Fano 3-fold hypersurfaces

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Abstract

We study the birational geometry of the 95 families of Fano 3-fold weighted hypersurfaces $X = X_d \subset \mathbb{P}(1, a_1, a_2, a_3, a_4)$, corresponding to the famous 95 families of K3 surfaces $X_d \subset \mathbb{P}(a_1, a_2, a_3, a_4)$ of Reid and Fletcher ([C3-f, §4] and [Fl, 13.3]). Our main aim is to prove a rigidity theorem for the general X_d in each family, by analogy with the famous theorem of Iskovskikh and Manin on the rigidity of the quartic 3-fold; on the way, we derive as much instruction and amusement as possible on topics in biregular and birational geometry of Fano 3-folds and Mori fibre spaces. While this paper uncovers an amazing wealth of new phenomena and methods of calculation, many of the basic questions remain open, and we spell some of these out.

Contents

1 Simple-minded introduction

Definition 1.1 In this paper a *Fano variety* (formerly $\mathbb{Q}$-*Fano variety*) is a projective variety X satisfying

(a) X has at worst $\mathbb{Q}$-factorial terminal singularities;

(b) $-K_X$ is ample;

(c) $\operatorname{Pic} X$ has rank 1.

The "Italian" and "Russian" traditions usually study Fano 3-folds with (a) replaced by nonsingularity (or assumptions in projective geometry that imply Gorenstein). The class of varieties just defined arises in Mori theory as one of the possible end products of classification: it is the particular case $S = \text{pt.}$ of Mori fibre space $X \to S$; the other cases (for 3-folds) are fibrations of del Pezzo surfaces over a curve and conic bundles over a surface (see for example [CKM], Chapter 5, [Ko] or [R]).

Our study does not aspire to this generality; we usually work under the following concrete assumptions, in addition to (a–c):

(d) The plurianticanonical ring of X is a hypersurface:

$$R(X, -K_X) = \bigoplus_{n \geq 0} H^0(X, -nK_X) = k[x_0, \dots, x_4]/(F).$$

In other words, X is a member of one of the famous 95 families of weighted projective hypersurfaces $X_d \subset \mathbb{P}(1, a_1, a_2, a_3, a_4)$ of Fletcher and Reid (see [Fl], II.6.6).

(e) X has at worst the terminal quotient singularities $\frac{1}{r}(1, a, r-a)$ of [YPG], 5.2, where $r \geq 2$ and a is coprime to r; that is, X is quasismooth (see [Fl], I.3.3 and I.5).

Definition 1.2 A Fano variety X is *rigid* (or *birationally rigid*) if for every Mori fibre space $Y \to S$, the existence of a birational equivalence $\varphi\colon X \dashrightarrow Y$ implies that $Y \cong X$.

Although we do not include it as part of the definition, the only way of proving such a statement that we can conceive of is to show that any $\varphi\colon X \dashrightarrow Y$ factors as a chain of suitable selfmaps $X \dashrightarrow X$ followed by an isomorphism $X \cong Y$ (compare 7.4.1 below).

When it holds, rigidity provides a uniqueness statement in Mori theory: any minimal model program run on any variety birational to X terminates with X itself (up to isomorphism); in other words, the model $X \to \text{pt.}$ of X

as a Mori fibre space is unique up to isomorphism. In birational geometry, the striking thing about rigidity is that it replaces the *category* of birational maps from X to something else by the *group* of birational selfmaps. Note that Iskovskikh and Manin [IM] prove that a nonsingular quartic hypersurface $X_4 \subset \mathbb{P}^4$ is rigid (compare 7.1 below), and Iskovskikh and his school have extended this to many other classes of nonsingular Fano varieties.

The definition generalises to the strict Mori fibre spaces (with some pain, see [Co2], Definition 1.3), and the results of Sarkisov on conic bundles (see [Sa1]–[Sa2]) and Pukhlikov on del Pezzo fibre spaces (see [Pu2]) are rigidity statements. Our main aim is the following.

Main Theorem–Conjecture 1.3 *Let X be a Fano 3-fold satisfying (a–e) above. Then X is rigid.*

We prove this for the *general* member of each of the famous 95 families, and conjecture it for *all quasismooth* X (and prove some cases of it). It is definitely false if we allow other terminal singularities, even quite mild ones (see Example 7.5.1 below). We believe that our methods should also prove the remaining conjectural part, but the volume of calculations required for this seems disproportionate to the interest of the result; see 7.4.2 for an example of interesting nongeneral behaviour, and 7.8 for a discussion of what generality means in the theorem.

Remark 1.4 Our proof includes, and in fact depends on, a description of birational involutions $i\colon X \dashrightarrow X$ that generate the birational group $\operatorname{Bir} X$. A remarkable and quite beautiful fact is that these divide into just two types: Type I or *quadratic involutions* and Type II or *elliptic involutions*; see 2.6, Figure 1 for the Geiser and Bertini involutions of a cubic surface, 4.4 and 4.10 for the involutions of 3-folds, and the final discussion in 7.3. This dichotomy seems to extend also to other classes of rigid Fano 3-folds not treated here, and we speculate that it may be a completely general phenomenon of birational geometry.

The proof of the theorem involves hundreds of calculations, including many interesting special cases. A lot of this detail is summarised in the Big Table of Chapter 6. Rather than the statement of the theorem, which takes up just a single line, it is this wealth of fascinating information that we consider to be the essential content of the paper.

Our introductory discussion continues at a technically more sophisticated level in Chapter 3. However, we want to set the scene first with the case of a "minimal" cubic surface, which illuminates almost all the main ideas and methods of this paper in an elementary setting.

2 Cubic surface

A minimal nonsingular cubic surface $X = X_3 \subset \mathbb{P}^3$ is an example of a Fano variety as defined in Chapter 1: the conditions (a–c) specialise to the following:

(a) X is nonsingular;

(b) $-K_X = \mathcal{O}_X(1)$;

(c) $\operatorname{Pic} X = \mathbb{Z}$.

Here the final condition (c) means that X is defined over an algebraically nonclosed field K, and every curve C on X defined over K is the intersection of X with a hypersurface, $C = F \cap X$. We write $\overline{K}$ for the algebraic closure of K.

Theorem 2.1 *A minimal nonsingular cubic surface X is rigid. More precisely, given*

- *any nonsingular surface Y defined over K which is rational over $\overline{K}$ and minimal over K;*
- *any birational map $\varphi\colon X \dashrightarrow Y$ defined over K.*

Then φ is a composite of a chain of Geiser and Bertini involutions (defined in 2.6 below) followed by an isomorphism $X \cong Y$.

The result is essentially due to Castelnuovo around 1900, and has been reworked several times since then, most systematically by Iskovskikh and his school. We give a detailed treatment here, in order to illustrate our methods and ideas, and at the same time to compare and contrast different approaches to birational geometry, from classical projective geometry through to Mori theory and the Sarkisov program. For dramatic tension, we mention here that our argument has a little sting in its tail (see 2.11). The structure of the proof (and that of Main Theorem 1.3 for 3-folds in Chapters 3–5) consists of 4 parts.

(1) Set-up and the Noether–Fano–Iskovskikh inequalities (Theorem 2.3);

(2) Exclusion (see 2.4);

(3) Untwisting: construction of Geiser and Bertini involutions (see 2.6);

(4) Conclusion (see 2.10): untwisting decreases degree (Lemma 2.9.3) so that the theorem follows by induction.

Of these, the two outer steps apply to very general situations in birational geometry, while the inner two depend on specific properties of X_3; these two steps certainly do not work to give the same conclusion, say, in the case of a del Pezzo surface X_4 of degree 4, and indeed, X_4 is usually not rigid.

2.2 Set-up

A birational map $\varphi\colon X \dashrightarrow Y \subset \mathbb{P}^N$ is given by a mobile linear system $\mathcal{H}$ (that is, a linear system without fixed part), defined over K. Write $A = -K_X = \mathcal{O}_X(1)$ for the positive generator of $\operatorname{Pic} X$ (see assumption (c)), to avoid confusion about signs. Suppose $\mathcal{H} \subset |nA|$. If $n = 1$ then φ is linear, and hence must be an isomorphism; this is the basis of an induction.

Theorem 2.3 (Noether–Fano–Iskovskikh inequalities) *If φ is not an isomorphism then $K_X + \frac{1}{n}\mathcal{H}$ is not canonical.*

This result is completely general (see [Co1], Theorem 4.2 and compare [IM], §2): if a mobile linear system $\mathcal{H}$ on a Fano variety X defines a birational map $\varphi\colon X \dashrightarrow Y$ to another Mori fibre space, not an isomorphism, then $K_X + \lambda\mathcal{H}$ has a noncanonical singularity, where $\lambda \in \mathbb{Q}$ is defined by $K_X + \lambda\mathcal{H} = 0$: roughly, *not canonical* says that the base locus of $\mathcal{H}$ has some points with big multiplicity compared to the class of $\mathcal{H}$ (see Definition 3.1.1 below for the correct statement). In the case of surfaces, it takes on the very concrete form:

$$K_X + \frac{1}{n}\mathcal{H} \text{ is not canonical} \iff \begin{cases} \mathcal{H} \text{ has a base point of} \\ \text{multiplicity } m > n. \end{cases}$$

2.4 Exclusion

The aim is to impose restrictions on the maximal singularities of $\mathcal{H}$. Since $\mathcal{H}$ is defined over K, its base locus is also defined over K. Now, whereas a point over $\overline{K}$ is simply a geometric point, a point over K has a degree, which is the number of points into which it splits up over $\overline{K}$; for example, a point P with $\deg P = 2$ is a pair $P = P_1 + P_2$ of points with coefficients in $\overline{K}$, but conjugate over K.

Lemma 2.5 *A base point of $\mathcal{H}$ of multiplicity $m > n$ has degree 1 or 2.*

Proof Since $A^2 = \deg X = 3$ and $\mathcal{H} \subset |nA|$, we have $\mathcal{H}^2 = 3n^2$. On the other hand, since $\mathcal{H}$ is a mobile linear system, its selfintersection is at least its local selfintersection at the base points, so that $3n^2 = \mathcal{H}^2 \geq \deg P \cdot m^2 > \deg P \cdot n^2$. Therefore $\deg P < 3$. Q.E.D.

2.6 Construction of Geiser and Bertini involutions

The aim is to find some birational selfmap $X \dashrightarrow X$ to blow up any point P with $\deg P = 1$ or 2. We want to describe these constructions in three different languages.

2.6.1 "Italian"

We only need a birational map; the construction is geometric (see Figure 1). First, linear projection from a point P of degree 1 expresses X as a rational double cover of $\mathbb{P}^2$. The birational involution exchanging the sheets in this double cover is called the *Geiser involution* i_P.

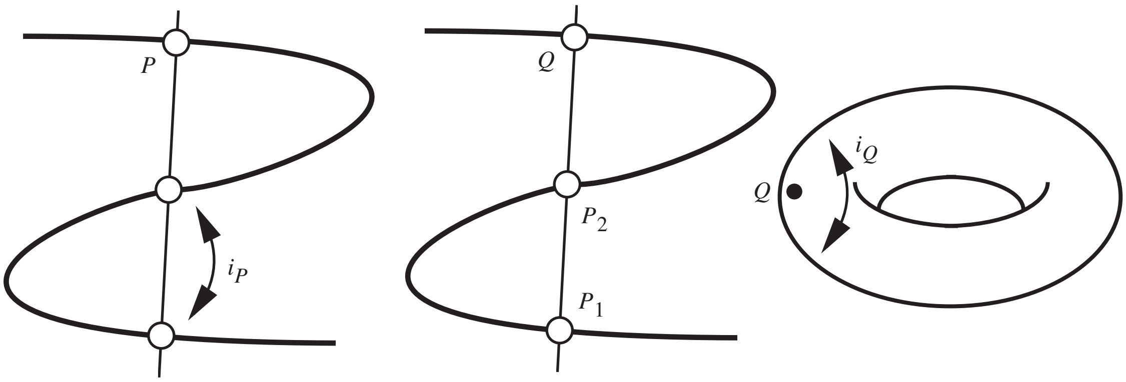

Figure 1: Geiser and Bertini involutions

Next, if P is a point of degree 2 in X then there is a unique line L of $\mathbb{P}^3$ through P, meeting X at a third point Q. Thus the linear system of plane sections of X through P is a pencil of elliptic curves, each with the marked point Q. The generic fibre F of the pencil has a group law with Q as the zero, which defines a reflection $x \mapsto -x$ of F, and hence of X. This involution is called the *Bertini involution* j_P.

