1}\right) + bc_{i,k}\left(\beta_{k+1} - \beta_k\right) \equiv i \pmod r.$$

Thus they depend only on a, b, r, s_k and s_{k+1}.

Let L_0 be the strict transform of the curve $(s^r = 0)$ shown in Figure 4.1 and $\widetilde{Q}$ a point in a neighbourhood of $L_0 \cap E_1$. Now $s_0 = t^r$ and $s_1 = s/t^\mu$ are local coordinates for $\widetilde{Q}$ for some μ. Since $t^\lambda \in \mathcal{M}_{i\widetilde{Q}}$ for some $1 \leq \lambda \leq r-1$ it follows that $u_i^r = s_0^\lambda$ and $L_0 \notin \operatorname{Supp} rM_i$. Similarly $L_1 \notin \operatorname{Supp} rM_i$.

Thus the local generator of the sheaf $\mathcal{M}_{i\widetilde{Q}}$ depends only on how g acts on a local analytic neighbourhood of the fixed point.

By continuing this process over all $\widetilde{Q} \in Y$ it follows that

$$rM_j = \sum_i -b_{i,j}A_i + \sum_{i,k} -c_{i,k,j}E_{i,k} = f^*(rD_i) + \sum_{i,k} -c_{i,k,j}E_{i,k}.$$

The terms $c_{i,k,j}$ depend only on how g acts on a local analytic neighbourhood of the points P_i.

Also $rK_Y = f^*(rK_X) - \sum d_{i,k}E_{i,k}$ and $d_{i,k}$ depends on only the local analytic nature of the singularites of X. By the projection formula $f^*A \cdot B =$

$A \cdot f_* B$ and so if B is exceptional then $f^* A \cdot B = 0$. Therefore

$$\begin{aligned}
\chi(\mathcal{M}_j) &= \chi(\mathcal{O}_Y) + \frac{1}{2}(M_j^2 - M_j K_Y) \\
&= \chi(\mathcal{O}_Y) + \sum_i \frac{b_{i,j}^2 A_i^2 - r b_{i,j} A_i f^* K_X}{2r^2} \\
&\quad + \sum_i \frac{\left(\sum_k c_{i,k,j} E_{i,k}\right)^2 - \left(\sum_k c_{i,k,j} E_{i,k}\right)\left(\sum_k d_{i,k} E_{i,k}\right)}{2r^2} \\
&= \chi(\mathcal{O}_Y) + \sum_{F \subset S^g} \mathrm{RR}(F, j)
\end{aligned}$$

where the sum runs through all irreducible components of the fixed locus. Again $E_{i,j} E_{i,k}$ depends only on the local analytic nature of the singularites. Hence if F is an isolated fixed point then $\mathrm{RR}(F, j)$ depends only on how g acts on a local analytic neighbourhood of the singularity.

Define $\mathrm{RR}(F) := \sum_j \varepsilon^j \, \mathrm{RR}(F, j)$. We complete the proof by computing these terms for each $F \subset S^g$.

If $F = C_i \subset S$ is a fixed curve then as shown above

$$\mathrm{RR}(C_i, j) = \frac{b_{i,j}^2}{2r^2} A_i^2 - \frac{b_{i,j}}{2r} A_i K_Y = \frac{b_{i,j}^2}{2r^2} B_i^2 - \frac{b_{i,j}}{2r} B_i K_X$$

where $b_{i,j}$ is defined such that $b_{i,j} a_i \equiv j \pmod{r}$.

For ease of notation assume $a_i = 1$ and so $b_{i,1} = 1$. If instead $b_{i,\lambda} = 1$ then replace ε with ε^λ in what follows. Thus

$$\mathrm{RR}(C_i) = \sum_{j=0}^{r-1} \frac{\varepsilon^j}{2r} \left(\frac{j^2}{r} B_i^2 + j B_i K_X \right).$$

By the ramification formula $K_S = \pi^* K_X + (r-1) \sum C_i$. For any divisors D_1 and D_2 in X such that D_1 does not pass through the singularities of X then $r D_1 D_2 = \pi^*(D_1) \pi^*(D_2)$. Thus

$$\begin{aligned}
\mathrm{RR}(C_i) &= \sum_{j=0}^{r-1} \frac{\varepsilon^j}{2r} \left(j^2 C_i^2 + j K_S C_i - j(r-1) C_i^2 \right) \\
&= -\frac{\varepsilon C_i^2}{(1-\varepsilon)^2} + \frac{1 - \mathrm{p_a}(C_i)}{1 - \varepsilon}. \qquad (4.10)
\end{aligned}$$

The final equality holds since

$$\sum_{i=1}^{r-1} i^2\varepsilon^i = \frac{-r^2}{1-\varepsilon} + \frac{-2r\varepsilon}{(1-\varepsilon)^2}.$$

Let $F = P_i$ be an isolated fixed point in S. Choose local analytic coordinates at P_i such that g acts by $g(s,t) = (\varepsilon^a s, \varepsilon^b t)$. We have shown above that $\mathrm{RR}(P_i)$ depends only on the action of g on a local analytic neighbourhood of P_i. Thus $\mathrm{RR}(P_i)$ can be computed for a point $P_i \in C \times C$ where C is a smooth curve acted on by $\mathbb{Z}_r$.

