Twenty five years of 3-folds – an old person's view

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1 Introduction

This paper is an expanded version of a historical and autobiographical talk given at the closing conference of the Warwick 3-folds activity in December 1995 (at the time still entitled "Twenty years ... "). It discusses the development of 3-folds since the 1970s, particularly my work on canonical singularities and Mori theory since around 1975.

Castelnuovo and Enriques's work establishing the foundational results on algebraic surfaces took place over something like 12 years around 1900. The classification of surfaces has undoubtedly progressed since then, and remains a vital area of research, but the foundations of the theory were laid during this relatively short period. The comparable foundational work on 3-folds took place over the 20 years between 1970 and 1990, from the first papers of Iitaka and Ueno on Kodaira dimension through to Mori's solution of the flip problem and his Fields medal at the Kyoto international congress. While future generations will undoubtedly simplify and extend the theory of 3-folds, and find new applications, it seems to be a matter of historical record that the primary issues were settled during the 1970s and 1980s.

A survey based on autobiography may be a relatively painless way to gain insight into the subject, compared to the investment of effort involved in working through a more serious account of Mori theory. The narrative form allows me to stress that, while the subject is not without its technical aspects, the key issues are mostly very simple once grasped; like other revolutionary ideas, canonical singularities and Mori theory build on classical foundations, and, with the benefit of hindsight, stand out clearly as inevitable developments. Looking through my old letters and papers for a retrospective

*It takes a long time to make a big pot.

sketch of my early career in 3-folds has given me some new insights, a bit like preparing the index to Volume 1: Foundations. Talk of "inevitable lines of thought" prompts the hope that some bolshy youngsters will mount their own revolution on my generation's orthodoxy (but I don't see them coming).

I don't really need to apologise for the self-indulgence of my undertaking: a whole generation of algebraic geometers now work in the classification of varieties, and it is hard to read a paper or hear a lecture in the subject without recognising concepts and terminology that I had a hand in shaping.

History It is widely appreciated that mathematicians usually treat history in a curiously dishonest way, rewriting the history of the subject as it should have been discovered, aiming to exploit past experience in building a modern worldview. The essential difficulty seems to be that the story in strictly chronological order will not make sense to anyone; the writer wants to give an explanation based on the logical layout of the subject, whatever violence it does to historical truth. Writing this paper has certainly confirmed this point for me: given the choice between saying things chronologically or according to the divisions of the subject matter, my sense of style always leads me to the latter. Another inevitable slant of autobiographical writing is to exaggerate the significance of the author's discoveries, and I also follow this tradition.

My first introduction to algebraic geometry took place in my first year as a graduate student at Cambridge in 1969–70 under the benevolent and liberal care of Peter Swinnerton-Dyer. I learned many things directly from him: the factorisation of birational maps between del Pezzo surfaces, Hodge theory and its application to Abelian varieties. A particular item that stands out is the derivation of the Weierstrass model of an elliptic curve from Riemann–Roch: the graded ring $R(E,P) = k[x,y,z]/(z^2 = y^3 + ax^4y + bx^6)$ and its Proj, the weighted hypersurface $E_6 \subset \mathbb{P}(1,2,3)$, has served as the model for all my graded ring calculations.

On Swinnerton-Dyer's advice I studied Mumford's "little red book" in a joint seminar with Jean-Louis Colliot-Thélène and Barry Tennison. My work on canonical 3-folds could reasonably be traced back to Mumford's question ([red], Chapter Three, §8; this follows a treatment of normalisation as a method of resolving surface singularities):

> *The existence of such a simple way of "making" every variety normal is one of the reasons why normal varieties are an important class. Life would be much simpler if there were an analogous way of canonically constructing a nonsingular variety birationally equivalent to any given variety.*

2 Canonical models are inevitable

The main discovery of my early career in 3-folds was the realisation that singularities arise as an inevitable part of higher dimensional life. They turn up in two different ways: by *equations* and by *cyclic quotient constructions*. Already the simplest cases are fun, and understanding them properly takes us a long way into the substantial issues. The ordinary quadric cone arises both as the simplest hypersurface singularity $X : (xy = z^2) \subset \mathbb{C}^3$ and as the simplest quotient singularity, the quotient of $\mathbb{C}^2$ by $(u, v) \mapsto (-u, -v)$. Its resolution $Y \to X$ is the ordinary blowup, having exceptional locus $E \cong \mathbb{P}^1$ with $E^2 = -2$ (a -2-curve), and it is well known that the canonical class of Y is trivial. Because $K_Y = K_X = 0$ this is a crepant resolution of a strictly canonical singularity. The 3-fold ordinary double point $(x^2+y^2+z^2+t^2 = 0)$ or $(xy = zt)$ discussed in Section 2.1 behaves in wholly different ways: it is not a finite quotient singularity, it is terminal, its blowup does not have trivial canonical class, it has the two small resolutions discussed in Section 3.2 that are related by the classic flop. It is completely unrelated to the simplest 3-fold quotient singularity $\frac{1}{2}(1, 1, 1)$, the quotient of $\mathbb{C}^3$ by $(u, v, w) \mapsto (-u, -v, -w)$ or the cone over the Veronese surface discussed in Section 2.2.

The next point concerns canonical rings. As I explain in Section 2.3, my interpretation of Enriques' work on surfaces of general type underlines the importance of working with pluricanonical graded rings, rather than just the 1-canonical or the individual m-canonical linear systems. Zariski and Mumford's proof of the finite generation of canonical rings of surfaces and Grothendieck's definition of Proj frees us from the obligation of working only with varieties with a fixed embedding in projective space and graded rings generated by elements of degree 1. The first canonical 3-folds generalise Enriques' models in a natural way. They make clear, for example, that the birational invariant K^3 that controls the asymptotic growth of the plurigenera is a rational number. Once this point is recognised, there is really no turning back to the nonsingular category.

2.1 Lefschetz theory and the ordinary double point

My first main point is the following:

> *The ordinary double point has entirely different behaviour and meaning in different dimensions.*

The ordinary double point (ODP)

$$X : \Big(\sum_{i=0}^{n+1} x_i^2 = 0\Big) \subset \mathbb{C}^{n+1}.$$

is the simplest n-fold singularity. The function $\mathbb{C}^{n+1} \to \mathbb{C}$ given by $f(\underline{x}) = \sum x_i^2$ has a *nondegenerate critical point* at 0: critical because all $\partial f/\partial x_i = 0$, and nondegenerate because the Hessian matrix of second derivatives is a nondegenerate quadratic form. Over $\mathbb{R}$, these are just the properties of a Morse function. Lefschetz theory is a complex analog of Morse theory. However, in many contexts of classification, we may be more interested in the singular n-fold X than in the function $f(\underline{x})$.

The basic idea of Lefschetz theory is to argue on the neighbouring fibre $X_t = f^{-1}(t)$ for t close to 0. Let's work in a small neighbourhood $\sum |x_i|^2 \leq \delta$ of the singular point, and take $t = \varepsilon e^{i\theta}$ where $0 < \varepsilon \ll \delta \ll 1$. Then from the C^∞ point of view, X_t and X_0 practically coincide outside the δ-ball, and $M = X_t \cap \delta$-ball (the *Milnor fibre* of the singularity) is given by quadratic equations. Scaling and separating real and imaginary parts puts the equations of M in the form

$$\sum u_i^2 = 1 \text{ and } \sum u_i v_j = 0, \quad \text{where} \quad x_i = (\text{modulus}) \times e^{i\theta} \times (u_i + iv_i).$$

First, the things that are common to all dimensions: the Milnor fibre is the tangent bundle to S^n (see Figure 2.1.1), the vanishing cycle is $\delta = S^n$, the monodromy as we follow the diffeomorphism of the fibres around $e^{i\theta}$ is given by the Picard–Lefschetz formula $x \mapsto x \pm (x \cdot \delta)\delta$.

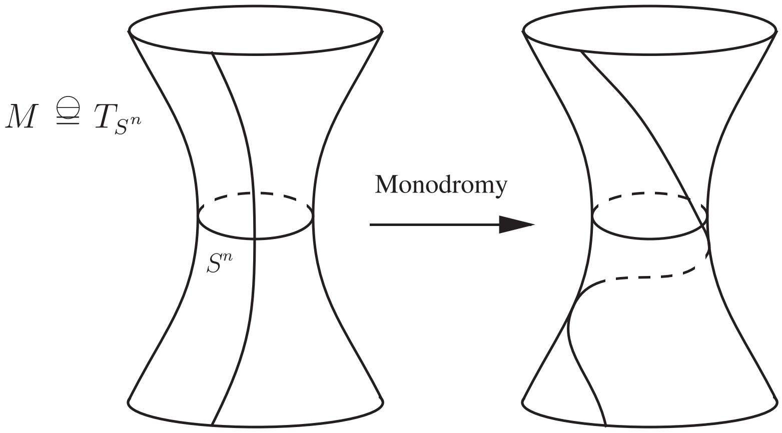

Figure 2.1.1: The ODP according to Lefschetz: the Milnor fibre is diffeomorphic to the tangent bundle T_{S^n} of the vanishing cycle $\delta = S^n$, and the monodromy action is given by the Picard–Lefschetz formula $x \mapsto x \pm (x \cdot \delta)\delta$

For the purposes of birational geometry, we want to resolve the singularity; then each dimension is quite different. In the curve case, the ODP is the node, consisting of two branches meeting only at a point. The resolution of singularities or normalisation consists simply of separating the two branches.