(The terms Geiser and Bertini involution are traditionally used to describe birational involutions of $\mathbb{P}^2$ corresponding to the system of cubics through 7 or 8 marked points. For cubic surfaces, the Geiser involution is an obvious projective construction, but the Bertini involution is considerably more subtle. Geiser was a German working around 1860, whereas Bertini was an Italian who continued writing into the 1920s.)

2.6.2 "Russian"

We want to add to the birational map information telling us which linear system defines it, and its effect on the curves and points of X. The Geiser involution i_P is defined by the linear system $\mathcal{H} = |2A - 3P| = |\mathcal{I}_P^3 \mathcal{O}_X(2)|$.

Let $Y \to X$ be the blowup of P, and E the exceptional -1-curve. Then i_P defines a biregular automorphism of Y, interchanging E and the birational transform of the curve $C = T_P X \cap X$, the unique curve in $|A - 2P|$. Thus the effect of i_P on $\operatorname{Pic} Y$ is given by

$$\begin{matrix} A \mapsto 2A - 3E, \\ E \mapsto A - 2E, \end{matrix} \quad \text{that is, by the matrix} \quad \begin{pmatrix} 2 & -3 \\ 1 & -2 \end{pmatrix}.$$

Exercise 1 Check the following:

$$(2A - 3E)^2 = 3, \quad (A - 2E)^2 = -1, \quad (2A - 3E)(A - 2E) = 0,$$

a curve in $|2A - 3E|$ is elliptic, a curve in $A - 2E$ is rational, and finally, the matrix $\left(\begin{smallmatrix} 2 & -3 \\ 1 & -2 \end{smallmatrix}\right)$ has square $= 1$.

The Bertini involution j_P is just a bit more tricky. It is given by $\mathcal{H} = |5A - 6E|$, and takes P into $C = |4A - 5E|$, which is a curve over K. Over $\overline{K}$, it splits as $C = C_1 + C_2$, where C_1, C_2 are rational curves respectively in $|2A - 3P_1 - 2P_2|$ and $|2A - 2P_1 - 3P_2|$, conjugate over K, and meeting only at $P = P_1 + P_2$. The corresponding action on $\operatorname{Pic} Y$ is given by

$$\begin{matrix} A \mapsto 5A - 6E, \\ E \mapsto 4A - 5E, \end{matrix} \quad \text{that is, by the matrix} \quad \begin{pmatrix} 5 & -6 \\ 4 & -5 \end{pmatrix}.$$

Exercise 2 Make up your own exercise by analogy with the above.

2.6.3 Our approach: calculating the anticanonical ring of Y

To express our constructions in their natural generality, in this section 2.6.3 only, we suppose that X is a (not necessarily minimal) cubic surface, and $P \in X$ a point of degree 1 or 2, or $P = P_1 + P_2$ a pair of points; if $\deg P = 2$, we assume that P is not contained in a line of X. Write $\sigma\colon Y \to X$ for the blowup of P. Consider the *anticanonical ring* (or plurianticanonical ring) of Y, defined by

$$R(Y, -K_Y) = \bigoplus_{n \geq 0} H^0(Y, -nK_Y).$$

Lemma 2.7 *The anticanonical linear system $|A| = |\mathcal{O}_X(1)|$ contains an irreducible curve E passing through P and nonsingular there. Moreover:*

> *If* $\deg P = 1$ *then* $R(Y, -K_Y) = k[x_1, x_2, x_3, y]/F_4$, *where* $\deg x_i = 1$ *and* $\deg y = 2$*; that is,* $\operatorname{Proj} R(Y, -K_Y)$ *is a weighted hypersurface* $Z_4 \subset \mathbb{P}(1, 1, 1, 2)$.

If $\deg P = 2$ *then* $R(Y, -K_Y) = k[x_1, x_2, y, z]/F_6$, *where* $\deg x_i = 1$, $\deg y = 2$ *and* $\deg z = 3$; *that is,* $\operatorname{Proj} R(Y, -K_Y)$ *is a weighted hypersurface* $Z_6 \subset \mathbb{P}(1, 1, 2, 3)$.

Thus in either case, a multiple $|nA|$ *of the anticanonical system defines a birational map* $f \colon Y \dashrightarrow Z$ *(in fact, a morphism) to a weighted hypersurface with a naturally occurring biregular automorphism* i_Z. *We deduce a birational involution* i_X *of* X, *the conjugate of* i_Z *by* $X \dashrightarrow Y \dashrightarrow Z$; *this is the Geiser or Bertini involution of* X *corresponding to* P. How to use this involution i_X to untwist any birational map $\varphi \colon X \dashrightarrow Y$ is explained in 2.10 below.

Proof The linear system $|A - P| = |\mathcal{I}_P \mathcal{O}_X(1)|$ of hyperplane sections of X through P is mobile, so by Bertini's theorem its general element is irreducible. For any point P' of $X(\overline{K})$ defined over $\overline{K}$, there is exactly one element of $|A|$ singular at P', namely the tangent hyperplane section $T_{P',X} \cap X$. Since $|A-P|$ has dimension ≥ 1, its general element E is not the tangent hyperplane section for any $P' \in P$, and so E is nonsingular at P.

The divisor $A_{|E}$ is a divisor of degree $3 - \deg P = 2$ or 1 on an elliptic curve, so we know how to calculate the graded ring

$$R(E, A_{|E}).$$

Also $R(Y, A) \twoheadrightarrow R(E, A_{|E})$ is surjective with kernel the principal ideal generated by the equation x_1 of E in Y. This completes the proof. Q.E.D.

Remark 2.8 In complete generality, we don't know too much about an anticanonical ring – it is not birationally invariant, and there is no particular reason for it to be finitely generated. However, in this paper, the only method we use to get a birational involution is to calculate the anticanonical ring of a suitable blowup $Y \to X$, and to observe that, as here, it naturally involves a quadratic extension of rings. We obtain birational maps $X \leftarrow Y \to Z$, where Z is a Fano hypersurface having a biregular involution. Compare Chapter 4 for many examples.

2.9 Geiser and Bertini involutions as Sarkisov links, untwisting

Let X be a minimal cubic surface and P an irreducible point of degree 1 or 2. If $\deg P = 2$ then the minimality assumption (c) implies that the line of $\mathbb{P}^3$ through P is not contained in X, so that the construction of the Geiser or Bertini involution in 2.6 applies. Write $Y \to X$ for the blowup of P. We now describe the involutions in terms of Mori theory on Y, and explain in what sense they *untwists* P as a maximal centre of a linear system $\mathcal{H}$.

Lemma 2.9.1 *$-K_Y$ is ample.*

It follows that $Y = Z$ (in the notation of Lemma 2.7), and has a biregular involution i_Z.

Proof First, $-K_Y^2 = 2$ or 1. Also $K_Y = -A + E$ and $\mathcal{H}_1 = nA - mE$ gives

$$-K_Y = A - E = \frac{m-n}{m}A + \frac{1}{m}\mathcal{H}_1.$$

Now A and $\mathcal{H}_1$ are both nef, $A\Gamma = 0$ if and only if $\Gamma = E$ is the exceptional curve of $Y \to X$, and $\mathcal{H}_1 E > 0$. Q.E.D.

Lemma 2.9.2 *$i_Z(E)$ is an extremal rational curve distinct from E, so that the Mori cone $\overline{\mathrm{NE}}\,Y$ is spanned by two extremal rays as in Figure 2.*

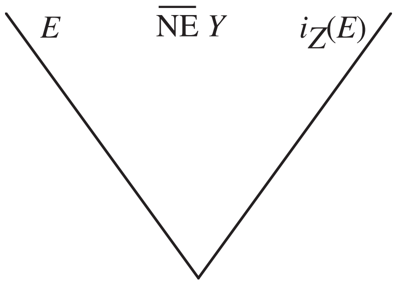

Figure 2: The Mori cone $\overline{\mathrm{NE}}\,Y$

Proof Because i_Z is a biregular involution, and E is an extremal rational curve, so is $i_Z(E)$. The divisor $E + i_Z(E)$ is a pullback from an ample divisor of weighted projective space, so has positive selfintersection. Thus $i_Z(E) = E$ is impossible. Q.E.D.

Lemma 2.9.3 *Let $\mathcal{H} \subset |nA|$ be a mobile linear system on X having P as a base point of multiplicity $m > n$, and write $\mathcal{H}' = (i_P)_*\mathcal{H}$ for its birational transform under the involution i_P. Then $\mathcal{H}' \subset |n'A|$ with $n' < n$, or in other words,* $\deg\mathcal{H}' < \deg\mathcal{H}$.

Proof Write $\mathcal{H}_1$ for the mobile linear system on Y obtained as the birational transform of $\mathcal{H}$. Then $K_X + \frac{1}{n}\mathcal{H} = 0$ on X, but

$$K_Y + \frac{1}{n}\mathcal{H}_1 = -\frac{m-n}{n}E < 0$$

is not quasieffective. In other words, the quasieffective threshold for $\mathcal{H}_1$ is certainly $> \frac{1}{n}$:

$$\lambda_0 := \inf\{\lambda \in \mathbb{R} \mid K_Y + \lambda\mathcal{H}_1 \text{ is quasieffective}\} > \frac{1}{n}. \tag{1}$$

We may view our involution $X \dashrightarrow X$ as the end product of a MMP on Y directed by $\mathcal{H}_1$. For this, start from $K_Y + \lambda\mathcal{H}_1$ with $\lambda \gg 0$, and decrease λ until we hit a ray Q of $\overline{\mathrm{NE}}\,Y$ with $(K_Y + \lambda\mathcal{H}_1)Q = 0$; necessarily $Q \neq R$, since

$$(K_Y + \frac{1}{m}\mathcal{H}_1)R = 0 \quad \text{and} \quad \lambda > \frac{1}{n} > \frac{1}{m}.$$

In the present case, this ray is the exceptional curve $i_Z(E)$, and we contract it to get back to the surface X. This contraction does not change the quasieffective threshold λ_0, so that $K_X + \lambda_0\mathcal{H}' = 0$, or $\mathcal{H}' \subset |n'A|$ with $n' = 1/\lambda_0 < n$. Q.E.D.

Remark 2.9.4 Of course, the lemma also follows from the Russian matrix of 2.6.2: it is an easy exercise to see that the degree n' of $\mathcal{H}'$ and its multiplicity m' at the contracted point P are

$$\begin{cases} n' = 2n - m \text{ and } m' = 3n - 2m & \text{for the Geiser involution,} \\ n' = 5n - 4m \text{ and } m' = 6n - 5m & \text{for the Bertini involution;} \end{cases}$$

here $m' = \mathcal{H}_1 \cdot i_Z(E)$, where $i_Z(E) = A - 2E$ (respectively $4A - 5E$) is the exceptional curve determining the nef threshold $\lambda = 1/m'$ in the above argument. Note that $m > n > n' > m'$, so that after the involution, $\mathcal{H}'$ is canonical at P: the involution untwists $\mathcal{H}$ both by lowering its degree, and by diluting its noncanonical singularity.