Let $\mathbb{Z}_r$ act on $\mathbb{P}^2$ by $h = \frac{1}{r}(1,1,0)$. We showed in Example 4.17 that ${\mathbb{P}^2}^h = \{L_2, P_2\}$. Let C_r be a smooth $\mathbb{Z}_r$-invariant curve of degree r in $\mathbb{P}^2$ which does not pass through the point $P_2 = (0:0:1)$. For example let $C_r\colon (x_0^r + x_1^r + x_2^r = 0)$. Then h fixes the r points where C_r meets the line $L_2\colon (x_2 = 0)$. At each of these points the induced action on the cotangent space is $\mathrm{d}h\mathrm{d}t_P = \varepsilon \mathrm{d}t_P$. Thus by equivariant Riemann–Roch for curves

$$\begin{aligned}
\mathrm{Tr}\left(h^a\colon \bigoplus_{j\geq 0} (-1)^j\,\mathrm{H}^j(C_r, \mathcal{O}_{C_r})\right) &= \sum_{P\in C_r^h} \frac{1}{1-\mathrm{d}h_P^a} \\
&= \frac{r}{1-\varepsilon^a}.
\end{aligned}$$

Let $g' = (h^a, h^b) \in \mathbb{Z}_r \times \mathbb{Z}_r$ act on $C_r \times C_r$ so that the induced action on the cotangent space is given by $\mathrm{d}g(\mathrm{d}s, \mathrm{d}t) = (\varepsilon^a \mathrm{d}s, \varepsilon^b \mathrm{d}t)$. This action fixes r^2 points in $C_r \times C_r$. The Künneth formula says that

$$\mathrm{H}^i(C_r \times C_r, \mathcal{O}_{C_r\times C_r}) = \sum_{j+k=i} \mathrm{H}^j(C_r, \mathcal{O}_{C_r}) \otimes \mathrm{H}^k(C_r, \mathcal{O}_{C_r}).$$

Thus

$$\begin{aligned}
&\mathrm{Tr}\left(g\colon \bigoplus_{i\geq 0} (-1)^i\,\mathrm{H}^i(C_r \times C_r, \mathcal{O}_{C_r\times C_r})\right) \\
&= \mathrm{Tr}\left(h^a\colon \bigoplus_{j\geq 0} (-1)^j\,\mathrm{H}^j(C_r, \mathcal{O}_{C_r})\right) \mathrm{Tr}\left(h^b\colon \bigoplus_{k\geq 0} (-1)^k\,\mathrm{H}^k(C_r, \mathcal{O}_{C_r})\right) \\
&= \frac{r}{1-\varepsilon^a}\frac{r}{1-\varepsilon^b}.
\end{aligned}$$

Since there are r^2 fixed points in $C_r \times C_r$ each point contributes $\frac{1}{(1-\varepsilon^a)\left(1-\varepsilon^b\right)}$ to $\operatorname{Tr}\left(g\colon \bigoplus_{i\geq 0}(-1)^i \operatorname{H}^i(C_r \times C_r, \mathcal{O}_{C_r\times C_r})\right)$ and so

$$\operatorname{RR}(P_i) = \frac{1}{(1-\varepsilon^a)(1-\varepsilon^b)} = \frac{1}{\det(I - \mathrm{d}g_{P_i})}. \tag{4.11}$$

Together Equations 4.10 and 4.11 imply

$$\begin{aligned}\operatorname{Tr}\left(g\colon \bigoplus_{p\geq 0}(-1)^p \operatorname{H}^p(S, \mathcal{O}_S)\right) &= \sum_{i=0}^{r-1} \frac{\varepsilon^i}{2} M_i(M_i - K_Y) = \sum_{F\subset S^g} \operatorname{RR}(F) \\ &= \sum_{P\in S^g} \frac{1}{\det(I-\mathrm{d}g_P)} + \sum_{C\subset S^g} \frac{1-\mathrm{p_a}(C)}{\det(I-\mathrm{d}g_C)} - \frac{\mathrm{d}g_C C^2}{\det(I-\mathrm{d}g_C)^2}.\end{aligned}$$

□

4.4.3 Proof of Theorem 4.1 for any invertible G-sheaf

Let $\mathcal{F}$ be an invertible G-sheaf on S. By Lemma 4.9 we may assume that $\mathcal{F} \subset k(S)$ and so $\mathcal{F} = \mathcal{O}_S(F)$. Then $\mathcal{F}$ is also an H-sheaf and so F is an H-invariant divisor by Lemma 4.10. If P, an isolated fixed point, belongs to $\operatorname{Supp} F$ then we can replace F by a linearly equivalent H-invariant divisor F' where $P \notin \operatorname{Supp} F'$. Then $\mathcal{F} \cong \mathcal{F}' \otimes V_i$ where $\mathcal{F}' = \mathcal{O}(F')$ and V_i is an irreducible representation of H. Since

$$\operatorname{Tr}\left(g\colon \bigoplus_{i\geq 0}(-1)^i \operatorname{H}^i(S, \mathcal{F})\right) = V_i(g) \operatorname{Tr}\left(g\colon \bigoplus_{i\geq 0}(-1)^i \operatorname{H}^i(S, \mathcal{F}')\right)$$

and for any fixed point P

$$\operatorname{Tr}(g\colon \mathcal{F}_P) = V_i(g) \operatorname{Tr}(g\colon \mathcal{F}'_P)$$

it is sufficient to prove the theorem when $\mathcal{F} = \mathcal{O}_S(F)$ and no isolated fixed point belongs to the support of F.