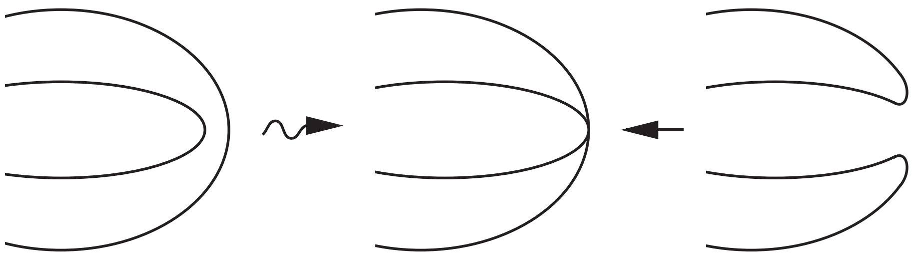

Figure 2.1.2: Picture of a curve losing 1 from its genus by acquiring an ODP, then disconnecting

This leads to the standard picture of genus change in a degeneration of Riemann surfaces (see Figure 2.1.2).

Something beautiful and completely unique happens in the surface case: the Milnor fibre M is diffeomorphic to the resolution $Y \to X$. The ODP is the ordinary quadratic cone in $\mathbb{C}^3$. Its resolution is a neighbourhood of a copy of $\mathbb{P}^1_{\mathbb{C}} = S^2_{\mathbb{R}}$ with self-intersection -2. At the level of topology, the diffeomorphism is clear from Figure 2.1.1, because the Milnor fibre is the tangent bundle to the vanishing cycle $S^2_{\mathbb{R}}$, which also has self-intersection -2.

The 3-fold ODP demonstrates originality by having a nontrivial analytic class group: up to the obvious change of coordinates, it is the cone over the quadric surface $Q \cong \mathbb{P}^1 \times \mathbb{P}^1 : (xy = zt) \subset \mathbb{P}^3$, and the two linear systems of generators of Q correspond to Weil divisors that are not factorial; for example, the principal divisor $x = 0$ breaks up as the union of two planes $x = z = 0$ and $x = t = 0$, neither of which is principal. The local class group can be viewed topologically as the second cohomology of the link of the singularity. As I discuss in Section 3.3, the two rulings of Q give rise to the two small resolutions of the 3-fold ODP and the flop between them.

It is also interesting to run through a parallel discussion of the A_2 singularity $\sum_{i=1}^{n+1} x_i^2 + x_0^3 = 0$. In the curve case, this singularity is a cusp, which is locally irreducible. The Milnor fibre has two vanishing cycles, but in the 3-fold case, they have nondegenerate intersection pairing, and the resulting singularity is factorial; in fact, it is a homology manifold. It has no small modification. Its blowup is a nonsingular 3-fold containing an exceptional surface E isomorphic to the quadric cone $Q \subset \mathbb{P}^3$, and with $\mathcal{O}_E(-E) \cong \mathcal{O}_Q(1)$. See Mirel Caibăr's thesis [Caibăr] for a much more detailed discussion of the relation between Milnor fibre, class group, and geometry of the resolution.

Problem on higher dimensional ODPs I propose as a challenge to try to understand the new characteristic features of 4-fold and 5-fold ODPs. For example, the 5-fold ODP is the cone over the Plücker quadric $Q_6 = \mathrm{Grass}(2, 4) \subset \mathbb{P}^5$, and the analog of the class group is the two tautological vector bundles of Grass(2, 4) or the spinor bundles of Q_6. My reason for saying this is that a number of people have looked for simple-minded extensions to higher dimensions of features such as simultaneous resolution of Du Val surface and small resolutions of 3-fold cDV points, but I really don't know of any reasonable class of higher dimensional examples where this is likely to give useful results. For anyone wanting to study 4-fold or 5-fold analogs of the theory of surfaces and 3-folds, my problem seems a more profitable direction.

History The ODP $y_1y_2 = y_3y_4$ and its two rational functions $\frac{y_1}{y_3} = \frac{y_4}{y_2}$ and $\frac{y_1}{y_4} = \frac{y_3}{y_2}$ occur in Mumford [red], Chap. I, §4, Remark IV after Proposition 1 in the discussion of rational functions. I was a graduate student at the IHES between 1970 and 1972, and heard about the ideas of Picard–Lefschetz from lectures by my great teacher Pierre Deligne, around the time he was preparing Part II of SGA7 for publication [D].

The first project that Deligne offered me, global Torelli and the surjectivity of the period map for K3 surfaces of degree 2 or 4, turned out to be beyond my technical capabilities, but I learned a vast amount in my first year by studying it, among other things: classification of surfaces, moduli spaces and G.I.T., Hodge theory including variation and degeneration of Hodge structures, Satake compactifications. Deligne taught me about Du Val singularities (including the name, in case you think I use it out of patriotism to the UK or to Trinity College, Cambridge). An important point in the Torelli project was the idea that K3s with only Du Val singularities should be viewed as points "at finite distance" in the moduli space, since they have monodromy of finite order, and their resolutions are still K3s. Equally well, you could start learning to live with the singular variety.

2.2 The Veronese cone

I change tack, to discuss the first type of canonical singularity arising as a quotient. Many classical construction of surfaces, such as those of Kummer surfaces, Enriques surfaces or Godeaux surfaces, involve a group action having no fixed points, or only involving ODPs.

> *Simple constructions for surfaces involving quotients by a group action often have 3-fold analogs having quotient singularities, for example, the Veronese cone singularity.*

The first 3-fold quotient singularity to come to my attention was the Veronese cone. As everyone knows, the Veronese surface $V_4 \subset \mathbb{P}^5$ is the image of $\mathbb{P}^2$ under the embedding given by all quadratic monomials. To see it by equations, we take coordinates x_1, x_2, x_3 on $\mathbb{P}^2$ and write down

$$M = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ u_{12} & u_{22} & u_{23} \\ u_{13} & u_{23} & u_{33} \end{pmatrix} \quad \text{where} \quad u_{ij} = x_i x_j.$$

Here M is the generic symmetric 3×3 matrix. The Veronese surface is defined by $\operatorname{rank} M \leq 1$. On the other hand, it's obvious that the projective coordinate ring of V_4 is $k[V_4] = k[x_1^2, x_1x_2, \dots, x_3^2]$. This is, of course, the ring of even polynomials in x_1, x_2, x_3, or equivalently, the ring of invariants of $\{\pm 1\}$ acting by $(x_1, x_2, x_3) \mapsto (-x_1, -x_2, -x_3)$. Its Spec, the affine cone X over V_4, is the quotient singularity $\frac{1}{2}(1, 1, 1)$ in modern terms. If you resolve it by a blowup $f\colon Y \to X$, you obtain a 3-fold Y containing an exceptional surface $E \cong \mathbb{P}^2$ with normal bundle $\mathcal{O}_E(E) \cong \mathcal{O}_{\mathbb{P}^2}(-2)$; by the adjunction formula, K_Y restricted to E is $\mathcal{O}_{\mathbb{P}^2}(-1)$.

Since K_Y has negative degree on E, all sections of K_Y in a neighbourhood of E vanish along E; it is an easy exercise in the same style to see that also all sections of nK_Y vanish along E with multiplicity $\geq n/2$. Thus

$$f_*\mathcal{O}_Y(nK_Y) = f_*\mathcal{O}_Y\Bigl(nK_Y - \frac{n}{2}E\Bigr) \quad \text{and} \quad 2K_Y = f^*(2K_X) + E.$$

If we start from an Abelian surface A and pass to the quotient surface $\overline{S} = A/\{\pm 1\}$ by the action of $\{\pm 1\}$, then $\overline{S}$ has 16 ODPs at the fixed points of the group action, that is, the 2-torsion points of A. Resolving these leads to -2-curves on the minimal nonsingular model $S \to \overline{S}$, which is then a K3 surface, the *Kummer surface* of A. In contrast, the Kummer variety $\overline{X} = A/\{\pm 1\}$ of an Abelian 3-fold A has 64 Veronese cone singularities at the 2-torsion points of A, and resolving these by the blowup $X \to \overline{X}$ is not necessarily such a good thing to do: you achieve nonsingularity at the expense of spoiling the property $2K_{\overline{X}} = 0$, replacing it by $2K_X = \sum E_i$ where E_i are the exceptional $\mathbb{P}^2$s over the singularities.

The Veronese cone appears in many similar contexts: for example, a popular construction of the Enriques surface (possibly originally due to Burniat) is to divide a complete intersection of 3 quadrics $S = Q_1 \cap Q_2 \cap Q_3 \subset \mathbb{P}^5$ by the free action of $\mathbb{Z}/2$ acting by $\operatorname{diag}(1, 1, 1, -1, -1, -1)$ (think of diagonal quadrics for clarity). We can make S avoid the two fixed planes $\mathbb{P}^2_{\pm}$ because we have three quadrics Q_i to spare. If we try to generalise this construction to a 3-fold complete intersection of 3 quadrics in $\mathbb{P}^6$, one of the fixed components Π must be $\mathbb{P}^3$ (at least), and we can't avoid having the 8 points $(Q_1 \cap Q_2 \cap Q_3) \cap \Pi$ as fixed points. This is one construction for Enriques–Fano 3-folds, that typically have 8 Veronese cone singularities.