We want to take this opportunity to stress that this traditional and oft-repeated numerical argument is completely unnecessary. While coarser and less precise, the purely adjunction theoretic argument of Lemma 2.9.3 uses essentially no specific geometric properties of X, is more general, and extends essentially without change to Sarkisov links in higher dimensions. Compare 4.2 for the higher dimensional version and 7.6 for more discussion.

2.10 Conclusion of the proof of Theorem 2.1

Under the assumptions of Theorem 2.1, a birational map $\varphi\colon X \dashrightarrow Y$ is given by a linear system $\mathcal{H} \subset |nA|$. By the NFI inequalities (Theorem 2.3), if φ is not an isomorphism then $\mathcal{H}$ has multiplicity $m > n$ at a base point P of degree 1 or 2. Write i_P for the Geiser or Bertini involution. The composite $\varphi \circ i_P\colon X \dashrightarrow Y$ is defined by the linear system $(i_P)_*\mathcal{H}$, which by Lemma 2.9.3 has degree $n' < n$. By induction on n, we can assume that $\varphi \circ i_P$ is a composite of Geiser or Bertini involutions, hence so is φ. Q.E.D.

2.11 Discussion: special position and bad links

The Geiser or Bertini involutions in 2.6 are defined as rational maps even for P in *special position*, for example, if P is a geometric point lying on a line of X_3; compare, for example, [Isk]. Special position usually contradicts minimality (say, by singling out one or more of the lines of X_3), but not always: it can happen, for example, that X is minimal over K, but has an Eckardt point P defined over K, with 3 conjugate lines out of it. At first sight, it might appear that our proof goes wrong in such cases; at least, it certainly does not work to give the same conclusion: indeed, $-K_Y$ is not ample, the anticanonical model $Y \to Z$ contracts an extremal ray spanned by -2-curves, and the two rays of $\overline{\mathrm{NE}}\,Y$ are not at all symmetric. Nevertheless, i_Z is defined, and hence it induces a birational selfmap of Y.

The solution to our puzzle is striking: for a minimal cubic surface X, a Geiser or Bertini involution centred at a point P in special position is *biregular*. The point is that the composite rational map

$$i_Y \colon Y \to Z \to Z \leftarrow Y$$

is clearly biregular outside the -2-curves of Y, and it is also biregular at the -2-curves by the usual argument from absolute minimal models of surfaces. The involution i_Y acts by the identity on $\operatorname{Pic} Y$ and on $\overline{\mathrm{NE}}\,Y$.

The fact that the Geiser involution centred at an Eckardt point is biregular is easily verified by a direct calculation in $\mathbb{P}^3$, and was probably known classically; the earliest reference we know is B. Segre [S], pp. 145 ffoll. (Section XIV, Autoprojective nonsingular cubic surfaces). It was rediscovered by Iskovskikh in the 1960s [Isk]. However, we first became aware of it in the context of the current argument via Mori theory.

Remark 2.12 In this case, our proof implies that such a point P is not a maximal singular centre of any linear system $\mathcal{H}$: if it were, by the Sarkisov program, we could start untwisting it by taking the blowup $Y \to X$ and running a MMP on Y; however, the shape of $\overline{\mathrm{NE}}$ means that this can only lead back to X via the identity or the Geiser or Bertini involution i_P, which is biregular, so doesn't untwist anything. This is a contradiction.

The argument here is an "exclusion" argument quite different from that of 2.4. Recall that in 2.4, if P is a maximal singularity, we calculated intersection numbers to prove that $\deg P \le 2$. In contrast, the argument just given is not at all numerical; it proceeds by attempting to construct a Sarkisov link, then arriving at a contradiction. The only possible ray of $\overline{\mathrm{NE}}\,Y$ to contract is a -2-curve, which is not a Mori extremal ray; contracting it would take us outside the Mori category of varieties with terminal singularities. We will see this principle in action in much less trivial cases in the "bad link" section 5.5 below. Compare the discussion in 7.6

3 Plan of proof of Theorem 1.3

We now give a concise statement that implies our main result Theorem 1.3. We first recall some definitions from the Sarkisov program [Co1]–[Co2] so far only hinted at in Chapters 1–2.

3.1 Basic notions

Definition 3.1.1 (canonical threshold) X is a variety, $\mathcal{H}$ a mobile linear system, and $\widetilde{X} \to X$ a resolution with exceptional divisors E_i. As usual, we write

$$K_{\tilde{X}} = K_X + \sum a_i E_i,$$
$$\widetilde{\mathcal{H}} = \mathcal{H} - \sum m_i E_i,$$

to define the *discrepancies* $a_i = a_{E_i}(K_X)$ of the exceptional divisors E_i and their *multiplicities* m_i in the base locus of $\mathcal{H}$. For $\lambda \in \mathbb{Q}$, we say that $K_X + \lambda\mathcal{H}$ is *canonical* if all $\lambda m_i \le a_i$, so that $K_{\tilde{X}} + \lambda\widetilde{\mathcal{H}} - (K_X + \lambda\mathcal{H})$ is effective (≥ 0). Then we define the *canonical threshold* to be

$$\begin{aligned} c = c(X, \mathcal{H}) &= \max\{\lambda \mid K + \lambda\mathcal{H} \text{ is canonical}\} \\ &= \min_{E_i}\{a_i/m_i\}. \end{aligned}$$

This is well defined, independently of the resolution $\widetilde{X}$. In all the cases we're interested in, $K_X + \frac{1}{n}\mathcal{H} = 0$ and $c < \frac{1}{n}$.

Definition 3.1.2 (maximal singularity) Now suppose that $K_X + \frac{1}{n}\mathcal{H} = 0$ and $K_X + \frac{1}{n}\mathcal{H}$ is not canonical, so that $c < \frac{1}{n}$. We make the following definitions:

(1) a *weak maximal singularity* of $\mathcal{H}$ is a valuation v_E of $k(X)$ for which $m_E(\mathcal{H}) \ge n a_E(K_X)$;

(2) a *strong maximal singularity* is an extremal extraction $Y \to X$ in the Mori category (see Definition 3.4.1) having exceptional divisor E with $c = a_E(K_X)/m_E(\mathcal{H})$.

In either case, the image of E in X, or the centre $C(X, v_E)$ of the valuation v_E, is called the *centre* of the maximal singularity E.

Remark 3.1.3 (a) Weak maximal singularity is the traditional definition used by the Moscow school, and is birational in spirit, whereas strong maximal singularity is a biregular construction. It is a seed that always grows into a Sarkisov link (by applying the MMP as in [Co1], Section 5).

The logic of Definition 3.1.2, and the part it plays in the paper is as follows:

(b) If a birational map $\varphi\colon X \dashrightarrow Y$ between Mori fibre spaces is not an isomorphism, the NFI inequalities give the existence of a *weak* maximal singularity.

(c) If a weak maximal singularity exists, the argument of [Co1], Proposition 2.10 (based on running a MMP on a resolution $\widetilde{X}$) constructs a *strong* maximal singularity.

(d) The definition of strong maximal singularity contains precise biregular information, in particular on the Mori cone $\overline{\mathrm{NE}}\,Y$ of the extraction $Y \to X$, that can be used as the basis of biregular methods of argument.

(e) To summarise, we distinguish three points of view: a *centre* is a point or curve $\Gamma \subset X$; a *weak singularity* is a valuation of $k(X)$; a *strong singularity* is an extremal extraction $(E \subset Y) \to (\Gamma \subset X)$.

Definition 3.1.4 (Sarkisov link) A *Sarkisov link of Type II* between two Fano 3-folds X and X' (Definition 1.1) is a birational map $f\colon X \dashrightarrow X'$ that factors as

$$\begin{array}{ccc} X_1 & \dashrightarrow & X_1' \\ \downarrow & & \downarrow \\ X & \overset{f}{\dashrightarrow} & X' \end{array}$$

where

(a) $X_1 \to X$ and $X_1' \to X'$ are extremal divisorial contractions in the Mori category, and

(b) $X_1 \dashrightarrow X_1'$ is a composite of inverse flips, flops and flips (in that order), and in particular, is an isomorphism in codimension 1.

Definition 3.1.5 (degree of φ) Suppose that X is a Fano 3-fold with the property that $A = -K_X$ generates the Weil divisor class group: $\mathrm{WCl}\,X = \mathbb{Z}\cdot A$ (this holds in our case by Lemma 3.5). Let $\varphi\colon X \dashrightarrow Y$ be a birational map to a given Mori fibre space $Y \to S$, and fix a very ample linear system $\mathcal{H}_Y$ on Y; write $\mathcal{H} = \mathcal{H}_X$ for the birational transform $\varphi_*^{-1}(\mathcal{H}_Y)$.

The *degree* of φ, relative to the given Y and $\mathcal{H}_Y$, is the natural number $n = \deg\varphi$ defined by $\mathcal{H} = nA$, or equivalently $K_X + \frac{1}{n}\mathcal{H} = 0$.

Definition 3.1.6 (untwisting) Let $\varphi\colon X \dashrightarrow Y$ be a birational map as above, and $f\colon X \dashrightarrow X'$ a Sarkisov link of Type II. We say that f *untwists* φ if $\varphi' = \varphi \circ f^{-1}\colon X' \to Y$ has degree smaller than φ.

Remark 3.1.7 The Sarkisov program factors an arbitrary birational map between Mori fibres spaces as a chain of more general types of links, using a more complicated inductive framework. See [Co1], Definition 3.4 for the general definition of a *Sarkisov link* $f\colon X \dashrightarrow X'$, and [Co1], Definition 5.1 for the *Sarkisov degree* of a birational map $\varphi\colon X \dashrightarrow Y$ between Mori fibre spaces. The above Definitions 3.1.4–3.1.6 are special cases that are sufficient for our purposes in this paper. We can get away with this because we start from a Fano 3-fold X, and blessed fortune has decreed that we only perform untwistings returning always to X (our aim is to prove rigidity), so that we never really venture beyond the first link of a Sarkisov chain.

The following result is our concise statement:

Theorem 3.2 *X is a* general *element of one of the famous* 95 *families of Fano hypersurfaces ("general" is discussed in 7.8). For every point P or curve C of X, we can do one of the following:*

(I) Either: *construct a birational involution $X \dashrightarrow X$ that is a Sarkisov link of Type II, untwisting P (or C) (see Definitions 3.1.4–3.1.6) as a strong maximal centre of any linear system on X.*

(II) Or: *exclude the possibility that any linear system $\mathcal{H}$ on X can have a strong maximal singularity E centred at P or C.*

All curves and nonsingular points of X are excluded. The Big Table of Chapter 6 documents the fate of the singular points (which by assumption are terminal quotient singularities $\frac{1}{r}(1,a,r-a)$): those marked Q.I. or E.I. are untwisted, and the remainder are excluded. For more details, see the introduction to Chapter 6.

This theorem summarises the conclusions of a whole series of calculations carried out for Part I in Chapter 4 and for Part II in Chapter 5. Its proof ends on page 232. Each of these chapters has an introduction expanding on a number of points concealed by the concise nature of our definitions and statement.

3.3 Theorem 3.2 implies Theorem 1.3

This is standard, and is the same as the proof in Chapter 2. If X is Fano and $Y \to S$ a Mori fibre space, a birational map $\varphi\colon X \dashrightarrow Y$ is defined by a mobile linear system $\mathcal{H}$. By the NFI inequalities ([Co1], Theorem 4.2

or [Pu1], Theorem 2.11), if φ is not an isomorphism then $\mathcal{H}$ has a maximal centre P or C (see Definition 3.1.2), hence a strong maximal centre by [Co1], Proposition 2.10. This centre can't be in Part II of Theorem 3.2, so by Part I there is a birational involution $i\colon X \dashrightarrow X$ that is a Sarkisov link, and untwists the maximal centre P or C, so that $\varphi \circ i$ has smaller degree. Thus after a number of steps, we arrive at $X \cong Y$. Q.E.D.