Let $\pi_*\mathcal{F} = \bigoplus \mathcal{F}_i$ and $\mathcal{E}_i = f^*\mathcal{F}_i/(\text{torsion})$. As before let $\pi_*\mathcal{O}_S = \bigoplus \mathcal{L}_i$ and $\mathcal{M}_i = f_*\mathcal{L}_i/(\text{torsion})$. We prove Theorem 4.1 when F is an irreducible

G-divisor this includes the case that $F = \sum_{i=1}^{r} F_i$ where each F_i is irreducible and $F_{i+1} = gF_i$.

Let f be the generator of $\mathcal{F}_P$ as an $\mathcal{O}_P$-module and u_i a generator of $\mathcal{M}_{i\widetilde{Q}}$ as an $\mathcal{O}_{\widetilde{Q}}$-module for some $\widetilde{Q} \in \pi^{-1}(P)$. If $P \notin \operatorname{Supp} F$ then $\mathcal{E}_{i\widetilde{Q}}$ is also generated by u_i. In particular this implies that $\mathrm{RR}_{\mathcal{F}}(P) = \mathrm{RR}_{\mathcal{O}}(P)$ for any of the isolated fixed points.

Let $P \in \operatorname{Supp} F$. By Lemma 4.10 we can choose a generator of $\mathcal{F}_P$ such that $g^*f = \varepsilon^\lambda f$ so $\pi_*\mathcal{F}_Q$ is generated by $fu_1, \ldots, fu_r$ as an $\mathcal{O}_Q$-module. If $\lambda \equiv 0$ then $f \in k(X)$ and so $\mathcal{E}_i = \mathcal{M}_i\,(f^*\pi(F))$. In this case

$$\mathrm{RR}_{\mathcal{F}}(C_j) = \sum_{k=0}^{r-1} \frac{\varepsilon^k}{2r}\left(\frac{k^2}{r}A_j^2 - kA_jf^*\pi(F) + r\,(f^*\pi(F))^2 \right.$$
$$\left. + kA_jK_Y - rf^*\pi(F)K_Y\right) = \mathrm{RR}_{\mathcal{O}_S}(C_j) + \frac{C_jF}{1-\varepsilon}.$$

If $\lambda \not\equiv 0$ then $f \notin k(X)$ but $f^r \in k(X)$ and so $fu_{i-\lambda} \in \mathcal{F}_{iQ}$. In this case $\mathcal{E}_i^{\otimes r} = \mathcal{O}_Y\left(f^*\pi(F) + r\sum_j c_{i,j}A_j\right)$ where $c_{i,j} = b_{i,j}$ if A_j does not meet $f^*\pi(F)$ and $c_{i,j} = b_{i-\lambda,j}$ otherwise. Again if A_j does not meet $f^*\pi(F)$ then $\mathrm{RR}_{\mathcal{F}}(C_j) = \mathrm{RR}_{\mathcal{O}_S}(C_j)$. If $f^*\pi(F)$ meets some A_j then

$$\mathrm{RR}_{\mathcal{F}}(C_j) = \sum_{k=0}^{r-1} \frac{\varepsilon^k}{2r}\left(\frac{(k-\lambda)^2}{r}A_j^2 - \frac{k-\lambda}{r}A_jf^*\pi(F) + (f^*\pi(F))^2 \right.$$
$$\left. + kA_jK_Y - f^*\pi(F)K_Y\right)$$
$$= \varepsilon^\lambda \sum_{k=0}^{r-1} \frac{\varepsilon^k}{2r}\left(\frac{k^2}{r}A_j^2 - \frac{k}{r}A_jf^*\pi(F) + f^*\pi(F)^2 + kA_jK_Y - f^*\pi(F)K_Y\right)$$
$$= \varepsilon^\lambda\left(\mathrm{RR}_{\mathcal{O}_S}(C_j) + \frac{C_jF}{1-\varepsilon}\right) = \operatorname{Tr}\left(g\colon \mathcal{F}_{|C}\right)\left(\mathrm{RR}_{\mathcal{O}_S}(C_j) + \frac{C_jF}{1-\varepsilon}\right).$$

Note this shows that if $f^*\pi(F)$ does not meet any of the A_j, or equivalently F does not meet any of the fixed curves then, F does not affect $\operatorname{Tr}\left(g\colon \bigoplus(-1)^p\,\mathrm{H}^p\,(S,\mathcal{F})\right)$.