History I shared an office for a few weeks with Bernard Saint-Donat on arriving at the IHES in 1970, and we both visited the Warwick Symposium ran by David Mumford in 1970–1971; I gave a Cambridge Part III course around 1974 based on his paper on projective models of K3 surfaces [S-D]. This contains the example of the K3 surface $S_5 \subset \mathbb{P}(1^3, 2) \subset \mathbb{P}^6$ embedded in the Veronese cone; its general hyperplane section is a canonical curve of genus 6 with a g_5^2, that is, the Veronese embedding of a plane quintic, and so it is an exceptional case for which the projective image is not an intersection of quadrics (see [S-D], Theorem 7.2, (ii)). It has the $\mathbb{Q}$-Fano extension $X_5 \subset \mathbb{P}(1^4, 2)$ having the singularity $\frac{1}{2}(1,1,1)$, and when following Iskovskikh's work, I was at first surprised to find it excluded from the company of nonsingular Fanos ([I], Part II, Prop. 1.7 and §2).

As another example, in the context of Fano's study of anticanonical models of 3-folds $V_{2g-2} \subset \mathbb{P}^{g+1}$, Iskovskikh asked for an example of a nonsingular V_{2g-2} containing a linearly embedded plane $\mathbb{P}^2$. If E is such a plane then, by the adjunction formula, its normal bundle is given by $\mathcal{O}_E(-E) = \mathcal{O}_{\mathbb{P}^2}(2)$, so that E should contract to a Veronese cone point. Examples such as $\mathbb{P}(1^3, 2)$, $X_3 \subset \mathbb{P}(1^4, 2)$ and $X_4 \subset \mathbb{P}(1^3, 2, 2)$ show that these really occur. (I wrote a letter to Iskovskikh in August 1978 attempting to prove that this cannot happen; but I found the examples soon after this.)

The 1970–71 Warwick symposium I followed Deligne to Warwick in the summer of 1971, and met Artin, Bombieri, Mumford, C.P. Ramanujam, Seshadri and many others; for example, I learnt more about G.I.T. from standing in a lunch queue with Seshadri than from several weeks' study of the book earlier that year. I took part in a seminar on etale cohomology organised by Saint-Donat and Anders Thorup, and had the embarrassing task of giving half-prepared lectures on the etale base change theorem with Mike Artin in the audience.

When Elaine Greaves threw me out of the old typewriter room (two clapped-out old mechanical typewriters, reserved for the use of graduate students), neither of us suspected that she would be my trusted henchman throughout the planning and organisation of the two subsequent Warwick algebraic geometry symposiums in 1982–83 and 1995–96. Several letters and rough drafts for [C3-f] were written on exactly the same typewriters when I arrived at Warwick in 1978. (In the meantime, our equipment budget has improved somewhat.)

2.3 Canonical rings and the first canonical 3-folds

There are many places in Enriques' work where he constructs a canonical surface by what seems to be an extraordinarily ingenious argument based on the geometry of the canonical or bicanonical map, but that becomes much more transparent when viewed in terms of graded rings.

> *Graded rings such as the canonical ring of a variety of general type contain more information than individual linear systems (for example, m-canonical maps).*

The simplest illustration for my purposes is Enriques' construction of a surface S with $p_g(S) = 4, q(S) = 0, K_S^2 = 6$ as the normalisation $S \to \overline{S}_6$ of a sextic surface $\overline{S}_6 \subset \mathbb{P}^3$ having an ordinary double curve along a plane cubic curve $C \subset \mathbb{P}^3$. My interpretation is as follows: the canonical ring of S needs 4 generators $x_i \in H^0(K_S)$ in degree 1, then (at least) 1 more generator $y \in H^0(2K_S)$ in degree 2, simply because quadratic monomials $x_i x_j$ only provide 10 elements of $H^0(2K_S)$, whereas by Riemann–Roch

$$h^0(2K_S) = \chi(\mathcal{O}_S) + K^2 = 11.$$

Now we easily find that we need 2 relations in degree 3 and 4 between these generators, giving S as the complete intersection $S_{3,4} \subset \mathbb{P}(1^4, 2)$. No ingenuity here, this is just the general complete intersection; we recover Enriques' model by assuming that the two relations are $x_0 y = a_3$ and $y^2 = b_4$, and eliminating y, giving the equation of the sextic as $\overline{S}_6 : (a_3^2 = x_0^2 b_4)$.

One of Enriques' most remarkable example of this kind is his construction of a surface of general type with $p_g(S) = 2, K_S^2 = 1$; in this case it is easy to see that the canonical system $|K_S|$ is a pencil of curves of genus 2 with a single transverse base point P, which is a Weierstrass point on each $C \in |K_S|$. Then $\mathcal{O}_C(2K_S) = K_C$, so that $|2K_S|$ maps each C as a double cover of $\mathbb{P}^1$. Since the image $\mathbb{P}^1$s all contain the image of P, the image is the quadric cone $\varphi_{2K_S}(S) = Q \subset \mathbb{P}^3$, and S is the double cover of Q branched in the intersection of C with a quintic and over the vertex. The modern construction is the weighted hypersurface $S_{10} \subset \mathbb{P}(1, 1, 2, 5)$.

Putting together the idea of canonical ring derived from Enriques' constructions and the experience of the Veronese cone described in Section 2.2, it was natural to look for examples such as $X_{14} \subset \mathbb{P}^4(1, 1, 2, 2, 7)$. This example can perfectly well be treated from the point of view of classical geometers: it has a canonical pencil $|K_X|$ consisting of surfaces $S_{14} \subset \mathbb{P}(1, 2, 2, 7)$. Each such surface has the equation $z^2 = f_7(x^2, y_1, y_2)$. If you substitute $X = x^2$ and $Z = xz$, and divide all the degrees by 2, this becomes the hypersurface $S_8 \subset \mathbb{P}(1, 1, 1, 4)$ (that is, $\mathbb{P}(2, 2, 2, 8)$) defined by $Z^2 = X f_7(X, y_1, y_2)$. In other words, my 3-fold has a pencil of surfaces with $p_g = 3$, $K^2 = 2$, each a double cover of $\mathbb{P}^2$ whose octic branch curve happens to split up as a septic plus a line, meeting of course in 7 nodes.

With what we know today, it is immediate that X_{14} has 7 Veronese cone singularities along the (y_1, y_2)-axis. The striking thing here is that $K_X^3 = \frac{1}{2}$; in other words, the plurigenera $P_n(X)$ grow like $\frac{1}{6}n^3 \times \frac{1}{2}$. On the other hand, a 3-fold of general type that has a nonsingular model with K_X nef has

K_X^3 a positive and even integer, so $K_X^3 \geq 2$. The plurigenera are manifestly birational invariants, and on a minimal model, we can use vanishing to deduce that P_n grows like $\frac{1}{6}n^3 \times K_X^3$. Thus my example has no nonsingular model with K_X nef. (Compare also Ueno [U].)

It might be a fun exercise to find a treatment in the old Italian style of examples such as $V_{6,10} \subset \mathbb{P}(1, 2^3, 3, 5)$, which has $p_g = 1$, $K^3 = 1/2$, and 15 Veronese cone points. The main point, however, is that canonical rings already give a simple construction of dozens of examples, with features that would be hard to study in other ways.

History The inevitability of working with the canonical ring comes out clearly in the study of algebraic surfaces such as Godeaux surfaces, which I studied around 1974, following Bombieri's paper on canonical models and his lectures at the IHES and the Warwick symposium in 1970. For me, the pleasure of seeing how simple constructions with graded rings generate effortlessly many classes of algebraic surfaces was the main motivation for trying to do the same in dimension 3. It is clear that there is still a whole lot more mileage to be got out of these ideas.

After two years of my research fellowship at Christ's College, Cambridge, I applied to the British Council for a second year on their Soviet exchange, from September 1975. The Soviet ministry didn't come up with my placement by the due starting date for their own reasons, and when they did, it turned out to be at Minsk. It's not plausible that they excluded me from Moscow because of the bad company I kept during my first year, and much more likely that this was a tit-for-tat because a Cambridge department had refused to take some party stooge.

In October 1975, after finishing my paper on elliptic Gorenstein surface singularities [EllGor], and while hanging around Cambridge, waiting for my placement, I got interested in 3-fold examples of the above type, and even wrote a primitive computer program (in Fortran!) to search for examples of canonical 3-folds as hypersurfaces $X_d \subset \mathbb{P}^4(a_0, \dots, a_4)$ with all $a_i \mid d$ and $\sum a_i = d + 1$. Unfortunately, the only singularity I really believed in and was confident of being able to resolve was the Veronese cone, so that although my program found a few examples such as [Fl], 15.1, No. 14: $X_{18} \subset \mathbb{P}(1, 2, 2, 3, 9)$, which has $2 \times \frac{1}{3}(1, 1, 2)$ quotient singularities along the z, t axis, I didn't recognise them as valid examples, and missed out on discovering terminal singularities of higher index. What a fool I was, Watson!