3.4 Divisorial contractions

We start with a short preliminary discussion of 3-fold divisorial contractions, fixing the terminology, and stating a theorem of Kawamata used repeatedly in Chapters 4–5. A maximal singularity (Definition 3.1.2) has centre $\Gamma \subset X$ that is one of

(1) a curve; (2) a nonsingular point; (3) a singular point;

our treatment and results are quite different in the three cases.

Definition 3.4.1 First, let $P \in X$ be the germ of a 3-fold terminal singularity. A *divisorial contraction* is a proper birational morphism $f\colon Y \to X$ such that

(1) Y has terminal singularities,

(2) the exceptional set of f is an irreducible divisor $E \subset Y$,

(3) $-K_Y$ is relatively ample for f.

An *extremal* divisorial contraction $f\colon Y \to X$ is an extremal divisorial contraction in the Mori category. In other words, Y has $\mathbb{Q}$-factorial terminal singularities, f is the contraction of an extremal ray R of $\overline{\mathrm{NE}}\,Y$ satisfying $K_Y \cdot R < 0$, and the exceptional set $\operatorname{Exc} f = E \subset Y$ is a divisor in Y. Its image $\Gamma = f(E)$ is a closed point or a curve of X, and we usually write $f\colon (E \subset Y) \to (\Gamma \subset X)$. Here X is not necessarily a germ, but $Y \to X$ is a divisorial contraction in the above sense above the germ around any point $P \in \Gamma$. Viewed from X, we also say that f is an *extremal extraction*, or that it *extracts* the valuation $v = v_E$ of $k(X)$ from its *centre* $\Gamma = C(X, v_E) \subset X$.

The classification of 3-fold divisorial contractions to a point is a beautiful open problem, the answer to which is only known in a few special cases, of which the following is important to us. The result is simple to state and prove, and very useful.

Theorem–Definition 3.4.2 (Kawamata [Ka]) *Let*

$$P \in X \cong \frac{1}{r}(1, a, r-a) \quad (\text{with } r \geq 2 \text{ and } a \text{ coprime to } r)$$

be the germ of a 3-fold terminal quotient singularity, and

$$f\colon (E \subset Y) \to (\Gamma \subset X)$$

a divisorial contraction such that $P \in \Gamma$. Then $\Gamma = P$ and f is the weighted blowup with weights $(1, a, r-a)$. (Note that we don't need to assume that the contraction f is *extremal* here.) $\square$

We refer to it as the *extremal divisorial contraction*, the *extremal blowup* or the *Kawamata blowup* of P.

Corollary 3.4.3 *Suppose that X is a 3-fold with only terminal quotient singularities. If a curve $\Gamma \subset X$ is the centre of a divisorial extraction $f\colon (E \subset Y) \to (\Gamma \subset X)$ then $\Gamma \subset \operatorname{NonSing} X$ (and f is the blowup of $\mathcal{I}_\Gamma$ over the generic point of Γ).*

For if Γ passed through a terminal quotient point P, Theorem 3.4.2 would imply that $\Gamma = P$, a contradiction. Q.E.D.

Remark 3.4.4 (curve as centre) The trinity of viewpoints discussed in Remark 3.1.3, (e) all boil down to more-or-less the same thing for a curve centre. Let $C \subset X$ be a curve in a 3-fold. The valuation of $k(X)$ above C that could be a weak maximal singularity is obviously unique. There is *at most* one extraction $Y \to X$ which is the blowup above the generic point of C. In our case, when X has only terminal quotient singularities, this extraction only exists in the Mori category if $C \subset \operatorname{NonSing} X$ and C has restricted singularities, for example, planar singularities.

We use the following notation and basic results many times in what follows, sometimes without explicit warning.

Notation 3.4.5 For a divisorial contraction $f\colon (E \subset Y) \to (\Gamma \subset X)$, we adopt the shorthand notation $A = -K_X$, $B = -K_Y$. As before, this avoids confusion about signs and unnecessary subscripts. When f is the Kawamata blowup of a terminal quotient singularity $P \in X \cong \frac{1}{r}(1, a, r-a)$, that is, the $(1, a, r-a)$-blowup, the adjunction formula for f is

$$B = A - \frac{1}{r}E.$$

In this context, we sometimes use B without referring to f explicitly.

Proposition 3.4.6 *(1) Let $(E \subset Y) \to (P \in X)$ be the Kawamata blowup of a 3-fold terminal quotient singularity $P \in X \cong \frac{1}{r}(1, a, r-a)$. Then $K_Y = K_X + \frac{1}{r}E$ (or $B = A - \frac{1}{r}E$), and*

$$\frac{1}{r^2}E^3 = \frac{1}{a(r-a)}\,.$$

(2) Write $\mathbb{C}^3$ for the local index 1 cover of $P \in X$ with ξ, η, ζ eigencoordinates for the $\mathbb{Z}/r$ action. Then ξ, η, ζ can be viewed as local analytic sections of $\mathcal{O}_X(1), \mathcal{O}_X(a), \mathcal{O}_X(r-a)$ respectively, and lifted to Y, they vanish on E with multiplicities exactly $\frac{1}{r}, \frac{a}{r}, \frac{r-a}{r}$, so that

$$\xi \in \mathcal{O}_Y(A - \frac{1}{r}E) = \mathcal{O}_Y(B), \quad \eta \in \mathcal{O}_Y(aA - \frac{a}{r}E) = \mathcal{O}_Y(aB),$$
$$\zeta \in \mathcal{O}_Y\Big((r-a)A - \frac{r-a}{r}E\Big) = \mathcal{O}_Y((r-a)B)$$

in a neighbourhood of $E \subset Y$.

Proof More-or-less standard exercises in toric blowups. For (1), it is easy to see that $E \cong \mathbb{P}(1, a, r-a)$ and $\mathcal{O}_E(-E) \cong \mathcal{O}_{\mathbb{P}(1,a,r-a)}(-r)$. To make sense of (2), one can for example take pullback of the invariant functions $\xi^r, \eta^r, \zeta^r \in \mathcal{O}_X$. $\square$

Lemma 3.5 *Let $X_d \subset \mathbb{P}(a_0, a_1, \ldots, a_n)$ be a quasismooth weighted hypersurface of dimension ≥ 3 (that is, $n \geq 4$). Then every Weil divisor class on X_d is linearly equivalent to $\mathcal{O}_X(i)$ for some $i \in \mathbb{Z}$.*

Proof Write $C_X \subset \mathbb{C}^5$ for the affine cone over X, that is, the hypersurface singularity given by the same equation as X_d. Then by definition of quasismooth, $0 \in C_X$ is an isolated singularity, and is factorial by a famous theorem of Grothendieck[1] (see Call and Lyubeznik [CL] or [SGA2], Exp. XI, Théorème 3.13). Thus every Weil divisor on X is the $\mathbb{C}^*$ quotient of a $\mathbb{C}^*$-invariant principal divisor on C_X. Q.E.D.

[1] In topology, this result is completely elementary: by Lefschetz theory, the link of an isolated hypersurface singularity $0 \in V \subset \mathbb{C}^n$ with $n \geq 5$ is 2-connected. Grothendieck's theorem and its reworking in [CL] concerns arbitrary local rings; the case of graded rings that we need is easier.

4 Untwisting

4.1 Introduction to involutions

Part I of Theorem 3.2 involves two types of untwisting: 54 quadratic involutions (see 4.4–4.9) and 8 elliptic involutions (see 4.10–4.13). In either case we construct a birational involution $i_P \colon X \dashrightarrow X$ that factors as follows:

$$\begin{array}{ccccccc} & Y & & & & Y & \\ {}^{f}\swarrow & & \searrow^{g} & & {}^{g}\swarrow & & \searrow^{f} \\ X & & Z & \xrightarrow{i_Z} & Z & & X \end{array}$$

Here

(1) $f \colon Y \to X$ is the Kawamata blowup of a terminal quotient singularity $P \in X$ of type $\frac{1}{r}(1, a, r-a)$;

(2) $-K_Y$ is nef, and the anticanonical model of Y defines a morphism $g \colon Y \to Z$ contracting finitely many flopping curves of Y to points of Z;

(3) as an anticanonical model, Z is a 3-fold with terminal singularities and ample $-K_Z$, but be warned: it is *not in the Mori category*, since it has local class group $\mathbb{Z}$ at each flopping point (thus $X \dashrightarrow Z$ is *not* a morphism of the Sarkisov category);

(4) Z has a biregular involution i_Z that acts by -1 on the local class group at every flopping singularity of Z;

(5) the birational involution $i_Y \colon Y \dashrightarrow Y$ is the standard flop of the curves contracted by $Y \to Z$;

(6) finally,

$$i_P = f \circ g^{-1} \circ i_Z \circ g \circ f^{-1}.$$

The key point of the proof is thus the calculation of the anticanonical ring of the Kawamata blowup $Y \to X$. Here $R(Y, B) \subset R(X, A)$ is the subring whose graded elements are sections $H^0(\mathcal{O}_X(dA))$ that vanish d/r times along E (in the sense of Proposition 3.4.6), so that they remain plurianticanonical on Y. (Recall that $A = -K_X$ and $B = -K_Y = A - \frac{1}{r}E$.) Note the close parallel with the treatment of Geiser and Bertini involutions in 2.6.3, and compare 7.3 for passing to the subring $R(Y, B)$ as a "projection".

Lemma 4.2 *A Type II link $X \dashrightarrow X$ untwists P as a strong maximal centre of any linear system $\mathcal{H}$.*

Proof This is essentially the same as the proof of Lemma 2.9.3. Suppose that P is a maximal centre of a linear system $\mathcal{H} \subset |nA|$. Then the Kawamata blowup $Y \to X$ is a strong maximal singularity of $|\mathcal{H}|$, because it is the only divisorial extraction centred at P (see Theorem 3.4.2). Hence

$$K_Y + \frac{1}{n}\mathcal{H}_Y = -\frac{m - na}{n}E < 0$$

The rest is as for Lemma 2.9.3. Q.E.D.

Remark 4.3 (Russian matrix) Since i_Z takes $B = A - \frac{1}{r}E$ to itself, it is not hard to see that i_Z acts on divisors of Y by $A \mapsto sA - \frac{s+1}{r}E$ and $E \mapsto r(s-1)A - sE$ for some s, that is, by the matrix

$$\begin{pmatrix} s & -(s+1)/r \\ r(s-1) & -s \end{pmatrix}.$$

Since we know the image of E, it is an easy exercise to calculate s (but we have never felt any inclination to do it). Compare 2.6.2 and Remark 2.9.4.

4.4 Quadratic involutions

The untwisting column of the Big Table contains 54 quadratic involutions (indicated by Q.I.). When properly understood, they all follow the same simple pattern: in suitable coordinates, the defining equation can be written

$$F = \eta\xi^2 + a\xi + b, \tag{2}$$

where $\xi = x_i$, $\eta = x_j$ for some $i \neq j$, and ξ does not appear in a, b. Here the centre we want to untwist is $P_i = (0, \dots, 1, 0, \dots)$. As a birational involution, i_{P_i} just interchanges the two roots of (2). We work this out in biregular terms in several examples and in general in Theorem 4.9. We use slightly different manoeuvres to achieve the form (2):

(i) In most cases, no extra assumption is needed (beyond quasismooth), and the choice of coordinates is obvious.

(ii) In 9 of our cases, the presence of the monomial $x_j x_i^2$ is an extra "generality" assumption on X_d, indicated by the asterix $*x_j x_i^2$ in the Big Table (see Example 4.6); as we explain in 7.4.2, when it does not appear, the corresponding point P_i is excluded as a maximal centre.

(iii) In a further 10 cases, the choice of coordinates is an obvious little trick (see Example 4.8).