The argument for any invertible G-sheaf is identical. $\square$

4.4.4 Proof of Theorem 4.1 when the action is not coprime

In this case there will be curves and points which are fixed by some subgroup of $\langle g \rangle = H$. There are the following five possibilities.

1. P is a point which is not fixed by any element of the group.
2. P is a point on a curve C which is fixed by some nontrivial subgroup of H and $\operatorname{Stab}_H C = \operatorname{Stab}_H P$.
3. P is a point in the intersection of two curves C_1 and C_2 which are fixed by nontrivial subgroups of H. Now $\operatorname{Stab}_H C_1 \cap \operatorname{Stab}_H C_2 = 1$. Suppose $\operatorname{Stab}_H C_1 \times \operatorname{Stab}_H C_2 = \operatorname{Stab}_H P$.
4. As in Case 2 but $\operatorname{Stab}_H C \neq \operatorname{Stab}_H P$.
5. As in Case 3 but $\operatorname{Stab}_H C_1 \times \operatorname{Stab}_H C_2 \neq \operatorname{Stab}_H P$.

In the first three cases the point $\pi(P) = Q$ is smooth.

We will only show the proof when $\mathcal{F} = \mathcal{O}_S$ but the case of a general invertible sheaf follows easily. First compute the generator of $\mathcal{M}_{i\widetilde{Q}}$ for $\widetilde{Q} \in f^{-1}Q$ in each of the above cases.

In Case 1 it follows as before that u_i^r is a unit and so $P \notin \operatorname{Supp} rD_i$.

Case 2 is analogous to the case of curves when $C^g \neq C^H$. We show that $\mathrm{RR}(C) = 0$. Let C be a curve fixed by g^a say, where a is the least such a, $\pi(C) = B$ and $a' = r/a$. Consider the short exact sequence

$$0 \to \mathbb{Z}_{a'} \to H \to \mathbb{Z}_a \to 1.$$

Decompose the morphism $\pi\colon S \to X$ as follows:

$$S \to S_1 := S/\mathbb{Z}_{a'} \to X = S_1/\mathbb{Z}_a.$$

Now $\pi_1\colon S \to S_1$ is ramified along C and $\pi_{1*}\mathcal{O}_S = \bigoplus_{i=0}^{a'-1} \mathcal{L}'_i$. The morphism $\pi_2\colon S_1 \to X$ is unramified along $\pi_1(C)$. Let u be a generator of $\mathcal{L}_{i+ja'}{}_Q$ and v a generator of $\mathcal{L}_{i+j'a'}{}_Q$. Then $u^r = v^r$ so if $a'|i-i'$ then $\mathrm{RR}(C,i) = \mathrm{RR}(C,i')$ and so

$$\begin{aligned}
\mathrm{RR}(C) &= \sum_i \varepsilon^i \,\mathrm{RR}(C,i) \\
&= \left(1 + \varepsilon^{a'} + \cdots + \varepsilon^{(a-1)a'}\right) \mathrm{RR}(C,0) \\
&\quad + \left(\varepsilon + \varepsilon^{1+a'} + \cdots + \varepsilon^{1+(a-1)a'}\right) \mathrm{RR}(C,1) \\
&\quad + \cdots \\
&\quad + \left(\varepsilon^{a'-1} + \varepsilon^{a'-1+a'} + \cdots + \varepsilon^{a'-1+(a-1)a'}\right) \mathrm{RR}(C,a-1) \\
&= 0.
\end{aligned}$$

Simplify Case 3 by assuming that P is fixed by H. Again this assumption is to simplify the notation. Let g^a be the generator of $\mathrm{Stab}_H C_1$ and g^b the generator of $\mathrm{Stab}_H C_2$. This case is similar to the above but $\pi_2\colon S_1 \to X$ is ramified over $\pi_1(C_2)$. Let $a' = r/a$ and $b' = r/b$. Choose local coordinates for the point so that $g(s,t) = (\varepsilon^{a'\lambda}s, \varepsilon^{b'\mu}t)$ and assume, for ease of notation, that $\lambda = \mu = 1$. Thus

$$\begin{aligned}
&\mathrm{RR}(P) \\
&= \frac{1}{2}\sum_{j=0}^{a-1}\sum_{k=0}^{b-1} \frac{\varepsilon^{a'j+b'k}}{r^2}\left(j^2A_1^2 + 2jkA_1A_2 + k^2A_2^2 + jrA_1K_Y + krA_2K_Y\right) \\
&= \left(\sum_{j=0}^{a-1} \frac{j}{r}\varepsilon^{a'j}\right)\left(\sum_{k=0}^{b-1} \frac{k}{r}\varepsilon^{b'k}\right) \\
&= \frac{1}{1-\varepsilon^a}\frac{1}{1-\varepsilon^b}.
\end{aligned}$$

Also simplify Case 4 by assuming that the point P is fixed by g. Let $\mathrm{Stab}_H C$ be generated by g^a and simplify further by assuming that $g(s,t) = (\varepsilon^{a'}s, \varepsilon t)$ where s and t are the local coordinates at the point P. Let $A = f^*\pi(C)$.