Notes written during my first month at Warwick in 1978 contain 21 cases of canonical complete intersections with Veronese cone points as their only singularities, including the example $V_{6,6,6} \subset \mathbb{P}(2^4, 3^3)$ with $p_g = 0$ discussed in [YPG], Example 2.9. The same notes and letters from the period conjectured falsely and repeatedly that the index of 3-fold canonical singularities is always ≤ 2. In a letter around 1977, Ueno suggested I look at $\frac{1}{3}(1, 1, 1)$ for an index 3 singularity. Of course this has index 1, and I ignored the suggestion. But it is possible that what Ueno wrote was a misprint for $\frac{1}{3}(1,1,2)$; in any case, I would have stumbled on this somehow if I had interpreted his suggestion more sympathetically. Nick Shepherd-Barron discovered the singularities $\frac{1}{r}(1,1,r-1)$ of index r in early 1979 in response to my false conjecture, and Dave Morrison provided the general case of terminal quotient singularities $\frac{1}{r}(1,a,r-a)$ in a letter of April 1980. I originally proposed to Nick that we should write [C3-f] as a joint paper, but he didn't want to, although he contributed several of the key ideas to it, in particular the first conjectural statement that the hyperplane section of a Gorenstein canonical singularity is either rational or elliptic. I realised at the Angers conference in July 1979 that we had a simple proof of this conjecture using nothing much more than the adjunction formula.

In the year after Angers I spent a couple of weeks in Bonn at the invitation of Van de Ven and Hirzebruch, where I discovered that I knew how to write out formally the "famous 95" list of K3 hypersurfaces.

3 Towards the definition of minimal model

The logic behind minimal models is much less clear-cut than for canonical models, and we arrived at the definition by interpolating between the canonical model and birational nonsingular models, with Mori and me working from opposite ends. (Compare [M3], Section 9 for a complementary historical discussion.) Whereas the preceding chapter argued the inevitability of canonical models, this chapter has the more modest task of putting together some considerations that make the definition of minimal model look at least reasonable. The material here does not use the Mori cone, and is mostly simpler than the eventual solution of the minimal model problem by Kawamata, Kollár, Mori, Shokurov and others (see Chapter 4).

Section 3.1 discusses Zariski decomposition of a divisor on a surface to explain the nef condition on a divisor. Section 3.2 goes back to the material of Section 2.1 to discuss small resolutions of ODPs and the classic flop, and Section 3.3 plays with examples of the construction of [C3-f] and [Pagoda] constructing partial resolutions of canonical singularities, in order to give some feeling for the condition that K is nef. Section 3.4 discusses the special case of relative minimal models of surface fibrations.

3.1 Zariski's paper and nef divisors on surfaces

Zariski's paper [Z] on the asymptotic form of Riemann–Roch for a divisor on a surface forms a crucial bridge between the Italian tradition of surfaces and modern work on 3-folds. My attention was drawn to it by Mumford's appendix, referred to in Bombieri [B] for the proof of the finite generation of the canonical ring of a surface of general type. The paper stresses the notion of a divisor D being *eventually free* (that is, the linear systems $|nD|$ is free for some $n > 0$), and its consequence *numerical eventually free* or *nef* (that is, $D\Gamma \geq 0$ for every curve $\Gamma \subset S$). Zariski introduces $\mathbb{Q}$-divisors, the idea of Zariski decomposition, and the related characterisation of quasieffective in terms of the existence of Zariski decomposition.

The degree of a divisor on a curve C is an integer, so the notion of positive degree is not hard to find, and when D has large degree, RR takes the simple form $h^0(C, D) = 1 - g + \deg D$. Zariski's aim is to study the Riemann–Roch space $H^0(S, nD)$ for a divisor on a surface, especially for $n \gg 0$; there is not much problem when D is ample, so the whole point is to deal with negative features of D.

Assume that some multiple of D is effective, say $mD = \sum d_i\Gamma_i$ for some $m > 0$ with $d_i > 0$. If $D\Gamma < 0$ for some curve on S then Γ must be one of the Γ_i, say Γ_1, and necessarily $\Gamma_1^2 < 0$. Then the fixed part of the linear system $|nD|$ contains Γ_1 with multiplicity $\geq na_1$, where $a_1 = (-D\Gamma_1)/(-\Gamma_1^2)$. In other words, decreasing D down to $D_1 := D - a_1\Gamma_1$, where a_1 is chosen so that $D_1\Gamma_1 = 0$, leaves all the H^0 unchanged: $H^0(nD) = H^0(nD_1)$ for all $n > 0$. We thus find ourselves playing games with the quadratic intersection form on the components of Γ_i, with the flavour of orthogonal complement or Gram–Schmidt orthogonalisation. The theory proceeds systematically via the *Zariski decomposition* of D, defined to be an expression $D = P + N$ where

(i) P is a nef $\mathbb{Q}$-divisor;

(ii) N is effective, negative definite and orthogonal to P.

The point is that the base part of the linear system $|nD|$ must be at least nN for any $n > 0$, so that $h^0(nD) = h^0(nP)$. For any divisor D, the existence of a Zariski decomposition is an important dichotomy: it exists if and only if D is quasieffective, and then $D\Gamma < 0$ happens for at most finitely many curves having a negative definite intersection pairing.

It seems to be a common misapprehension that just defining a Zariski decomposition in terms of orthogonal complement somehow makes it a one-off process in bilinear algebra, so I emphasise the following point:

> *Calculating the Zariski decomposition of a divisor on a surface is an inductive process, and is a kind of minimal model program.*

Continuing the argument from above, if $D_1\Gamma_2 < 0$ we have to subtract off a multiple of Γ_1 and Γ_2 to make D_2 orthogonal to both Γ_1 and Γ_2, and so on by induction as long as D_j is not nef. We don't know whether or not Γ_2 will fall in the negative part just by looking at D and Γ_2 without running the inductive process. (I stress the point, at the risk of repetition: whether a rational curve $\Gamma \subset S$ on a surface with self-intersection $\Gamma^2 \leq -2$ is contracted on passing to the minimal model is not determined just by a neighbourhood of Γ, but by how many contracted -1-curves it meets in the course of $S \to S_1 \to \cdots$.)

If S is a surface with $p_g > 0$ and $D = K_S$, then calculating the Zariski decomposition of D is exactly the same thing as running a minimal model program on S. The first Γ_1 is a -1-curve because $K_S\Gamma_1 < 0$ and $\Gamma_1^2 < 0$. We subtract off a multiple of Γ_1, and from that point on always work in the orthogonal complement to Γ_1, which is equivalent to working only with the pullback of divisors from the contraction $S \to S_1$ of Γ_1. The negative part of the Zariski decomposition is the *discrepancy* $K_S - K_{S_{\min}}$ of the morphism to the minimal model $S \to S_{\min}$. Kawamata's work on log minimal models of log surfaces (or "noncomplete" surfaces) is a straightforward application of Zariski decomposition of log canonical divisors $K + D$.

History I first visited Moscow for one year 1972–73 under the British Council exchange. By that time, Soviet citizens who talked to foreigners were no longer imprisoned or shot (or at least, not very much), but it was still not easy for a fairly ignorant and frivolous foreigner to get into serious conversation with people (except taxi drivers). Although I spent a lot of time with Moscow mathematicians, it was not until I got home that I realised that several of my colleagues had distinguished dissident connections. During my second visit in 1975–76 things were different: I was less ignorant, having read Solzhenitsyn and Robert Conquest on the Stalin terror and the even more terrible collectivisation holocaust. But more, I believe that most Russians were well informed and cynical about their political system, and keen to criticise it and to hear about the West. Even people who were party members would crack jokes such as "Is socialism scientific? No – they would have tried it out on animals first".

Minsk in 1975 was different; I was more or less the first capitalist seen in the town, and there were various confused reactions: some of my neighbours in the students' hostel were ordered by their komsomol organisers (communist youth league) not to consort with me. The foreign office in the university was splendidly friendly and incompetent, and cheerfully signed orders to the visa office to give me permission to travel to Moscow, where I stayed for several months, illegally as it turned out. The student hostel on Oktyabr'skaya Ulitsa, next to the vodka factory, was a real masterpiece of Mr Brezhnev's 5 year plans – the doors and windows didn't fit, and the elevator had to be turned off after 6 pm (in a 13 storey building) because of the inconvenience of calling out the repair man at night when it broke down. Over many months in Minsk, I searched everywhere to buy a teapot, but in vain; while making myself into a 3-folder by studying Zariski's paper, weighted projective spaces, the Riemann–Roch formula for 3-folds and simultaneous resolution of Du Val singularities, I made tea in a jam jar, and tried to pour it into the glass without scalding my hands. The jam jar was the big kind you got 5 kopecks back on; at the time vodka was 4 roubles and 12 kopecks a half-litre, with 12 kopecks back on the bottle; these prices were stable over several decades.

1976–77 my first visit to Japan I was aware of young Japanese algebraic geometers at the AMS summer institute at Arcata in 1974, and I wrote to Kodaira to invite myself to Tokyo. He put me in touch with Miyaoka, who was also doing Godeaux surfaces. I was very lucky with my contacts, and made many friends at once. As my host at Tokyo University, Iitaka looked after me with great kindness; he introduced me to his young graduate student Kawamata, then working on log varieties. Miyaoka had worked on my notes on Bogomolov's inequality $c_1^2 \le 4c_2$, and lectured on his proof of the famous inequality $c_1^2 \le 3c_2$ during my first couple of weeks in Tokyo. Among many other kindnesses, Iitaka advised me to go to the March 1977 spring meeting of the Japan Math. Soc. at Kyoto "to meet new people"; this was when I first met my wife Nayo.

3.2 Simultaneous resolution and the classic flop

As mentioned in Section 2.1, an easy topological argument shows that the Milnor fibre M and the minimal resolution Y of the surface ordinary double point $xy + z^2 = 0$ are diffeomorphic. In the 1950s, Atiyah [A] discovered a much more convincing reason for the diffeomorphism $M \cong Y$, with an argument that was to have far-reaching consequences for 3-fold geometry.