The conclusion of the construction (and the information documented in the Big Table of Chapter 6) is stated as Theorem 4.9. *Let us play:*

Example 4.5 No. 2, $X_5 \subset \mathbb{P}(1,1,1,1,2)$, centre $P = P_4 = \frac{1}{2}(1,1,1)$

In this case the preliminary massaging of the equation F_5 is very familiar. Write $x_0, \dots, x_3, y$ for homogeneous coordinates, of which only y is nonzero at P_4. Suppose that the equation of X_5 is F_5; then y only appears quadratically in F_5, accompanied by a nonzero linear form, which we can take to be x_0. Thus

$$F_5 = x_0 y^2 + a_3 y + b_5 = 0, \tag{3}$$

where $a_3, b_5 \in k[x_0, x_1, x_2, x_3]$ are homogeneous of the indicated degrees. The birational involution i_P just exchanges the roots of F_5.

Now to see this as the biregular procedure of 4.1. To specify the midpoint Z, we eliminate y, replacing it with the new variable $w = x_0 y$, which is integral over $k[x_0, \dots, x_3]$: indeed, multiplying equation (3) by x_0 gives

$$g_6 = w^2 + a_3 w + b_5 x_0 = 0.$$

This is the equation of a Fano hypersurface $Z_6 \subset \mathbb{P}(1,1,1,1,3)$.

The transformation $X_5 \dashrightarrow Z_6$ is obtained by performing the Kawamata blowup $Y \to X$ of P_4, followed by a flopping contraction morphism $Y \to Z_6$. Indeed, P_4 is a singularity of type $\frac{1}{2}(1,1,1)$ with local coordinate x_1, x_2, x_3, and locally around P_4, the Kawamata blowup $Y \to X$ is the graph of the rational map to $\mathbb{P}^2$ given by x_1, x_2, x_3. The divisor $B = A - \frac{1}{2}E$ is eventually free and defines the morphism $Y \to Z_6$; in fact, in the present case, $|B| = |x_0, x_1, x_2, x_3|$ is itself a free linear system.

What does $Y \to Z_6$ contract? It is clear from what we have just said that the exceptional locus E of the blowup $Y \to X$ is mapped biregularly; thus any contracted curves are present in the original X given by (3), and they are the lines L_i through P given by $x_0 = a_3 = b_5 = 0$. We now prove that this is a finite set of lines (15 for generic X): write $\overline{a}_3, \overline{b}_5 \in k[x_1, x_2, x_3]$ for reduction modulo x_0; if these two have a common factor c (say) then the original hypersurface X_5 contains the surface $x_0 = c = 0$, contradicting quasismooth (because F can be rewritten $\alpha x_0 + \beta c$, which is singular at $\alpha = \beta = x_0 = c = 0$).

Example 4.6 No. 5, $X_7 \subset \mathbb{P}(1,1,1,2,3)$, centre $P = P_3 = \frac{1}{2}(1,1,1)$

Write x_0, x_1, x_2, y, z for homogeneous coordinates; only y is nonzero at P. In this case, P is untwisted by a quadratic involution i_P if and only if the monomial zy^2 appears in F.

The point is the following: quasismoothness at P involves a monomial ηy^m with $\eta = x_i$ or z. Because $\operatorname{wt} y = 2$, a priori, F_7 is a cubic in y, say

$x_0y^3 + \gamma zy^2 + \cdots$. However, if $\gamma \neq 0$, a change of coordinates $z \mapsto z' = \gamma z + x_0y + \cdots$ kills any other monomials divisible by y^2 in F. After this, we arrive at the quadratic equation

$$F_7 = zy^2 + a_5y + b_7, \tag{4}$$

in y, where $a_5, b_7 \in k[x_0, x_1, x_2, z]$. The remainder of the construction is word-for-word as in Example 4.5: the involution i_P interchanges the two roots of the quadratic (4). The midpoint of the link is $Z_{10} \subset \mathbb{P}(1,1,1,3,5)$, obtained by eliminating y in favour of $w = yz$, and with equation $g_{10} = w^2 + a_5w + b_7w$. The map $X \dashrightarrow Z_{10}$ is the Kawamata blowup $Y \to X$ followed by the flopping contraction morphism $Y \to Z$; this is a morphism because Y is the graph of $x_0 : x_1 : x_2$, locally over P. As before, $Y \to Z_{10}$ is flopping because quasismooth implies that modulo z, the residues $\overline{a}_5, \overline{b}_7 \in k[x_0, x_1, x_2]$ have no common factor.

Remark 4.7 (starred monomial $*\eta\xi^2$) In this case, the possibility that $zy^2 \notin F$ does not contradict quasismooth, and what happens is also interesting: we will see later that P is then excluded as a maximal centre (see 7.4.2). The Big Table commemorates this extra generality assumption by starring the key monomial $*zy^2$. The same dichotomy involving a starred monomial $*\eta\xi^2$ happens in 8 other cases, Nos. 12, 13, 20, 25, 31, 33, 38, 58.

Example 4.8 No. 4, $X_6 \subset \mathbb{P}(1,1,1,2,2)$, centre $P \in P_3P_4 = 3 \times \frac{1}{3}(1,1,2)$ and No. 6, $X_8 \subset \mathbb{P}(1,1,1,2,4)$, centre $P \in P_3P_4 = 2 \times \frac{1}{2}(1,1,1)$

The new trick in these cases is simply to arrange that P is one of the coordinate points. In No. 4, P_3P_4 is the (y_1, y_2)-line, and the restriction of F_6 is a cubic in (y_1, y_2). Since P is one the roots, we can take $P = (0,1)$ and $F = y_1y_2^2 + \cdots$. (This monomial must occur by quasismoothness at P.) Then

$$F_6 = y_1y_2^2 + a_4y_2 + b_6,$$

and we proceed as before. Nos. 9, 17, 27 involve the same trick.

In No. 6, P_3P_4 is the line $\mathbb{P}(2,4)$, a badly-formed $\mathbb{P}^1$, with coordinates y^2, z. Quasismoothness at P_4 implies that $P_4 \notin X$, that is, $z^2 \in F$. Thus F_8 restricted to P_3P_4 factors as $(z - c_1y^2)(z - c_2y^2)$ with $c_1, c_2 \in k$. A change of coordinates $z \mapsto z' = z - c_1y^2$ takes P to $P_3 = (0,0,0,1,0)$ (with only y nonzero), and the monomial zy^2 is forced by quasismoothness. Thus

$$F_8 = zy^2 + a_6y + b_8,$$

and we proceed as before. Nos. 15, 30, 41, 42, 68 involve the same trick.

Theorem 4.9 *Suppose that $X_d \subset \mathbb{P}(1, a_1, a_2, a_3, a_4)$ is given by*

$$F_d = x_{i_3}\xi^2 + a_l\xi + b_d = 0, \tag{5}$$

where ξ, x_{i_3} are two of the coordinates, and a_l, b_d do not involve ξ. Write P_ξ for the coordinate point at which $\xi \neq 0$ and all the other $x_i = 0$. Then interchanging the roots of (5) defines a birational involution i_{P_ξ} of X_d. It is a Sarkisov link of Type II in the sense of Definition 3.1.4, and untwists P_ξ as a maximal centre.

This construction provides all the quadratic involutions marked Q.I. in the Big Table. The assumption of the theorem includes the presence of the key monomial $*x_jx_i^2$. If the starred monomial is not present, the centre is excluded (see 7.4.2).

The Big Table lists the following information: the key monomial $x_jx_i^2$ (here a $*$ indicates that the presence of the monomial is an extra assumption), and the weighted degrees of a, b. For example, No. 33, centre P_3, $*ut^2, 12, 17$ means that the presence of the monomial ut^2 is assumed, $X_{17} \subset \mathbb{P}(1, 2, 3, 5, 7)$ is defined by

$$F_{17} = ut^2 + a_{12}t + a_{17},$$

and the centre P_3 is untwisted by the birational involution i_{P_3}.

Proof Set $\deg \xi = r$. Write $x_{i_0}, x_{i_1}, x_{i_2}$ for the other 3 variables x_i in addition to ξ, x_{i_3} and $a_{i_j} = \deg x_{i_j}$ for their degrees. Since P_ξ is a singularity of type $\frac{1}{r}(1, a, r-a)$, with local coordinates $x_{i_0}, x_{i_1}, x_{i_2}$, it follows that $a_{i_0}, a_{i_1}, a_{i_2}$ are congruent to $1, a, r-a$ modulo r (in some order). We claim the following little coincidence: the degrees $a_{i_0}, a_{i_1}, a_{i_2}$ are actually *equal* to $1, a, r-a$. Indeed, if $\deg x_{i_3} = s$ then, in view of the term $x_{i_3}\xi^2 \in F_d$, it follows that $d = 2r + s$; on the other hand, $d = \sum a_i + 1 = r + s + \sum_{j=0}^{2} a_{i_j} + 1$, so that $a_{i_0}, a_{i_1}, a_{i_2}$ add to $r + 1$.

Now the Kawamata blowup $Y \to X$ of P_ξ is the $(1, a, r-a)$ weighted blowup, so locally at P_ξ it is the graph of the rational map to $\mathbb{P}(1, a, r-a)$ given by $x_{i_0}, x_{i_1}, x_{i_2}$. It follows easily from Proposition 3.4.6, (2) that $x_{i_0}, x_{i_1}, x_{i_2}$ and x_{i_3} belong to the anticanonical ring of Y, so that it is of the form

$$R(Y, B) = k[x_{i_0}, x_{i_1}, x_{i_2}, x_{i_3}, w]/(g),$$

where $w = \xi x_{i_3}$ and $g = w^2 + a_lw + x_{i_3}b_d$.

As in Example 4.5, quasismoothness implies that $x_{i_3} = a_l = b_d = 0$ is a finite number of lines through P_ξ. Therefore the anticanonical morphism $Y \to Z$ is a flopping contraction. Q.E.D.

4.10 Elliptic involutions

There are eight elliptic involutions, indicated by E.I. in the Big Table. The computation is the same in each case, although considerably more amusing than for the quadratic involutions. It involves *two* different key monomials: the centre we want to untwist is $P = P_\xi$, and the monomial defining the tangent space there is $x_{i_2}\xi^3$; however, the same variable ξ also takes part in a monomial $\xi\zeta^2$ defining the tangent space at another point $Q = Q_\zeta$. Thus

$$F = \xi\zeta^2 + \cdots + x_{i_2}\xi^3 + \cdots .$$

Here $x_{i_0}, x_{i_1}, x_{i_2}, \xi, \zeta$ (in some order) are coordinates on $\mathbb{P}$, and necessarily $\deg\zeta > \deg\xi$. Typically, $\xi = x_3$ and $\zeta = x_4$, so that $P = P_3$ and $Q = P_4$.

Whereas our treatment of the quadratic involution was based on eliminating one variable, for the elliptic involution, we eliminate *both* variables ξ and ζ at once, replacing them by somewhat complicated terms

$$v = \zeta^2 + x_{i_2}\xi^2 + \cdots \quad \text{and} \quad w = \zeta^3 + x_{i_2}\xi^2\zeta + \cdots .$$

These are designed to be plurianticanonical on Y (that is, vanish enough times on the exceptional divisor E of $Y \to X$), and it turns out that, together with the other coordinates $x_{i_0}, x_{i_1}, x_{i_2}$, they generate the anticanonical ring of Y, and satisfy a relation of the form

$$w^2 + Aw = v^3 + Bv^2 + Cv + D \tag{6}$$

with $A, B, C, D \in k[x_{i_0}, x_{i_1}, x_{i_2}]$. This equation defines the midpoint of the link (see 4.1), which is a (weak) Fano hypersurface $Z_{6e} \subset \mathbb{P}(1, b_1, b_2, 2e, 3e)$ having a biregular involution i_Z coming from interchanging the two roots of the quadratic equation (6).