Figure 4.2: Exceptional locus for the blowup of Q

Figure 4.2 shows the exceptional locus of Y in a neighbourhood of $f^{-1}(Q)$. If $\widetilde{Q} \in E_1$ then as in the prime case $u_i = t^i$ and so $L_0 \notin \mathrm{Supp}\, M_i$. If $\widetilde{Q} \in E_n$, then $u_i^r = \left(t^l/s^m\right)^{\lambda_i} (s^a)^{\mu_i}$ and again if $a'|i - i'$ then $\lambda_i = \lambda_{i'}$.

Again $rK_Y = f^*(rK_X) - \sum d_{i,k}E_{i,k}$. Therefore

$$\begin{aligned}\mathrm{RR}(P) &= \sum_i \frac{\varepsilon^i}{2r^2}\left(\left(\sum_{j=1}^n a_{i,j}E_j + \lambda_i A\right)^2 + r\sum_{j=1}^n a_jE_jK_Y + r\lambda_i AK_Y\right)\\ &= \sum_i \frac{\varepsilon^i}{2r^2}\left(\left(\sum_{j=1}^n a_{i,j}E_j\right)^2 + 2\lambda_i a_{n,i} - \sum a_{i,j}E_j \sum d_kE_k\right)\end{aligned}$$

where $a_{i,j}$, λ_i, E_jE_k and d_j depend only on the local analytic nature of the singularity $Q = \pi(P)$. Thus $\mathrm{RR}(P)$ is determined only by how g acts in a local analytic neighbourhood of the point P.

Let C_r be the curve in $\mathbb{P}^2$ of degree r defined previously. Let the action of $\mathbb{Z}_r$ on $\mathbb{P}^2$ be generated by $h = \frac{1}{r}(0,0,1)$ and $g = (h, h^a)$ acts on $C_r \times C_r$. Then $g^{a'}$ fixes the one of the curves C_r and g fixes r^2 points on $C_r \times C_r$. One of these fixed points P' contributes

$$\mathrm{RR}(P') = \frac{1}{1-\varepsilon^a}\frac{1}{1-\varepsilon}.$$

to Tr $\left(g\colon \bigoplus(-1)^i \operatorname{H}^i(C_r \times C_r, \mathcal{O}_{C_r\times C_r})\right)$. Since $\mathrm{RR}(P)$ depends only on how g acts in a local analytic neighbourhood of the point P it follows that

$$\mathrm{RR}(P) = \mathrm{RR}(P') = \frac{1}{1-\varepsilon^a}\frac{1}{1-\varepsilon}.$$

Exactly as in the previous case, for the final possibility we can show that $\mathrm{RR}(P)$ depends only on how g acts on the local analytic neighbourhood of P. So compute $\mathrm{RR}(P)$ for $P \in C_r \times C_r$ and $g = \left(h^a, h^b\right)$. Thus

$$\mathrm{RR}(P) \quad = \quad \frac{1}{1-\varepsilon^a}\frac{1}{1-\varepsilon^b}.$$

□

4.5 Orbifold Riemann–Roch

Let X be a orbifold with at worst isolated cyclic quotient singularities. Let P be a singular point of X such that there exists a neighbourhood U of P such that $U \cong V/\mathbb{Z}_r$ where $\mathbb{Z}_r$ is generated by $\frac{1}{r}(a_1, \dots, a_n)$ with a_i coprime to r. Let $\mathcal{L} = \mathcal{O}(D)$ be a divisorial sheaf on X such that $\mathcal{L}_{|U}$ is isomorphic to the ith eigensheaf of $\pi_*\mathcal{O}_V$. In this case we say P with the sheaf $\mathcal{L}$ is of type $i\left(\frac{1}{r}(a_1, \dots, a_n)\right)$.

Let $f\colon Y \to X$ be a log resolution of the pair (X, D). The sheaf $\mathcal{M} = f^*\mathcal{L}/(\text{torsion})$ is the torsion free rank one sheaf on Y such that $f_*\mathcal{M} = \mathcal{L}$ by Lemma 4.5. The sheaf $\mathcal{M}^{**}$ is reflexive and hence invertible. The proof of Lemma 4.6 implies that $f_*(\mathcal{M}^{**}) = f_*(\mathcal{M})$. Hence from now on we replace $\mathcal{M}$ with $\mathcal{M}^{**}$.

Now $\mathcal{M} = \mathcal{O}(f^*D + \sum b_iE_i)$ and $K_Y = f^*K_X + \sum a_iE_i$. Thus

$$\chi(Y, \mathcal{M}) = \int_Y \mathrm{ch}(f^*D + \sum b_iE_i)\,\mathrm{td}(Y) = \int_X \mathrm{ch}(D)\,\mathrm{td}(X) + \sum_P c'_P(D)$$

where $c'_P(D)$ depends only on the local analytic nature of the singularity. The expression $\int_X \mathrm{ch}(D)\,\mathrm{td}(X)$ is interpreted as follows. Take $\mathrm{td}(X)$ on the nonsingular locus of X and intersect with D. This makes sense because some multiple of D is Cartier so can be moved away from the singularities.