For this, think of the family as $\mathcal{X} : (xy + z^2 = t) \subset \mathbb{C}^3 \times \mathbb{C}$, with $\mathcal{X} \to \mathbb{C}$ given by t. Replace the deformation parameter t by $\sqrt{t} = \tau$, thus replacing $\mathcal{X}$ by $\mathcal{X}' : (xy + z^2 = \tau^2) \subset \mathbb{C}^3 \times T$. The fibres of $\mathcal{X}' \to T$ are the same fibres as before, but now each fibre X_t with nonzero t occurs twice as $X'_{\pm\tau}$. The new total space $\mathcal{X}'$ is the 3-fold ODP $xy + z^2 - \tau^2 = 0$, or $xy = zt$ after the trivial change of coordinates to $\tau \pm z$. Now the graph of the rational function $x/z = t/y$ provides a resolution of singularities $\mathcal{Y} \to \mathcal{X}'$, and moreover, one in which X_τ moves in a smooth 1-parameter family together with the minimal resolution $Y_0 \to X'_0$. (It is an easy exercise to see that the blowup of a line through a surface ODP has the same effect as the blowup of the point itself.) This is called a *simultaneous resolution* of the surface singularity.

Now consider the 3-fold ODP $X : (xy = zt)$, forgetting the family of surfaces for the moment. Let $Y \subset X \times \mathbb{P}^1$ be the graph of the rational function $x/z = t/y$. Then $f \colon Y \to X$ is a birational morphism with fibre $L \cong \mathbb{P}^1$ over 0. It is easy to check that Y is nonsingular, and that the normal bundle to L in Y is $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. This is an example of a *small resolution*: the exceptional locus contains no divisors.[1]

We could have paired the factors x, y and z, t in the other order, and considered instead the rational function $x/t = z/y$. This gives a second resolution $f'\colon Y' \to X$ with exactly the same properties as f. It might seem at first sight that $Y = Y'$, but this is not so. Indeed, since $Y \subset X \times \mathbb{P}^1$ and the ratio $(x : z)$ is the coordinate on Y, the divisors of zeros and poles of $x/z = t/y$ are two disjoint surfaces cutting $L = \mathbb{P}^1$ transversally at 0 and ∞;

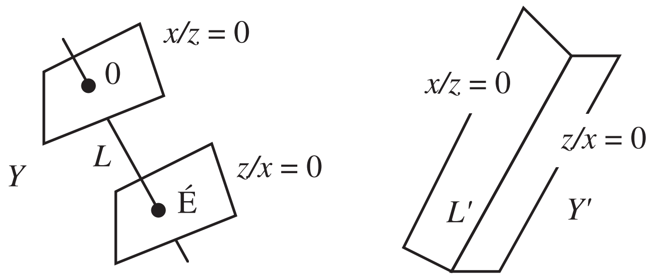

Figure 3.2.1: The flop $Y \dashrightarrow Y'$.

these are the birational transform on Y of the planes $x = t = 0$ and $z = y = 0$ of X (see Figure 3.2.1). On Y' however, the same rational function $x/z = t/y$ has divisor of zeros and poles that meet transversally along L', marking the two factors of the normal bundle $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. The "same surfaces" (more precisely, their birational transforms) have intersection number $+1$ with L and -1 with L'.

The birational map $\varphi\colon Y \dashrightarrow Y'$ is the *classic flop*. To think of it as a correspondence, consider its closed graph $W \subset Y \times Y'$. Then W fits into the traditional picture

$$\begin{tikzcd} & W \arrow[dl] \arrow[dr] & \\ Y \arrow[rr, dashed] \arrow[dr] & & Y' \arrow[dl] \\ & X & \end{tikzcd}$$

[1]The small morphism $f\colon Y \to X$ is closely related to the class group: $\mathcal{O}_X$ is not a UFD because $xy = zt$; the function $x/z = t/y$ can be written with two essentially different denominators, which was the point of the reference to [red] in Section 2.1.

An n-fold point $P \in X$ is $\mathbb{Q}$-*factorial* if every codimension 1 irreducible subvariety of X through P has a multiple that is a principal divisor. Although the $\mathbb{Q}$-factorial property may seems somewhat arcane at first sight, it is very simple, and plays a central role in Mori theory. To understand it, please do the following elementary exercise: if $P \in X$ is $\mathbb{Q}$-factorial, then for every partial resolution $f\colon Y \to X$ every component of the exceptional locus of f has pure dimension $n - 1$. (This is called *van der Waerden purity*.)

with $W \to X$ the blowup of the ordinary double point having exceptional surface the quadric $Q = \mathbb{P}^1 \times \mathbb{P}^1$ with normal bundle $\mathcal{O}(-1,-1)$. The two sides $W \to Y$ and $W \to Y'$ are the respective blowups of $L \subset Y$ and $L' \subset Y'$ or the contractions of Q along its two rulings.

Flops relate closely to the nef condition: let $\varphi\colon Y \dashrightarrow Y'$ be a classic flop as above. Because φ is an isomorphism outside a set of codimension ≥ 2, a divisor D on Y has a birational transform D' on Y'. In other words, it is natural to identify the divisors on Y and Y'. On the other hand, as we saw in Figure 3.2.1, if $DL > 0$ then $D'L' < 0$.

On a surface, Zariski had proved ([Z], Theorem 6.1) that if $|D|$ is a *mobile* linear system (that is, $|D|$ has no fixed part) then some multiple $|nD|$ is *free* (that is, $|nD|$ has no base points). I tried to prove the same thing for 3-folds until Iskovskikh and Bogomolov told me it was obvious nonsense, because of flops. The conclusion from this is that if you're hoping for Zariski decomposition for a divisor D on a 3-fold X, you can't expect it to take place on the 3-fold itself. If it makes sense at all, you first subtract off some "negative part" N so that $(D-N)\Gamma < 0$ holds for only finitely many curves Γ_i, and then hope to "flip" these by a birational modification $Y \dashrightarrow Y'$ so that the Γ_i are replace by Γ'_i with $(D-N)\Gamma'_i > 0$.

The classic flop is also central to the questions of the Weil divisor class group and projectivity of 3-folds. For example, let Π be a plane in $\mathbb{P}^4$, and X_d a general 3-fold hypersurface of degree $d \geq 3$ containing Π. The Weil divisor class group of X_d is generated by Π and the hyperplane class. Then X_d can be written $xA + yB$ where $\deg A, B = d-1$, and in general has $(d-1)^2$ ODPs at the points $x = y = A = B = 0$. Taking the graph of the rational map $x/y = -B/A$ amounts to blowing up $\mathbb{P}^4$ along Π, and is a projective small resolution of X_d that introduces a flopping curve L above each node. In terms of local coordinates, each curve L can be flopped, but carrying out a flop of any strict subset of the $(d-1)^2$ lines L results in a nonprojective variety.

It is known by work of Brieskorn [B] and Tyurina [Tyu] in the late 1960s that simultaneous resolution applies to any family of surfaces with at worst Du Val singularities (after passing to a suitable cover of the base, like taking the root $\tau = \sqrt{t}$ above). For example, a 1-parameter family of surfaces having nonsingular fibres when $t \neq 0$ and a Du Val singularity of type A_{n-1} above $t = 0$ pulls back on taking the nth root $\tau = \sqrt[n]{t}$ to

$$xy = z^n - \tau^n = \prod_{i=1}^{n} z_i, \quad \text{where} \quad z_i = z - \varepsilon_i \tau$$

(and ε_i runs through the roots of unity). The minimal resolution $Y_0 \to X_0$ of the central fibre has a chain on $n-1$ copies of $\mathbb{P}^1$, and the family has

$n!$ different simultaneous resolutions corresponding to ordering the factors z_i. You pass between them by flops, corresponding to transpositions of the z_i.

The idea of simultaneous resolution means that you can allow ODP in families of surfaces (K3 surfaces, surfaces of general type, del Pezzo surfaces, etc.), in the knowledge that they can be resolved in families. The moral for 3-folds, and an important reason underlying the success of my program in [C3-f] and [Pagoda] is that by taking cyclic covers, a canonical model is always only a finite distance away from a nonsingular minimal model.

History Local Torelli says that moduli of nonsingular K3s is locally isomorphic to the period space, but the moduli problem is not locally separated near surfaces with extra -2-curves. This phenomenon is related to the need to make a cover to kill the monodromy before obtaining a simultaneous resolution, and to the ambiguity in the choice of the simultaneous resolution that gives flops: on the Hodge theory side of Torelli, you need the extra information of the effective cone (in other words, the lattice spanned by -2-curves has a distinguished Weyl chamber consisting of effective curves). If you're really careful with the definitions, you get a Torelli isomorphism between the moduli and period stacks; it is likely that Deligne had this aspect of the Torelli problem in mind for me in my first year. The problem was solved soon after by Burns and Rapoport [BR], and their solution was an important background item in my work on [Pagoda]. Meiki Rapoport also studied with Deligne, and was my office mate for 2 years at the IHES

I always referred to the classic flop as *Atiyah's flop* during the 1970s and 1980s, until Gavin Brown pointed out to me that it occurred in Zariski papers in the 1930s. As soon as you start looking back into the past, you see that it occurs in factoring the standard Cremona transformation

$$T_{\mathrm{tet}} : \mathbb{P}^3 \to \mathbb{P}^3 \quad \text{given by} \quad (x_1, \dots, x_4) \mapsto \Big(\frac{1}{x_1}, \dots, \frac{1}{x_4}\Big),$$

which flops the 6 edges of the coordinate tetrahedron; it must have been well understood in this context by Cremona and Noether around 1870. (According to Hilda Hudson's bibliography, T_{tet} was studied by Magnus in 1837 and by Beltrami in 1863.)