The form of (6) makes clear that the argument depends at some level on the fact that the fibres of the rational map to weighted $\mathbb{P}^2$ given by $x_{i_0}, x_{i_1}, x_{i_2}$ are birationally elliptic curves with a section; we first deduced the existence of a birational involution in terms of these fibres. However, on X itself, these curves are quite singular at P_ξ: you can get the flavour by trying to understand why a general curve $E_{25} \subset \mathbb{P}(1, 7, 9)$ is birationally an elliptic curve with a section (the basic reason is that its Newton polygon has the shape of a Weierstrass cubic, compare (7)). Our anticanonical treatment is biregular, and seems to us to be easier and more convincing.

The reader not sharing our addiction to the luscious details of a delicious calculation may fast forward to the next chapter. *Let us play:*

Example 4.11 No. 61, $X_{25} \subset \mathbb{P}(1, 4, 5, 7, 9)$, centre $P_3 = \frac{1}{7}(1, 5, 2)$

Write x, y, z, t, u for coordinates on $\mathbb{P}$, so that only t is nonzero at $P = P_3$. The tangent monomial at P is necessarily yt^3, and P is of type $\frac{1}{7}(1, 5, 2)$.

However, t also appears in the tangent monomial tu^2 at the point $Q = P_4 = (0,0,0,0,1)$ of type $\frac{1}{9}(1,4,5)$.

Note that t is $\mathbb{Z}/7$-invariant and nonzero at P and in effect, we set $t = 1$ there. The Kawamata blowup $Y \to X$ of P is the $(1,5,2)$-weighted blowup, and locally at P it is the graph of the rational map to $\mathbb{P}(1,5,2)$ given by $(x : z : u/t)$; let $E = \mathbb{P}(1,5,2)$ be the exceptional surface, so that $-K_Y = B = A - \frac{1}{7}E$. By Proposition 3.4.6, (1), since $x, z, u/t$ are the local coordinates for the Kawamata blowup, $u \in H^0(X, \mathcal{O}_X(9A))$ only vanishes $\frac{2}{7}$ times at E, so that

$$u \in H^0(Y, 9A - \frac{2}{7}E) = H^0(Y, 9B + E), \quad \text{but} \quad u \notin H^0(Y, 9B);$$

that is, u is *not* anticanonical. On the other hand,

$$x \in H^0(Y, B), \quad z \in H^0(Y, 5B),$$

and we get $y \in H^0(Y, 4B)$ from the form of the equation $F = yt^3 + \cdots = 0$. To complete the calculation of the anticanonical ring of Y, we exhibit two further elements v, w of degree $18, 27$ satisfying a relation of the form $w^2 = v^3 + \cdots$ (7.2 explains how numbers such as $18, 27$ can readily be predicted in this kind of context). For v, we have to find an element in $H^0(X, 18A)$ with divisor of zeros $\geq \frac{18}{7}E$.

Start by viewing the equation of X as a quadratic in u, obtaining $F = tu^2 + \cdots$; any other terms divisible by tu in F can be killed by a change of variables $u \mapsto u' = u + t\alpha$. Thus we can assume that the coefficient of u does not involve t, so that F is of the form

$$\begin{aligned} F = {} & tu^2 + a_{16}u \\ & - yt^3 - b_{11}t^2 - c_{18}t + d_{25} \end{aligned} \qquad \text{with } a, b, c, d \in k[x, y, z]. \tag{7}$$

Note that X quasismooth implies that a, d don't have any common factor, that is, $(a = d = 0) \subset \mathbb{P}(1,4,5)$ is a finite set (20 points in general).

Filtering off the terms divisible by t, we define

$$v = u^2 - yt^2 - b_{11}t - c_{18}, \quad \text{so that} \quad F = vt + a_{16}u + d_{25}. \tag{8}$$

We claim that the divisor of zeros of v on Y is $\geq \frac{18}{7}E$, so $v \in H^0(Y, 18B)$. The point is that the final form of the relation $F = 0$ in (8) allows us to write $v = -(a_{16}u + d_{25})/t$, with t a unit at P; now all terms in d_{25} have divisor of zeros $\geq \frac{25}{7}E$, all terms in a_{16} have divisor of zero $\geq \frac{16}{7}E$, and u has divisor of zeros $\frac{2}{7}E$.

Next, we multiply the final form of F by u, and substitute for u^2 in terms of v. We get

$$\begin{aligned} uF &= uvt + a_{16}u^2 + d_{25}u \\ &= uvt + a_{16}(v + yt^2 + b_{11}t + c_{18}) + d_{25}u. \end{aligned}$$

We again filter off the terms divisible by t, and set

$$w = vu + a_{16}yt + a_{16}b_{11}, \tag{9}$$

so that

$$uF = wt + a_{16}(v + c_{18}) + d_{25}u. \tag{10}$$

As before, we use the final relation to write $w = -(a_{16}(v + c_{18}) + d_{25}u)/t$, with t a unit at P; all terms in $a_{16}(v + c_{18})$ have divisor of zeros $\geq \frac{34}{7}E$, and all the terms in $d_{25}u$ have divisor of zeros $\geq \frac{27}{7}E$, so that $w \in H^0(Y, 27B)$.

Notice that

$$\begin{aligned} -uw + (yt+b)F &= -u^2v - auyt - abu + (yt+b)f \\ &= -(v + yt^2 + bt + c)v - auyt - abu + (yt+b)(vt + au + d), \end{aligned}$$

and cancelling gives

$$(yt+b)F = wu + dyt - v(v+c) + db. \tag{11}$$

In order to eliminate t, u in favour of v, w, note that we can view (8–11) as inhomogeneous linear relations in t, u with coefficients in $k[x, y, z, v, w]$:

$$\begin{array}{rrccccccc}
(8) & F = & a_{16}u & + & vt & + & d_{25} & = & 0, \\
(10) & uF = & d_{25}u & + & wt & + & a_{16}(v + c_{18}) & = & 0, \\
(9) & \text{definition of } w: & vu & + & a_{16}yt & + & -w + a_{16}b_{11} & = & 0, \\
(11) & (yt+b)F = & wu & + & dyt & + & -v(v+c) + db & = & 0.
\end{array}$$

That is,

$$\begin{pmatrix} a & v & d \\ d & w & a(v+c) \\ v & ay & -w + ab \\ w & dy & -v(v+c) + db \end{pmatrix} \begin{pmatrix} u \\ t \\ 1 \end{pmatrix} = 0. \tag{12}$$

The equation relating v and w is

$$w^2 - abw = v^3 + v^2c - (bd + a^2y)v + (-a^2c + d^2)y. \tag{13}$$

It can be derived from (12) in several different ways: by premultiplying the matrix by $(dy, -ay, w, -v)$, or by calculating any of its 3×3 minors.

Write $Z_{54} \subset \mathbb{P}(1, 4, 5, 18, 27)$ for the hypersurface given by (13), and $S \subset Z_{54}$ for the locus

$$S : \operatorname{rank} \begin{pmatrix} v & w & ay & dy \\ a & d & v & w \end{pmatrix} \leq 1. \tag{14}$$

Solving (12) clearly gives t, u as rational functions on Z, regular outside S. This shows that the correspondence $(t, u) \mapsto (v, w)$ is birational.

Write $r = v/a = w/d =$ etc. for the common ratio. It is a rational section of $\mathcal{O}_S(2)$, regular outside $a = d = v = w = 0$ (a finite set). Also, $r^2 = y$. Thus the normalisation of S is $\mathbb{P}(1,5,2)$ with coordinates x, z, r, and S is the image of $\mathbb{P}(1,5,2)$ under the birational inclusion morphism $\mathbb{P}(1,5,2) \to \mathbb{P}(1,4,5,18,27)$ defined by

$$(x, z, r) \mapsto (x, y = r^2, z, v = ar, w = dr).$$

Here $\mathbb{P}(1,5,2)$ is the exceptional surface of the blowup $Y \to X$. The map $\mathbb{P}(1,4,5,7,9) \dashrightarrow \mathbb{P}(1,4,5,18,27)$ is a "projection" map, that takes $\mathbb{P}(1,5,2)$ to the nonnormal surface S. It identifies the pairs of points with $\pm r$ in the set $a(x, z, r^2) = d(x, z, r^2) = 0$ (there are 20 pairs).

The equations

$$(a = d = v = 0) \subset X_{25} \subset \mathbb{P}(1,4,5,7,9)$$

(where, as in (8), $v = u^2 - yt^2 - bt - c$) define a finite number of curves through P (in general 20) with birational transforms contracted by $Y \to Z_{54}$. For general X, we prove that these are the *only* contracted curves; as explained in 4.1, they are then exactly the flopping curves of a birational involution of X. For t, u are regular outside S by (12); moreover, (8) and (9) show that they are also regular, except possibly at points of S where $y = b = v = 0$, $u^2 = c$ and $au + d = 0$. But $(y = b = 0, d^2 = a^2c)$ defines the empty set in $\mathbb{P}(1,4,5)$ in general.

Example 4.12 No. 7, $X_8 \subset \mathbb{P}(1,1,2,2,3)$, centre $P \in P_2P_3 = 4 \times \frac{1}{2}(1,1,1)$

This case is very similar to Example 4.11, except for a couple of preliminary skirmishes. We choose coordinates x_0, x_1, y_1, y_2, z such that $P = P_3 = (0,0,0,1,0)$; the tangent monomial at P must be $y_1y_2^3$. The equation of the tangent plane at P_4 is $(\alpha y_1 + \beta y_2)z^2$. The general case is $\beta \neq 0$, and then we can change coordinates, to arrange that the tangent monomial at P_4 is y_2z^2. Thus we write $*y_2z^2 - y_1y_2^3$ in the Big Table, where, as before, the star reminds us that the presence of the monomial involves an extra generality assumption. Assuming y_2z^2 is present, any other multiple of y_2z in F can be killed by a coordinate change $z \mapsto z + e_3(x_0, x_1, y_1)$, so that

$$\begin{aligned} F = y_2z^2 &+ a_5z \\ &- y_1y_2^3 - b_4y_2^2 - c_6y_2 + d_8 \end{aligned} \qquad \text{with } a, b, c, d \in k[x_0, x_1, y_1].$$

From that point onwards, the calculation is identical to that of Example 4.11 on making the substitution

$$y \mapsto y_1, \quad t \mapsto y_2, \quad u \mapsto z.$$

We get $X_8 \dashrightarrow Z_{18} \subset \mathbb{P}(1,1,2,6,9)$.

Theorem 4.13 *In the notation of Section 4.10, let* $X_d \subset \mathbb{P}(1, a_1, a_2, a_3, a_4)$ *be a general hypersurface defined by*

$$F_d = \xi\zeta^2 + a\zeta - x_{i_2}\xi^3 - b\xi^2 - c\xi + d,$$

where $x_0, x_{i_1}, x_{i_2}, \xi, \zeta$ *are the* 5 *coordinates (in some order), and* $a, b, c, d \in k[x_0, x_{i_1}, x_{i_2}]$. *Then* P_ξ *is a quotient singularity of type* $\frac{1}{r}(1, a_{i_1}, r - a_{i_1})$, *where* $r = \deg \xi$, *having local coordinates* x_{i_0}, x_{i_1}, ζ *(in the sense explained in Proposition 3.4.6, (2)), and* $\deg \zeta = 2r - a_{i_1}$.

Let $Y \to X$ *be the Kawamata blowup of* P_ξ. *Then the anticanonical ring of* Y *is of the form* $R(Y, B) = k[x_0, x_{i_1}, x_{i_2}, v, w]/(G)$, *where,*

$$v = \zeta^2 - x_{i_2}\xi^2 - b\xi - c, \quad w = v\zeta + ax_{i_2}\xi + ab, \quad \text{and}$$
$$G = -w^2 + abw + v^3 + v^2c - (bd + a^2y)v + (-a^2c + d^2)y.$$

Write $Z = \operatorname{Proj} R(Y, B)$; *then* $Y \to Z$ *is a morphism and contracts finitely many curves in* Y *obtained as the birational transform of* $(a = d = v = 0)$. *The composite* $X \dashrightarrow Z$ *gives rise to an involution as described in 4.1.*

This construction provides all the elliptic involutions marked E.I. in the Big Table.