By Lemma 4.5 the sheaves $\mathcal{R}^i f_*(\mathcal{M})$ are supported only on the singularities so its space of global sections and hence $h^0(X, \mathcal{R}^i f_*(\mathcal{M}))$ depends only on the analytic nature of the singularities. Thus

$$\chi(X, \mathcal{L}) = \int_X \mathrm{ch}(D)\,\mathrm{td}(X) + \sum_P c_{i,P}$$

where $c_{i,P} = c'_{iP} + \sum_{i=1}^{n-2}(-1)^{i+1}\chi(P, \mathcal{R}^i f_*(\mathcal{M})_{|P})$.

To compute the term $c_{i,P}$ for a singularity of type $i(\frac{1}{r}(a_1, \ldots, a_n))$ with $a_1, \ldots, a_n$ coprime to r calculate them for a global orbifold Z/G having only singularities of this type.

We may assume that $\mathcal{F} \cong \mathcal{F}' \otimes V$ where $\mathcal{F}'$ is not supported at P and V is an irreducible representation of $\mathbb{Z}_r$. The contribution that P makes to $\mathrm{Tr}\left(g\colon \bigoplus_{p\geq 0}(-1)^p\,\mathrm{H}^p(Z, \mathcal{F})\right)$ depends only on how g acts on a local analytic neighbourhood of P. Therefore the contribution that P makes to $\mathrm{Tr}\left(g\colon \bigoplus_{p\geq 0}(-1)^p\,\mathrm{H}^p(Z, \mathcal{F})\right)$ equals the contribution a point $P' \in C_1 \times \cdots \times C_n$ makes to $\mathrm{Tr}\left(g'\colon \bigoplus_{p\geq 0}(-1)^p\,\mathrm{H}^p\left(C_1 \times \cdots \times C_n, V \otimes \mathcal{O}_{C_1\times\cdots\times C_n}\right)\right)$ for suitable g'.

As in the case of a surface each curve can be taken to be $C_r \subset \mathbb{P}^2$ a curve of degree r invariant under the action of $h = \frac{1}{r}(0,0,1)$ not passing through the point $(0:0:1)$. This curve has induced action $\frac{1}{r}(1)$ at each of the fixed points.

The Künneth formula implies that

$$\mathrm{H}^p(C_r \times \cdots \times C_r, \mathcal{O}_{C_1\times\cdots\times C_n}) = \bigoplus_{p_1+\cdots+p_n=p} \mathrm{H}^{p_1}(C_1, \mathcal{O}_{C_1}) \otimes \cdots \otimes \mathrm{H}^{p_n}(C_n, \mathcal{O}_{C_n}).$$

Let $g = (h^{a_1}, \ldots, h^{a_n})$ act on $C_r \times \cdots \times C_r$. This fixes r^n points in $C_r \times \cdots \times C_r$. Since each of the a_i is coprime to r it follows that $C_r^h = C_r^H$. Then Theorem 4.11 gives

$$\begin{aligned}
&\operatorname{Tr}\left(g\colon \bigoplus_{p\geq 0}(-1)^p \operatorname{H}^p(C_1\times\cdots\times C_n, V\otimes\mathcal{O}_{C_1\times\cdots\times C_n}\right)\\
= \;& \operatorname{Tr}(h^{a_1}\colon \bigoplus_{p\geq 0}(-1)^p \operatorname{H}^p(C_r, V\otimes\mathcal{O}_{C_r}))\prod_{i=2}^{n}\operatorname{Tr}(h^{a_i}\colon \bigoplus_{p\geq 0}(-1)^p \operatorname{H}^p(C_r, \mathcal{O}_{C_r}))\\
= \;& \left(\sum_{P\in C_r^h}\frac{\operatorname{Tr}\left(h^{a_1}\colon V\otimes\mathcal{O}_{C_r|P}\right)}{1-\mathrm{d}h_P^{a_1}}\right)\prod_{i=2}^{n}\left(\sum_{P\in C_r^h}\frac{\operatorname{Tr}\left(h^{a_i}\colon \mathcal{O}_{C_r|P}\right)}{1-\mathrm{d}h_P^{a_i}}\right)\\
= \;& r^n\frac{\operatorname{Tr}\left(h^{a_1}\colon V\otimes\mathcal{O}_{C_r|P}\right)}{1-\mathrm{d}h_P^{a_1}}\prod_{i=2}^{n}\left(\frac{\operatorname{Tr}\left(h^{a_i}\colon \mathcal{O}_{C_r|P}\right)}{1-\mathrm{d}h_P^{a_i}}\right)\\
= \;& r^n\frac{V(g)}{1-\varepsilon^{a_1}}\frac{1}{1-\varepsilon^{a_2}}\cdots\frac{1}{1-\varepsilon^{a_n}}.
\end{aligned}$$

Therefore $P\in Z$ contributes

$$\frac{V(g)}{(1-\varepsilon^{a_1})\cdots(1-\varepsilon^{a_n})} \text{ to } \operatorname{Tr}\left(g\colon \bigoplus(-1)^p\operatorname{H}^p(Z,\mathcal{F})\right).$$