3.3 From canonical models to minimal models

As discussed above, there is no choice in [C3-f] about the definition of canonical models and canonical singularities. You must simply take whatever comes as the Proj of the canonical ring of a variety of general type (assumed finitely generated). The point I want to make here is this:

> *A little knowledge on canonical singularities shows that they admit a restricted class of blowup called* crepant partial resolution. *Applying this to a canonical model leads to a minimal model. This procedure is a close analog of the minimal resolution of Du Val surface singularities.*

The ordinary double point, the A_2 double point and the Veronese cone point described in Sections 2.1–2.2 are among the simpler kind of canonical singularities called *terminal singularities*. Any blowup of them leads to exceptional divisors that are *discrepant.*[2] For example, as we saw in Section 2.2, the blowup $Y \to X$ resolving the Veronese cone point has $K_Y = K_X + \frac{1}{2}E$; the discrepancy divisor $\frac{1}{2}E$ prevents K_Y from being nef. See [YPG] for more details.

A simple example of a more typical canonical singularity would be a triple point such as $X : (x^3 + y^3 + z^3 + t^k = 0) \subset \mathbb{C}^4$ with $k \geq 3$. If $k = 3$, this is the cone over a nonsingular cubic surface E, a del Pezzo surface with $K_E = \mathcal{O}_E(-1)$. The blowup $X_1 \to X$ is nonsingular, and has E as a divisor with normal bundle $\mathcal{O}_E(E) = \mathcal{O}_E(-1) = \mathcal{O}_E(K_E)$, so that $K_{X_1} = K_X$; we say that $X_1 \to X$ is a *crepant* resolution, and E a crepant exceptional divisor. For higher k, the exceptional divisor E is a cubic cone, and X_1 has in turn a singularity $X : (x^3 + y^3 + z^3 + t^{k-3} = 0)$, and we can repeat.

Consider again the crepant resolution $Y \to X$ of the cone over a nonsingular cubic surface $S \subset \mathbb{P}^3$, and write $E \cong S$ for the exceptional surface. Any of the 27 lines L of E is a $(-1,-1)$ curve on Y, and has a flop $Y \dashrightarrow Y_1$ that takes $E \to E_1$ by contracting the -1-curve L and introducing a flopped curve L_1. Then Y_1 is an alternative resolution of X, and its exceptional locus consists of a del Pezzo surface E_1 of degree 4 together with a line L_1 that meets E_1 transversally. This illustrates the fact that flops occur all over the place, and that you almost never expect to get uniqueness of minimal models of 3-folds. In the present case, crepant resolutions $Y' \to X$ (that is, minimal models of X) correspond one-to-one with faces of the nef cone of the surface E, of which there are several hundred. For example, since there are 72 different contraction morphisms $E \to \mathbb{P}^2$ that contract 6 disjoint lines on E, there are 72 different crepant resolutions $Y \to X$ whose exceptional locus consists of the del Pezzo surface of degree 9 ($\mathbb{P}^2$ with normal bundle $\mathcal{O}(-3)$) and 6 flopping lines sticking out of it.

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[2] A linguistic point to help the non-English reader: *discrepancy* means a difference or contradiction, as when the honest cop finds a discrepancy between the politician's small declared income and large detected expenditure. Here it means the difference $K_Y - f^*K_X$, that is, $1/r$ times the divisor of zeros on Y of a local basis of rK_X. The word *crepant* is an English pun meaning not discrepant; it seems to have somewhat unpleasant reverberations in several European languages.

Problem We now know that flops account for the nonuniqueness of minimal models of 3-folds. Thus any two birational minimal models have the same Betti numbers, the same singularities, etc. All very tame stuff. But in dimension ≥ 4, who knows?

History The general result highlighted above is the main content of my two papers [C3-f] and [Pagoda]. In [C3-f], I showed that a suitable chain of blowup can be used to extract all the crepant divisors living above a canonical but not terminal point. In [Pagoda], which was one of the main outcomes of my 1981 visit to Kyoto, I used the Brieskorn–Tyurina theory of simultaneous resolution to resolve nonisolated cDV points.

The flop $Y \dashrightarrow Y_1$ is clearly related to the projection of the del Pezzo surface S_4 of degree 4 to a cubic surface S_3, and should be viewed as a kind of affine cone over it: X is the cone over S_3, and Y_1 contracts to a 3-fold X_1 having the cone over S_4 as its singularity, but with a projective line sticking through it. This picture is the original motivation for Francia's flip: consider the Veronese surface $V \subset \mathbb{P}^5$, and its projection from a point to the scroll $\mathbb{F}(1,2) \subset \mathbb{P}^4$. Write X for the affine cone over $\mathbb{F}(1,2)$; then X is the first flipping singularity. It has a ruling by planes that define a small resolution $X^+ \to X$ with exceptional locus a line L^+ having normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-2)$, so that $K_{X^+} \cdot L^+ = 1$. On the other hand, it has a small partial resolution X^- corresponding to the projection $V \dashrightarrow \mathbb{F}(1,2)$ where X^- has a Veronese cone point with a line L^- sticking through it, and $K_{X^-} \cdot L^- = -\frac{1}{2}$.

3.4 Minimal models via semistable degenerations of surfaces

Degeneration of surfaces is a special case of the problem of minimal models and classification of 3-folds that has attracted a lot of attention. Instead of just a 3-fold, we consider a 3-fold X with a fibre space structure $X \to B$ over a base curve B. This has a number of advantages:

1. For most purposes we can treat the generic fibre X_{gen} first, and assume that it is a minimal model, typically a K3 surface or a minimal surface of general type.

2. The minimal model problem then reduces to a neighbourhood of each degenerate fibre.

3. After making a base change by a ramified cover $B' \to B$ (replacing X by a model birational to $X' = X \times_B B'$ as in Section 3.2), we can

assume that $X \to B$ satisfies the conclusions of Mumford's semistable degeneration theorem.

For reasons of space, I will not say very much about this big and technically difficult topic. Mumford raised the problem in the early 1970s of modifying a semistable degeneration of K3s to one of several normal forms corresponding to the possible unipotent filtrations of monodromy; in other words, if we believe that Torelli for K3s extends to infinity, we can use the compactification of the period space as a model for how to compactify the moduli problem.

Mumford's problem was solved around 1976 by Viktor Kulikov [Ku], who proved the existence of a minimal model for the 3-fold total space of a K3 fibre space with semistable degeneration; this was the first time that 3-fold minimal models were proved to exist in substantial cases. Kulikov works in the nonsingular category, without projective assumptions. His main aim is to get rid of any surface component of the degeneration that appears in the canonical class with positive multiplicity, first using 3-fold flops to reduce it to a minimal ruled surface and then contracting it out. The main difficulty is that using a flop to contract a -1-curve on a component of the degeneration has the effect of introducing another -1-curves somewhere else, possibly even on a different fibre of the same ruled surface, like a serpent that bites its own tail. (Think of Hironaka's nonprojective examples. This kind of thing would be excluded by projective assumptions.) Kulikov has to work hard with graphs and combinatorics and so on to exclude this kind of loop. His paper contained major innovations, together with some minor difficulties of exposition and inaccuracies, and figuring it out led to a flurry of activity in the USA around 1980 ([PP], [deg]).

I heard Kulikov's lectures in Moscow in 1976 while his program was in the unfinished state, and still involved formidable technicalities. At this time my ideas on canonical and minimal models were just forming. His systematic use of flops predate my published work by several years, and must have influenced me. However, since my own approach is always to run away from difficult things wherever possible, and I had my own agenda involving canonical and minimal models, I did not get involved in the details of his proof, and my main conclusion was that I really preferred to wait until the general minimal model program could be sorted out.

Several years later, after Kawamata's log surfaces, canonical models and Mori theory were all on-line, Tsunoda proposed a more general program to construct minimal models of semistable degenerations of surfaces with K nef, by applying Mori theory to Kawamata's log surfaces. The experts seem to believe that Tsunoda's work over several years contains a more or less complete proof of semistable minimal models. However, in the intervening time Shokurov, Kawamata, Mori and most recently Corti have given more convincing versions of semistable flips and the semistable minimal model program. It remains a strong possibility that the future will see the easiest way to general Mori flips through semistable minimal models. (Compare Corti's proof given in [KM], Chapter 7.)

The December 1981 Kinosaki meeting Japanese algebraic geometers run a regular workshop at Kinosaki, a famous spa town in the north of Kyoto prefecture. The tradition apparently started because of a family connection with the owners of one of the ryokan (traditional Japanese inn), for whom December is the off-season. At Kinosaki, you're supposed to visit all the other baths, walking around the streets in the evenings in yukata (evening kimono) and wooden clogs, and holding traditional greased paper umbrellas if it's snowing. Mathematicians gather in the hotel lobby in the evenings to discuss math and other matters, and to sample generous quantities of the finest sake and Japanese whisky. Up to the late 1980s, the lectures were still held in the intimate environment of the ryokan's dining room.