Proof Write $r = \deg \xi$ and $s = \deg \zeta$. Then

$$d = a_{i_1} + a_{i_2} + r + s = r + 2s = a_{i_2} + 3r,$$

because $d = \sum a_i$ (for all our hypersurfaces), and because of the monomials $\xi\zeta^2, x_{i_2}\xi^3 \in F_d$. This gives $s = a_{i_1} + a_{i_2} = 2r - a_{i_1}$. The rest of the proof is the same as the calculations in Example 4.11, and we omit it. Q.E.D.

5 Excluding

This chapter proves the following theorem, which is just Theorem 3.2 chopped up into cases for ease of digestion (see Definition 3.1.2 for weak and strong maximal singularity):

Theorem 5.1 *Let* X *be a general hypersurface of our list.*

5.1.1 *No curve* $\Gamma \subset X$ *can be a strong maximal centre.*

5.1.2 *No nonsingular point* $P \in X$ *can be a weak maximal centre.*

5.1.3 *No singular point* $P \in X$ *with* $B^3 \le 0$ *can be a strong maximal centre (see 3.4.5 for the notation* $A = -K_X$, $B = -K_Y = A - a_E E$, *etc. for a divisorial extraction* $f\colon (E \subset Y) \to (\Gamma \subset X)$*).*

5.1.4 *Other than those untwisted in Chapter 4, no singular point* $P \in X$ *with* $B^3 > 0$ *can be a strong maximal centre.*

Remarks

(1) See 7.8 for a discussion of *general*; in any case, it always means "outside a Zariski closed subset". In particular we always assume that X is quasi-smooth. The central "exclusion" column of the Big Table of Chapter 6 contains data on the method we use to exclude each singularity.

(2) We guess that every curve on X is excluded as a weak maximal centre; but our methods do not prove this, even a posteori (except for cases when X has no links). It might happen, for example, that a linear system $\mathcal{H}$ with very high multiplicity along a curve C has a singular point $P \in C$ as a strong maximal centre, so that untwisting P flops C and dilutes the singularity along it.

(3) In our treatment, we distinguish between singular points $P \in X$ with $B^3 \leq 0$ and $B^3 > 0$, as explained in 5.4–5.5 (see especially Proposition 5.4.1). In the first case, we have the neat statement 5.1.3. Singular points with $B^3 > 0$ fall into two groups: some are untwisted by the quadratic or elliptic involutions described in Chapter 4, and are therefore strong maximal centres. The remainder are excluded by 5.1.4.

The proof divides into 6 sections:

Sections 5.2–5.5 make up the technical and conceptual core of the chapter, introducing the different methods for excluding: the test class method and variations on it, the $\Gamma = S \cap T$ method of determining $\overline{\mathrm{NE}}\,Y$, and the bad link method. At the same time, they prove Theorems 5.1.1–5.1.4, modulo some bulky calculations that we relegate to 5.6 and 5.7 to avoid interrupting the flow of ideas. As in the preceding chapter, rather than reproducing here all of the (several hundred) calculations involved, which would take a disproportionate amount of space, we introduce all the methods and work out detailed examples of each; the numerical data on each case is summarised in the Big Table.

In 7.6 we discuss and compare the different methods of exclusion, and how they are used. As we explain in the final section 7.9, it seems doubtful that a single method could cover all the cases we exclude.

5.2 The test class method, and curves

In this section we introduce the test class method in its pure form and use it to prove that no curve on X is a maximal centre (Theorem 5.1.1). The method is based on the following lemma and its corollary.

Lemma 5.2.1 *Let $\Gamma \subset X$ be a strong maximal centre, and $f\colon (E \subset Y) \to (\Gamma \subset X)$ a maximal extraction. Then the 1-dimensional cycle $B^2 \in N_1 Y$ lies in the interior of the Mori cone of Y:*

$$B^2 \in \operatorname{Int} \overline{\mathrm{NE}}\, Y$$

(in fact $\overline{\mathrm{NE}}\, Y = \mathrm{NE}\, Y$).

Proof The assumption means that there is a linear system $\mathcal{H} \subset |nA|$ with multiplicity m_E along E large compared with the discrepancy of E:

$$m_E(\mathcal{H}) > na_E(K_X).$$

We set $m = m_E(\mathcal{H})$ and $a = a_E(K_X)$. Let $\mathcal{H}' = f_*^{-1}\mathcal{H}$ be the birational transform, and write

$$\begin{aligned} B &= \frac{m - na}{m} A + \frac{a}{m} \mathcal{H}' \\ &= \alpha A + \beta \mathcal{H}' \quad \text{with } \alpha, \beta > 0. \end{aligned}$$

Squaring both sides gives:

$$B^2 = \alpha^2 A^2 + 2\alpha\beta A \cdot \mathcal{H}' + \beta^2 \mathcal{H}'^2 \quad \text{with } \alpha^2, \alpha\beta, \beta^2 > 0.$$

Now $\operatorname{rank} N^1 Y = 2$, and the classes A^2, $A \cdot \mathcal{H}'$ and $\mathcal{H}'^2$ all lie in $\overline{\mathrm{NE}}\, Y$, and are not all proportional, because $A - \varepsilon E$ is ample on Y for some ε. Thus any strictly positive combination of A^2, $A \cdot \mathcal{H}'$ and $\mathcal{H}'^2$ is in $\operatorname{Int} \overline{\mathrm{NE}}\, Y$, and hence $B^2 \in \operatorname{Int} \overline{\mathrm{NE}}\, Y$. Q.E.D.

Definition 5.2.2 Let $f\colon (E \subset Y) \to (\Gamma \subset X)$ be an extremal divisorial contraction in the Mori category. A *test class* is a nonzero nef class $M \in N^1 Y$.

Corollary 5.2.3 *Let $f\colon (E \subset Y) \to (\Gamma \subset X)$ be an extremal divisorial contraction in the Mori category. If $M \cdot B^2 \leq 0$ for some test class M, then E is not a maximal singularity.*

Proof The contradiction is obvious: if $M \geq 0$ on $\overline{\mathrm{NE}}$ and $M \cdot B^2 \leq 0$ then $B^2 \notin \operatorname{Int} \overline{\mathrm{NE}}$. Q.E.D.

We now explain the simplest and most common way of constructing a test class.

Definition 5.2.4 Let L be a Weil divisor class on a variety X and $\Gamma \subset X$ an irreducible subvariety of codimension ≥ 2. For an integer $s > 0$, consider the linear system

$$\mathcal{L}_\Gamma^s = |\mathcal{I}_\Gamma^s(sL)|,$$

where $\mathcal{I}_\Gamma$ is the ideal sheaf of Γ. We say that the class L *isolates* Γ, or is Γ-*isolating*, if

(a) $\Gamma \subset \operatorname{Bs} \mathcal{L}_\Gamma^s$ is an isolated component for some positive integer s; in other words, in a neighbourhood of Γ, the base locus of $\mathcal{L}_\Gamma^s$ is contained in Γ (as a set). See Lemma 5.6.4 for the point of the argument at which we sometimes need to use $s > 1$.

(b) In the case Γ is a curve, the generic point of Γ appears in $\operatorname{Bs} \mathcal{L}_\Gamma^s$ with multiplicity 1.

Lemma 5.2.5 *Suppose that L isolates $\Gamma \subset X$, and let s be as above. Then, for any extremal divisorial contraction*

$$f\colon (E \subset Y) \to (\Gamma \subset X)$$

the birational transform $M = f_*^{-1} \mathcal{L}_\Gamma^s$ is a test class on Y.

Proof First, $M = f^*sL - mE$ for some $m > 0$, and is obviously nonzero; we must prove that M is nef, that is, $MC \geq 0$ for every irreducible curve $C \subset Y$. There are 4 cases:

(1) C is contracted by f. Then $Cf^*L = 0$ and $C(-E) > 0$, so $CM > 0$.

(2) C is disjoint from E. Then $CM = Cf^*L = f(C)L > 0$.

(3) C meets E but is not contained in E. Then by assumption, $f(C)$ is not contained in the base locus of $\mathcal{L}_\Gamma^s$, so that C is not contained in the base locus of $|M|$. Therefore $CM \geq 0$.

(4) C is contained in E but not contracted. Then by Definition 5.2.4, (ii), C is not in the base locus of $|M|$, hence $C \cdot M > 0$. Q.E.D.

The test class method (pure form)

In a nutshell, the method is this: start with an extremal divisorial contraction $(E \subset Y) \to (\Gamma \subset X)$. To exclude E as a maximal singularity, we find a test class M on Y with $M \cdot B^2 \leq 0$, by constructing a Γ-isolating system on X or otherwise, and conclude by Corollary 5.2.3.

We use the method in this section to exclude curves, and in 5.4 to exclude some of the singular points. What these two cases have in common is that the centre has a unique extremal extraction, so that by excluding E (on that biregular model), we exclude the point or curve as a strong maximal centre. The method is not applicable as such to a nonsingular point $P \in \operatorname{NonSing} X$, because the extremal extractions from P are not yet completely classified (there are infinitely many of them – for a discussion, see [Co2], Section 3.3, which also contains a plausible conjecture as to their classification). The following section 5.3 develops two variations on this theme, which are used to exclude all nonsingular points as maximal centres.

Proof of Theorem 5.1.1: No curve on X is a strong maximal centre
The argument follows closely the original approach of Iskovskikh and Manin [IM], and is in two steps. The first uses a crude numerical argument to exclude the great majority of cases. In the second step, we use the test class method to exclude all the remaining cases.

STEP 1 Let $\Gamma \subset X$ be an irreducible curve. If Γ is a strong maximal centre, there is a linear system $\mathcal{H} \subset |nA|$ and a maximal extraction

$$f\colon (E \subset Y) \to (\Gamma \subset X)$$

centred in Γ whose exceptional divisor E has multiplicity $m > n$. In particular, f is an extremal extraction, and $\Gamma \subset \operatorname{NonSing} X$ by Corollary 3.4.3, hence its anticanonical degree is a strictly positive integer:

$$d = \deg \Gamma = A \cdot \Gamma \in \mathbb{N}.$$

Let s be large enough so that $|sA|$ is a very ample Cartier divisor, $S \in |sA|$ a general member, and $H_1, H_2 \in \mathcal{H}$ general members. Then

$$A^3 sn^2 = S \cdot H_1 \cdot H_2 \geq sm^2 \deg \Gamma > sn^2 \deg \Gamma,$$

so

$$A^3 > \deg \Gamma.$$

In particular, $A^3 > 1$, and a quick inspection of the list then reveals that X and $\deg \Gamma$ must be one of the following:

(a) $X_4 \subset \mathbb{P}^4$, $\deg \Gamma = 1, 2$ or 3;

(b) $X_5 \subset \mathbb{P}(1,1,1,1,2)$, $\deg \Gamma = 1$ or 2;

(c) $X_6 \subset \mathbb{P}(1,1,1,1,3)$, $\deg \Gamma = 1$;

(d) $X_6 \subset \mathbb{P}(1,1,1,2,2)$, $\deg \Gamma = 1$;

(e) $X_7 \subset \mathbb{P}(1,1,1,2,3)$, $\deg \Gamma = 1$.

STEP 2 In each of the remaining cases (a–e), we construct a test class M with $M \cdot B^2 \leq 0$. By Corollary 5.2.3, this contradicts the assumption that E is a maximal singularity and thus completes the proof. The method is the same in all cases, so we only do Case (b) as an illustration, leaving the others to the reader.