Thus if G is a cyclic group acting on a smooth variety Z of any dimension and having only isolated fixed points then

$$\begin{pmatrix} 1 & 1 & 1 & \cdots & 1\\ 1 & \varepsilon & \varepsilon^2 & \cdots & \varepsilon^{r-1}\\ \vdots & \vdots & \vdots & & \\ 1 & \varepsilon^{r-1} & \varepsilon^{r-2} & \cdots & \varepsilon \end{pmatrix}\begin{pmatrix}\chi(X,\mathcal{L}_0)\\ \chi(X,\mathcal{L}_1)\\ \vdots\\ \chi(X,\mathcal{L}_{r-1})\end{pmatrix} = \begin{pmatrix}\chi(Z,\mathcal{F})\\ \sum_{P\in Z^g}\frac{\operatorname{Tr}(g\colon(\mathcal{F}_P))}{\det(I-\mathrm{d}g_P)}\\ \vdots\\ \sum_{P\in Z^g}\frac{\operatorname{Tr}(g^{r-1}\colon(\mathcal{F}_P))}{\det(I-\mathrm{d}g_P^{r-1})}\end{pmatrix}. \tag{4.12}$$

Inverting the Vandermonde matrix gives a Riemann–Roch expression for a sheaf $\mathcal{L}_i$ on a global orbifold X.

$$\frac{1}{r}\begin{pmatrix} 1 & 1 & 1 & \cdots & 1\\ 1 & \varepsilon^{r-1} & \varepsilon^{r-2} & \cdots & \varepsilon\\ \vdots & \vdots & \vdots & & \\ 1 & \varepsilon & \varepsilon^2 & \cdots & \varepsilon^{r-1} \end{pmatrix}\begin{pmatrix}\chi(Z,\mathcal{F})\\ \sum_{P\in Z^g}\frac{\operatorname{Tr}(g\colon(\mathcal{F}_P))}{\det(I-\mathrm{d}g_P)}\\ \vdots\\ \sum_{P\in Z^g}\frac{\operatorname{Tr}(g^{r-1}\colon(\mathcal{F}_P))}{\det(I-\mathrm{d}g_P^{r-1})}\end{pmatrix} = \begin{pmatrix}\chi(X,\mathcal{L}_0)\\ \chi(X,\mathcal{L}_1)\\ \vdots\\ \chi(X,\mathcal{L}_{r-1})\end{pmatrix}$$

This proves the following theorem.

Theorem 4.18 (Orbifold Riemann–Roch) *Let X be a cyclic orbifold having at worst isolated quotient singularities. Let $\mathcal{L}$ be a divisorial sheaf such that $\mathcal{L}_{|U_P}$ is isomorphic to the ith eigensheaf of $\pi_*(\mathcal{O}_V)$. Then*

$$\chi(X, \mathcal{L}) = \int_X \mathrm{ch}(D)\,\mathrm{td}(X) + \sum_{P \in \mathrm{Sing}\, X} c_{i,P}.$$

If P is a point of type i $\left(\frac{1}{r}(a_1, a_2, \ldots, a_n)\right)$ for $i = 0, \ldots r-1$ then

$$c_{i,P} = \frac{1}{r}\sum_{l=1}^{r-1} \frac{\varepsilon^{-il}}{(1-\varepsilon^{la_1})\cdots(1-\varepsilon^{la_n})}.$$

By Equation 4.12 when $X = Z/G$ then

$$\int_X \mathrm{ch}(D)\,\mathrm{td}(X) = \frac{1}{r}\int_Z \mathrm{ch}(D')\,\mathrm{td}(Z)$$

where $D' = \pi^* D$.

This expression can be simplified to show that

$$\begin{aligned} c_{i,P} &= (-1)^{n+1}\left(\frac{(r-1)^n}{2^n r} - \sum_{(\lambda_1,\ldots,\lambda_n)\in\Lambda_{i,P}} \frac{\lambda_1\cdots\lambda_n}{r^n}\right), \qquad (4.13)\\ \Lambda_{i,P} &:= \left\{(\lambda_1,\ldots,\lambda_n) \mid \sum a_j\lambda_j \equiv i \pmod r,\ 0 \le \lambda_k \le r-1\right\}. \end{aligned}$$

Remark 4.19 The following two observations follow immediately from this definition.

- If $(\lambda_1, \ldots, \lambda_n) \in \Lambda_{i,P}$ then $(r-\lambda_1, \ldots, r-\lambda_n) \in \Lambda_{-i,P}$.
- $\sum_i c_{i,P} = (-1)^{n+1}\left(r(r-1)^n/(2^n r) - (1+\cdots+r-1)^n/r^n\right) = 0$.