Tsunoda announced his solution to the minimal model problem for semistable degenerations of surfaces at the Kinosaki meeting in December 1981. After the workshop, Tsunoda, Maehara, Kurke and I spent an afternoon visiting the famous beauty spot Ama-no-hashidate. The Buddhist temple Nari-ai-ji at the top of the hill contained a Boddhisatva that is a Yuigon Bosatsu, that is, has the unusual power of granting any request, even if only prayed for a single time (the normal practice is to require 10,000 repetitions, a bit like trying to get your kids off the phone). On hearing of this, Tsunoda and I both pressed our palms together, bowed our heads, and solemnly prayed for 3-fold canonical rings to be finitely generated. We forgot to ask help in solving the problem for ourselves.

4 The Mori cone and Mori minimal models

Castelnuovo and Enriques, together with every geometer up to Zariski and Mumford, expressed the classification of surfaces in terms of birational invariants, especially the plurigenera and irregularity. In these terms, two of the main aims of the theory are Castelnuovo and Enriques' criteria

$$\begin{aligned} P_2(S) = q(S) = 0 \quad &\Longrightarrow \quad S \text{ is rational; and} \\ P_{12} = 0 \quad &\Longrightarrow \quad S \text{ is ruled or rational.} \end{aligned}$$

Here the assumptions and the conclusions are birational, and for this reason, the classification of surfaces is often described as *birational classification.* This is one of several traditional preconceived notions you must unlearn if you wish to become a 3-folder.

The systematic approach to higher dimensional classification initiated by Iitaka and Ueno follows the same logic. This approach gives the right results (such as the Iitaka fibration) in cases when the plurigenera grow, or you have other information such as a nontrivial Albanese map. But if the plurigenera and irregularity are zero, it doesn't really give anything other than optimistic conjectures that the variety should be covered by rational curves, conjectures that are somewhat cheap because there is no plan for proving them. My approach described in the preceding chapter, although it set itself biregular aims, and was natural enough in view of my main interest in varieties of general type and their canonical rings, depended on even stronger assumptions than Iitaka's program: general type and finitely generated canonical ring.

Mori's interest seems to have started out from Fano varieties (varieties with $-K$ ample that classical geometers describe as "close to rational") and the existence of rational curves on them. The springboard for his first paper in minimal models [M1] was his earlier solution of the Hartshorne conjecture characterising $\mathbb{P}^n$ by ampleness of the tangent bundle. (I believe that this conjecture originated with a question of Deligne after Hartshorne's lecture on ample vector bundles at the 1970–71 Warwick symposium. I hope Mumford and Zeeman remembered to get it into the final SRC report.) A classical dimension count predicts that the space of rational curves is positive dimensional at any point, but we know that this type of argument cannot prove it is nonempty; Mori's key breakthrough is his extraordinary method of *bending-and-breaking* and *reduction to characteristic* p to prove the existence of rational curves.

The Kleiman–Mori cone described in Section 4.1 gives a strategy for minimal models based on contracting extremal rays (Section 4.2). In contrast to Iitaka's and my methods, these ideas work best of all for Fano varieties, whose Mori cone is a finite rational polyhedron with rays spanned by extremal rational curves. The contraction part can be done very generally by a circle of ideas that Kawamata's school call the *X method*, and modifications of its proof lead to the general cone theorem (Section 4.3). Flips were the last foundational problem to be solved by Mori in the late 1980s, and still lack a simple and wholly convincing argument.

4.1 Kleiman criterion and the Kleiman–Mori cone

The Riemann–Roch formula $\chi(\mathcal{O}_S(D)) = \frac{1}{2}D(D-K_S) + \chi(\mathcal{O}_S)$ for a divisor D on a nonsingular surface S is quadratic in D. It follows that for $D^2 > 0$, either $h^0(nD)$ or $h^0(-nD)$ grows quadratically with n. Grothendieck's proof of the algebraic Hodge index theorem follows easily from this: the index of the intersection form on the algebraic part of $H^2(S,\mathbb{R})$ is $(+1, -(\rho-1))$, and effective divisors with $D^2 > 0$ form a half-cone in $H^2(S,\mathbb{R})$ that we draw as a future light cone.

The Kleiman criterion [Kl] starts with the question of the cone of nef divisor classes $\operatorname{Nef} S \subset H^2(S,\mathbb{R})$. The class of an ample divisor is both in the interior of $\operatorname{Nef} S$, and of the $D^2 > 0$ cone; thus ample divisors form a subcone of the future light cone. Conversely, it turns out that a nef divisor is in the *closure* of the ample cone in $H^2(S,\mathbb{R})$: if D is nef and H is ample then the index polynomial $p(\lambda) = (D + \lambda H)^2$ takes positive values for $\lambda \gg 0$, while $D+\lambda H$ is nef for any $\lambda \geq 0$. But $p(\lambda)$ cannot go from being positive to being negative while $D + \lambda H$ is nef, and it follows that $D^2 \geq 0$ and $D + \varepsilon H$ is an ample $\mathbb{Q}$-divisor for any $\varepsilon \in \mathbb{Q}$, $\varepsilon > 0$.

Thinking about effective and nef divisors on surfaces tends to obscure a key point that comes out naturally for a higher dimensional variety V:

> *Rather than $H^2(S,\mathbb{R})$ and its* quadratic *intersection form, it is better to think of the* bilinear *pairing between the two dual vector spaces $H^2(V,\mathbb{R})$ and $H_2(V,\mathbb{R})$.*

The first of these contains the classes of $\mathbb{Q}$-Cartier divisors, and the second the cone of effective curves $\operatorname{NE} V$ and its closure $\overline{\operatorname{NE}}\, V \subset H_2(V,\mathbb{R})$, the *Kleiman–Mori cone*.

The Kleiman criterion says that a Cartier divisor L is ample on V if and only if its divisor class in $H^2(V,\mathbb{R})$ is strictly positive on $\overline{\operatorname{NE}}\, V \setminus \{0\}$. This condition is slightly stronger than saying $L\Gamma > 0$ for every curve Γ in V; it requires equivalently that $L - \varepsilon H$ is still nef for any ample H and $\varepsilon > 0$, or that $Lz > 0$ for z a *limit* of effective curves.

The Kleiman–Mori cone $\overline{\operatorname{NE}}\, X \subset H^2(X,\mathbb{R})$ is a closed convex cone by definition. You don't expect to be able to calculate it in general, but enough cases are known to suggest that both $\overline{\operatorname{NE}}\, X$ and the relation between $\operatorname{NE} X$ and its closure $\overline{\operatorname{NE}}\, X$ can be pretty well arbitrary in the half-space $K_X z \geq 0$. Mori's discovery concerns the half-space $K_X z < 0$: on a nonsingular n-fold $\overline{\operatorname{NE}}\, X \cap \big((K_X + \varepsilon H)z \leq 0\big)$ is finite rational polyhedral for any ample H and any $\varepsilon > 0$. The next two sections sketch two approaches to proving this: Mori's argument by bending-and-breaking, and the rational threshold argument related to Kodaira vanishing.

History I learned the idea of the positive cone implicitly from Zariski [Z], from working with Bogomolov on asymptotic RR, and more especially from Mumford's comments on my notes on Bogomolov.

The use of cones such as the positive cone, Nef and NE can reasonably be traced back to Nagata's papers on rational surfaces [N]. If you look for Kleiman's paper [Kl] in the library of the Kyoto University math department, you will find that Vol. 84 of Ann. of Math. is particularly dirty and dog-eared, as if held over the photocopier dozens of times. I presume that Nagata has set

Kleiman's paper as background reading for successive generations of students. (Both Hironaka and Mori were Nagata's M.Sc. students.)

4.2 Mori cone theorem and Mori contractions

Mori first introduced bending-and-breaking under very strong assumptions in the context of the Hartshorne conjecture, where he must prove the existence of a rational curve in a variety $\Gamma \subset V$ (that is to be a straight line $\Gamma = \mathbb{P}^1 \subset \mathbb{P}^n$, so that $-K_V\Gamma = n+1$). He extended the method in the proof of his original cone theorem, and later Kollár and Miyaoka generalised it further in several directions. I sketch Kollár's ideal form of the theorem.

> *Let $C_0 \subset X$ be a curve with $-K_XC_0 > 0$ on a nonsingular n-fold X. Then there exists a rational curve $\Gamma \subset X$ passing through any point $Q \in C_0$ with $0 < -K_X\Gamma \le n+1$.*

Step 1: Replace the embedded curve $C_0 \subset X$ by the map $f\colon C \to C_0 \subset X$ from a nonsingular curve, with $P \in C$ such that $f(P) = Q$.

Step 2: The deformation of the map f is controlled by the cohomology of the pullback f^*T_X of the tangent bundle of X, a rank n vector bundle on C of degree $-K_XC_0 > 0$. Suppose that this degree is not just positive, but very large compared to n and the genus of C. Then

bending: we can deform f as a morphism $F_0\colon C \times T_0 \to X$ over a parameter curve T_0 so that the image curve $f_t(C)$ sweeps out a surface, while fixing the value $f(P) = Q$.

breaking: let T be the projective completion of T_0, so that $F\colon C \times T \dashrightarrow X$ is a rational map; then F is *certainly not a morphism* in a neighbourhood of the section $\{P\} \times T$. Because this section has self-intersection 0, but F contracts it to a point in X.

We get a rational curve on X through P by resolving the indeterminacy of F by blowups and taking the image of a suitable exceptional curve, and we can arrange for it to have bounded degree and $0 < -K_X\Gamma \le n+1$ by rerunning an appropriate version of the same argument if necessary.