If $n = 1$ then there are no singular points but we can still compute $c_{i,P}$. In Section 4.3 we computed the degree of the sheaves $\mathcal{L}_j$ by calculating the degree of the divisors D_j as $\mathbb{Q}$-divisors. If we let $\mathcal{L}'_j = \mathcal{O}(\lfloor D_j \rfloor)$ then the difference between $\chi(X, \mathcal{L}_j)$ and $\chi(X, \mathcal{L}'_j)$ is the sum of the terms $c_{j,P}$ for all P appearing in $\operatorname{Supp} D_j$. For P a point of type $i\left(\frac{1}{r}(1)\right)$, Equation 4.13 implies

$$c_{i,P} = \frac{r-1}{2r} - \frac{i}{r}.$$

Let P be a singularity of type $i\left(\frac{1}{r}(1,-1)\right)$. Then

$$\Lambda_0 = \{(1,1),(2,2),\ldots,(r-1,r-1)\}$$

and

$$\begin{aligned} c_{0,P} &= \frac{1+4+\cdots+(r-1)^2}{r^2} - \frac{(r-1)^2}{4r} \\ &= \frac{r(r-1)(2r-1)}{6r^2} - \frac{(r-1)^2}{4r} = \frac{(r-1)(r+1)}{12r}. \end{aligned} \tag{4.14}$$

The following result is taken from Reid's "Young person's guide to canonical singularities" [Rei87]

$$c_{i,P}\left(\frac{1}{r}(a_1,\ldots,a_n)\right) - c_{i-a_n,P}\left(\frac{1}{r}(a_1,\ldots,a_n)\right) = c_{i,P}\left(\frac{1}{r}(a_1,\ldots,a_{n-1})\right).$$

Together with Equation 4.14 this implies

$$c_{i,P} = \frac{r^2-1}{12r} - \frac{i(r-i)}{2r}.$$

This agrees with [Rei87].

If P is a singularity of type $i\left(\frac{1}{r}(1,1)\right)$ then

$$\Lambda_{0,P} = \{(1,r-1),(2,r-2),\ldots,(r-1,1)\}$$

and in the same way it follows that

$$c_{i,P} = \frac{(r-1)(5-r)}{12r} + \frac{i(r-i-2)}{2r}.$$

Let G be any finite group such that Z^g is zero dimensional for all $g \in G$. Then we will obtain a similar matrix expression to Equation 4.12 where the square matrix will be the character table of G. In the same way inverting this matrix will imply an orbifold Riemann–Roch formula for any orbifold having isolated singularities.

Bibliography

[AS68] M. Atiyah and I. Singer. The index of elliptic operators. III. *Ann. of Math. (2)*, 87: pages 546–604, 1968.

[Ati67] M. Atiyah. *K-theory.* W. A. Benjamin, Inc., New York-Amsterdam, 1967.

[BKR01] T. Bridgeland, A. King, and M. Reid. The McKay correspondence as an equivalence of derived categories. *J. Amer. Math. Soc.*, 14: pages 535–554, 2001.

[Bou65] N. Bourbaki. *Algèbre commutative.* Hermann, 1965.

[CR99] A. Craw and M. Reid. How to calculate A-Hilb $\mathbb{C}^3$. Preprint math.AG/9909085, 1999.

[Cra01] A. Craw. *The Mckay correspondence and representations of the McKay quiver.* PhD thesis, University of Warwick, 2001.

[God73] R. Godement. *Topologie algébrique et théorie des faisceaux.* Hermann, 1973.

[GSV83] G. Gonzalez-Sprinberg and J. Verdier. Construction géométrique de la correspondance de McKay. *Ann. Sci. École Norm. Sup.*, 16: pages 409–449, 1983.

[Har77] R. Hartshorne. *Algebraic geometry.* Springer-Verlag, 1977.

[IN99] Y. Ito and I. Nakamura. Hilbert schemes and simple singularities. In *New trends in algebraic geometry (Warwick, 1996)*, pages 151–233. Cambridge Univ. Press, 1999.

[Ito95] Y. Ito. Crepant resolution of trihedral singularities and the orbifold Euler characteristic. *International Journal of Mathematics*, 6(1): pages 33–43, 1995.

[KM98] J. Kollár and S. Mori. *Birational geometry of algebraic varieties.* Cambridge University Press, 1998.

[Lan93] S. Lang. *Algebra.* Addison-Wesley Publishing Company, third edition, 1993.

[Mum85] D. Mumford. *Abelian varieties.* Oxford University Press, 1985.

[Nak01] I. Nakamura. Hilbert schemes of Abelian group orbits. *J. Alg. Geom.*, 10, 2001.

[Rei80] M. Reid. Canonical 3-folds. In *Journées de Géometrie Algébrique d'Angers, Juillet 1979/Algebraic Geometry, Angers, 1979*, pages 273–310. Sijthoff & Noordhoff, 1980.

[Rei87] M. Reid. Young person's guide to canonical singularities. In *Proceedings of Symposia in Pure Mathematics*, volume 46, pages 345–414. Amer. Math. Soc., 1987.

[Rei97] M. Reid. Mckay correspondence. *Proc. of algebraic geometry symposium (Kinosaki, Nov 1996)*, pages 14–41, 1997.

[Rei02] M. Reid. La correspondance de McKay. *Astérisque*, 276: pages 53–72, 2002.

[Sha94] I. R. Shafarevich. *Basic algebraic geometry 1*. Springer-Verlag, 1994.