Step 3: Now comes the really clever bit. If we are in characteristic p, we can replace f by its composite f^n with a suitable iteration of the Frobenius map. This allows us to pump up $\deg f^*T_X$ to be as large as we like, while fixing the genus. Thus the required rational curves exist in almost all characteristic p; an argument on finiteness of the Hilbert scheme implies that they also exist in characteristic 0. Q.E.D.

It follows from this result that any cycle in $z \in \overline{\mathrm{NE}}$ with $(K + \varepsilon H)z < 0$ necessarily splits off a rational curve of bounded degree. The Mori cone theorem (for a nonsingular variety) is a rather formal consequence of this.

The kind of contraction morphisms $\varphi\colon X \to Y$ we are interested in are determined by certain faces of $\overline{\mathrm{NE}}\,X$; namely if Y is projective and H ample on it, then $L = \varphi^*H$ is nef, and $L^\perp \cap \overline{\mathrm{NE}}\,X$ is the cone of effective curves contracted by φ, that is, the relative cone $\overline{\mathrm{NE}}(X/Y)$. A face of $\overline{\mathrm{NE}}$ is a relative cone for a contraction if and only if it has a rational supporting hyperplane corresponding to an eventually free divisor L, and the morphism $\varphi_{nL}\colon X \to Y$ for $n \gg 0$ (or rather, its Stein factorisation) is uniquely determined up to isomorphism by the face.

In [M2], Mori only contracted extremal rays on nonsingular 3-folds in the $Kz < 0$ half of the cone, and obtained the famous classification that I'm sure you know about.

History I want to stress that the Mori cone came as a huge surprise to everyone, and that even now his leap of imagination seems quite incredible. I first heard that Mori was working on 3-fold contractions in a letter from Mumford around 1980 on practicalities of a visit to Warwick. Just in passing, he scribbled a note, that I paraphrase from memory: Mori had given a Harvard seminar, proving that if K_X is not nef on a smooth 3-fold, there exists a contractible surface, either a scroll, or $\mathbb{P}^2$ with normal bundle $\mathcal{O}(-1)$ or $\mathcal{O}(-2)$, or a quadric surface. P.S. (clearly added as an afterthought) That is, unless we are in the uniruled case.

In other words, even after Mori had described his work at a seminar, none of us had really grasped the significance of his cone theorem, or the extra precision that his methods give us in the uniruled case. For about 3 years after this, I regularly wrote to Mori to bother him for more details, both for my own sake and for my Moscow correspondents. The research announcement [M1] only aroused our curiosity, and I clearly remember getting the preprint of [M2] and wondering what all this clever-clever stuff about cones had to do with contracting surfaces on 3-folds. My next reaction was to look among the varieties I knew for counterexamples to his conclusions for extremal contractions that were conic bundles or divisorial contractions to ODPs.

I learned Mori theory, and many of the details of Mori's paper [M2] from a series of lectures by Miyanishi at Ueno's seminar at Kyoto University in 1981, and its applications to Fanos from talking to Mukai who was at that time working with Mori [MM] on the classification of nonsingular Fano 3-folds with $B_2 \geq 2$. The 2-ray game plays an important role in their study (without the name). In 1982–83 I organised my first Warwick symposium. During the first half of this year, I got together with several long term symposium visitors (Beltrametti, Dolgachev, Palleschi and Tsunoda) to run a seminar on Mori's paper and subsequent developments.

4.3 The X method, contractions and thresholds

Two related issues presented themselves around 1981:

(i) If X is a 3-fold minimal model of general type (that is, at worst terminal singularities, K_X nef and $K_X^3 > 0$), can we prove that K_X is eventually free? This is equivalent to the finite generation of the canonical ring of a minimal model X.

(ii) If $F \subset \overline{\mathrm{NE}}\,X$ is a face of the Mori cone contained in the half-space $K_X z < 0$, can we prove that there exist a morphism $X \to Y$ contracting exactly the curves in F? Mori proved the contraction theorem for an extremal ray on a nonsingular 3-fold X, but his proof seems impossible to generalise, since it involves a detailed knowledge of X in a neighbourhood of the contracted divisor.

Both these questions were answered at the same time by Kawamata's theorem [Ka]: if D is a nef Cartier divisor and $D - \varepsilon K_X$ is ample for some $0 < \varepsilon \ll 1$ then D is eventually free. This statement only has content if D is a supporting hyperplane of $\overline{\mathrm{NE}}\,X$ defining a nontrivial face $D^\perp \cap \overline{\mathrm{NE}}\,X$ in the half-space $K_X z < 0$.

The method of proof is a complicated game that has subsequently been refined by many authors, notably Kawamata himself and Shokurov. Kawamata's graduate students, who have had to suffer this method more than most, have with characteristic Japanese linguistic inventiveness christened it the *X method*; the instruction manual is [KMM]. The method is powerful, we know how to use it pretty well in practice, and it's been around for 20 years without anyone thinking of a better one, but (in my opinion) it is still messy and not optimal, and we don't really understand it.

Kodaira vanishing is the theorem that if X is a nonsingular projective n-fold over a field of characteristic 0 and D and ample divisor, then

$$H^i(X, D + K_X) = 0 \quad \text{for all } i > 0.$$

Many generalisations are known, for example Kawamata–Viehweg vanishing: we can allow D to be an ample $\mathbb{Q}$-divisor with fractional part supported on a normal crossing divisor, and get the conclusion $H^i(X, \ulcorner D \urcorner + K_X) = 0$. When using the X method, we blow up the n-fold X together with any divisors we can see on it to a nonsingular variety V with a normal crossings divisor D, then manoeuver into position to apply vanishing at the level of the different strata of V, D, thus obtaining surjective restriction maps.

> *Kodaira vanishing and the X method give a reason of principle why life is simple around the K negative boundary of* $\overline{\mathrm{NE}}$. *These ideas apply to give the Mori cone theorem and the contraction theorem for very general class of singular varieties* X.

Very roughly, the argument is as follows: suppose K_X is not nef, and let D be an ample Cartier divisor. Consider $D + tK_X$ as t increases; then there is some critical value or *nef threshold* at which $D + tK_X$ stops being ample

$$t_0 = \sup\{t \mid D + tK_X \text{ is nef}\}.$$

just before we reached t_0, the divisor $D + tK_X$ is ample. But if we add K_X, then $D + (t+1)K_X$ is not nef; however, Kodaira vanishing applies to it, because $D + tK_X$ is ample. This implies that something very rational is going on, and, following [R], Kawamata and Kollár proved that t_0 is a rational number whose denominator can be bounded in terms of the dimension n and the index of X.

History During my 1981 stay at Kyoto University, I made extensive attempts to prove the contraction theorem for 3-folds of general type with K_X nef. I was hoping to prove, for example, that $|10K|$ is free. I tried to use ideas of the following type: if D is a divisor on X passing through a point P with multiplicity ≥ 3, and $\sigma\colon X_1 \to X$ the blowup of P with exceptional locus E, then $D^{(3)} = \sigma^* D - 3E$ is an effective divisor, and by exactly the same argument as in the Bombieri–Kodaira–Ramanujam method for surfaces,

$$P \in \operatorname{Bs}|D + K_X| \iff H^1(\mathcal{O}_{D^{(2)}}) \to H^1(\mathcal{O}_{D^{(3)}}) \text{ is not surjective.}$$

For example, if D is a reduced surface with an ordinary triple point at P, then P is a base point of $D + K_X$ if and only if the blowup of the elliptic singularity P increases the irregularity of D, rather than decrease the geometric genus (so that P "disconnects D", by analogy with a node disconnecting a curve in Bombieri's method). If D is a little bit ample, then this should be impossible by some kind of Lefschetz hyperplane section argument in topology. I'm not sure why this idea still can't be made to work.

At the July 1981 conference in Tokyo, in connection with my [Pagoda] lecture, I discussed these kind of problems with Kawamata. This was just before he went to Berkeley to take up his Miller fellowship. I received several of his preprints (including [Ka]) during the Warwick symposium, and lectured on them and started developing my own version of them [R]. A particular aim was to massage the statement of Kawamata's theorem on eventually freedom to imply contractions of faces of the Mori cone.

Over many months during the Warwick symposium, Tsunoda came to discuss attempts to prove the Mori cone by deforming curves. I considered these to be doomed to failure because of the Francia flip: in that case, there is just a single curve on which K is negative, and of course it doesn't deform. (I was wrong: a few years later, Kollár [Ko] found an ingenious argument involving Mori's idea of quotient stacks to extend the deformation argument to some singular n-folds, in particular those with hypersurface or quotient singularities.) I stumbled on the threshold proof of the Mori cone theorem in preparing lectures on [R], and in trying to find another way around the cone theorem to answer Tsunoda's questions. Shokurov, who discovered a version of the same argument independently of me [Sh], wrote in the summer of 1983 that he could do the nonvanishing trick that made these methods work for general n-folds. The cone and contraction theorems are now proved whenever it makes sense.

In the report to SRC on the 1982–83 year I wrote:

> *... the framework for the study of 3-folds and higher dimensional varieties is just in the process of construction; we have just got to the stage when conventional wisdom was that 'there is not satisfactory extension to higher dimensions of the theory of minimal models of surfaces, or of the details of the classification' to the point when everyone now believes that 'there is a good theory provided that we work in the right class of singular varieties'. It is quite clear that the Warwick year 1982/83 will go down as a key period when substantial cases of the general theory of higher dimensional varieties were first proved.*

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