Consider thus Case (b). If $\deg \Gamma = 1$, we can choose coordinates so that $\Gamma : (y = x_0 = x_1 = 0) \subset \mathbb{P}$. The blowup $f\colon (E \subset Y) \to (\Gamma \subset X)$ of the ideal sheaf of Γ is the only extremal extraction centred at Γ and, by what we just said, $2A$ is a Γ-isolating class and

$$M = f_*^{-1}|\mathcal{I}_\Gamma(2A)| = 2A - E$$

is a test class. An easy calculation, using $A^3 = \frac{5}{2}$, $A^2 \cdot E = 0$ (projection formula), $A \cdot E^2 = -\deg \Gamma = -1$ (also by the projection formula) and $E^3 = -\deg N_\Gamma X = -\deg \Gamma + 2 - 2p_a(\Gamma) = 1$, shows that

$$\begin{aligned} M \cdot B^2 &= (2A - E)(A - E)^2 \\ &= 2A^3 - (4+1)A^2 \cdot E + (2+2)A \cdot E^2 - E^3 = 5 - 5 \leq 0, \end{aligned}$$

and Γ is not a maximal centre by Corollary 5.2.3.

If $\deg \Gamma = 2$, then Γ is given by either $z = x_3 = x_0x_1 + x_2^2 = 0$ or $z^2 + a_4(x_0, x_1) = x_3 = x_2 = 0$. In the first case, $M = 2A - E$ is a test class, in the second case $M = 4A - E$ is a test class. In either case, $M \cdot B^2 < 0$ (we leave the numerics as an exercise to the reader) and, as above, Γ cannot be a maximal centre. Q.E.D.

5.3 Two variations on the test class method, and nonsingular points

This section introduces two technical results, Theorems 5.3.2 and 5.3.3, that are variations on the test class method, and applies them to prove Theorem 5.1.2. In simple-minded terms, the idea is to reduce to an intersection calculation on a surface to derive a strong numerical conclusion from a maximal singularity at a point $P \in \operatorname{NonSing} X$, without any reference to extremal extractions $Y \to X$; but in setting up this calculation, we make knowledgeable use of adjunction theoretic ideas to guide us to the most powerful conclusion.

Proof of Theorem 5.1.2: No nonsingular point on X is a maximal centre The proof involves a single exceptional case. Here the variety is No. 2, $X_5 \subset \mathbb{P}(1,1,1,1,2)$ studied in Example 4.5; in the notation given there, there is a quadratic involution i_{P_4} that untwists P_4 by blowing it up, then flopping the birational transform of the lines L_i (15 in general) of X_5 of degree $AL_i = \frac{1}{2}$ defined by $x_0 = a_3 = b_5 = 0$. We label this case the *renegade* for future reference:

$$\text{No. 2, } X_5 \subset \mathbb{P}(1,1,1,1,2), \text{ centre } P \in L_i \setminus P_4 \tag{$*$}$$

It is thus convenient to divide what we have to prove into

(A) *If X is general then, except possibly for the renegade case $(*)$, no nonsingular point on X can be a weak maximal centre.*

(B) *No nonsingular point in case $(*)$ can be a weak maximal centre.*

The bulk of the proof of (A) is contained in the following statement, the existence of a suitable P-isolating class on X (Definition 5.2.4). Its proof is a lengthy but mechanical calculation (not without its moment of excitement) that we postpone until 5.6 to avoid interrupting the flow of ideas.

Theorem 5.3.1 *Suppose that $X = X_d \subset \mathbb{P} = \mathbb{P}(1, a_1, a_2, a_3, a_4)$ is general, and let $P \in \operatorname{NonSing} X$ be a nonsingular point. Then except in the renegade case $(*)$, lA is a P-isolating class on X (Definition 5.2.4) for some $l \le 4/A^3$.*

Proof See 5.6. □

The remainder of this section carries out the additional work needed to deduce Theorem 5.1.2 from Theorem 5.3.1. We need some preparations. To understand these, to say that $P \in X$ is a weak maximal centre means by definition that there is a linear system $\mathcal{H} \subset |nA|$ and $P = C(X, v_E)$ is the centre of a valuation with multiplicity

$$m_E(\mathcal{H}) > n a_E(K_X).$$

Another way to say this is that

$$K_X + \frac{1}{n}\mathcal{H}$$

is not canonical at P (Definition 3.1.1). If $S \subset X$ is a surface passing through P then $K_X + S + \frac{1}{n}\mathcal{H}$ is not log canonical (we explain this below). This is the starting point for the following result.

Theorem 5.3.2 *Let $P \in X$ be the germ of a nonsingular 3-fold, and $\mathcal{H}$ a mobile linear system on X. Assume that*

$$K_X + \frac{1}{n}\mathcal{H}$$

is not canonical at P.

(1) If $P \in S \subset X$ is a surface and $\mathcal{L} = \mathcal{H}_{|S}$, then

$$K_S + \frac{1}{n}\mathcal{L}$$

is not log canonical.

(2) If $Z = H_1 \cap H_2$ is the intersection of two general members H_1, H_2 of $\mathcal{H}$, then

$$\operatorname{mult}_P Z > 4n^2.$$

Proof This is a special case of [Co2], Corollary 3.4. Here (2) follows from (1) and the next Theorem 5.3.3 (the proof is spelled out in [Co2], Section 3.2). We can prove (1) in a single formal sentence by saying that it is an immediate corollary of [FA], 17.7 and 17.4 (results of Shokurov, reworked by Kollár).

Since this is a key technical point, at which the literature is not excessively friendly, we digress briefly to expand on the proof of (1), for the benefit of the reader and two of the coauthors. What we're doing is a special case of Shokurov's *inversion of adjunction.* We want to prove

$$K_X + \frac{1}{n}\mathcal{H} \text{ not canonical at } P \implies K_S + \frac{1}{n}\mathcal{L} \text{ not log canonical.}$$

The obvious approach makes the result seem trivial: just blow up and count discrepancies. Let $(\widetilde{S} \subset \widetilde{X}) \to (S \subset X)$ be a resolution with exceptional locus having simple normal crossings, and $\widetilde{\mathcal{H}}$ free. Then

$$\begin{aligned} K_{\tilde{X}} &= K_X + \sum a_i E_i \\ \widetilde{S} &= S - \sum c_i E_i \\ \widetilde{\mathcal{H}} &= \mathcal{H} - \sum m_i E_i. \end{aligned}$$

Now $K_X + S + \frac{1}{n}\mathcal{H}$ not canonical at P means that $a_i - \frac{1}{n}m_i < 0$ for some E_i over P, and then $a_i - c_i - \frac{1}{n}m_i < -1$, because S is a Cartier divisor through P, so that $c_i \in \mathbb{Z}$, $c_i > 0$. This proves that $K_X + S + \frac{1}{n}\mathcal{H}$ is not log canonical at P. Restricting $K_X + S$ gives $S = K_S$ and $K_{\tilde{X}} + \widetilde{S}$ gives $K_{\tilde{S}}$; now if $E_i \cap \widetilde{S} \neq \emptyset$, it is an exceptional divisor on $\widetilde{S}$ at which $K_S + \frac{1}{n}\mathcal{H}$ has discrepancy < -1, which gives the required implication.

The trouble is that E_i could be disjoint from the birational transform $\widetilde{S}$. This is a traditional difficulty in all discrepancy arguments of this type (compare [C3-f], Theorem 2.6, II and [YPG], Remark 6.2, ii). However, we easily get round it by the *connectedness theorem* of Shokurov and Kollár ([FA], 17.4): the union of $\widetilde{S}$ and the E_i with $a_i - c_i - \frac{1}{n}m_i < -1$ is connected. Q.E.D.

The inequality for the multiplicity of Z in Theorem 5.3.2, (2) is enough to prove (A). The next result is used in [Co2], Section 3.2 for the proof of (1) implies (2), and we use it in the proof of (B) below.

Theorem 5.3.3 *Let $P \in \Delta_1 + \Delta_2 \subset S$ be the analytic germ of a normal crossing curve on a nonsingular surface (so that $\Delta_1 + \Delta_2 \subset S$ is isomorphic to the coordinate axes $(xy = 0) \subset \mathbb{C}^2$).*

Let $\mathcal{L}$ be a mobile linear system on S, and write $(\mathcal{L}^2)_P$ for the local intersection multiplicity $(L_1 \cdot L_2)_P$ of two general members $L_1, L_2 \in \mathcal{L}$ at P. Fix rational numbers $a_1, a_2 \geq 0$, and assume that

$$K_S + (1 - a_1)\Delta_1 + (1 - a_2)\Delta_2 + \frac{1}{n}\mathcal{L}$$

is not log canonical at P. Then

(1) If either $a_1 \leq 1$ or $a_2 \leq 1$, then

$$(\mathcal{L}^2)_P > 4a_1a_2n^2.$$

(2) If both $a_i > 1$, then

$$(\mathcal{L}^2)_P > 4(a_1 + a_2 - 1)n^2.$$

Proof Although the statement is fine-tuned for the most powerful result, the proof is quite elementary, and makes a worthwhile exercise. See [Co2], Section 3.1, Theorem 3.1 for the solution. □

We are now ready to prove (A) and (B) of 5.3.

Proof of (A) Let $P \in X$ be a nonsingular point not in the renegade case $(*)$. To say that $P \in X$ is a weak maximal centre means that $P = C(X, v_E)$ is the centre of a valuation, and there is a linear system $\mathcal{H} \subset |nA|$ with multiplicity

$$m_E(\mathcal{H}) > na_E(K_X).$$

Letting lA be the P-isolating class of Theorem 5.3.1, we use a general $S \in |\mathcal{I}_P^s(slA)|$ as a "test surface". Since $\operatorname{mult}_P S \geq s$, using Theorem 5.3.2, (2) gives

$$4sn^2 < (S \cdot H_1 \cdot H_2)_P.$$

On the other hand $l \leq 4/A^3$ by Theorem 5.3.1, so that

$$S \cdot H_1 \cdot H_2 = sln^2A^3 \leq 4sn^2.$$

Combining the 2 previous inequalities, we have a contradiction. Q.E.D.

Proof of (B) Assume that $P \in C \subset X_5 \subset \mathbb{P}(1,1,1,1,2)$, where $C = L_i$ is one of the lines of degree $\frac{1}{2}$ (see Example 4.5). Although $|\mathcal{I}_P(A)|$ does not isolate P (it has base locus C), we still choose a general surface

$$S \in |\mathcal{I}_P(A)|,$$

to use as a "test surface". Restricting $\mathcal{H}$ to S we obtain

$$\mathcal{H}_{|S} = cC + \mathcal{L},$$

where $\mathcal{L}$ is a mobile linear system on S and $c = \operatorname{mult}_C \mathcal{H} \geq 0$ the multiplicity of $\mathcal{H}$ along C.

The argument now proceeds much as before in 3 straightforward steps: (a) we show that two general members $L_1, L_2 \in \mathcal{L}$ have a large intersection number at P; (b) we estimate $\mathcal{L}^2 = L_1 \cdot L_2$ from above; (c) we derive a contradiction from the 2 previous steps.

Step A First, S is nonsingular at P; indeed if $F = x_0y^2 + f_3y + f_5$ is quasismooth at P the projection of P to $\mathbb{P}^3$ must be a nonsingular point of $(f_5 = 0)$. We now assume that P is a maximal centre, and apply Theorem 5.3.3 to $\Delta_1 = C$, $1 - a_1 = \frac{c}{n}$ and $1 - a_2 = 0$. Then Theorem 5.3.2, (1) implies that $K_S + \frac{c}{n}C + \frac{1}{n}\mathcal{L}$ is not log canonical and by Theorem 5.3.3,

$$(L_1 \cdot L_2)_P > 4\Big(1 - \frac{c}{n}\Big)n^2.$$

Step B S has an ordinary double point at $P_4 = (0,0,0,0,1)$ and

$$(K_S \cdot C)_S = (K_X + S \cdot C)_X = 0,$$

so that

$$(K_S + C)_{|C} = K_C + \operatorname{Diff} = K_C + \frac{1}{2}P_4,$$

and

$$(C \cdot C)_S = -2 + \frac{1}{2} = -\frac{3}{2}.$$

From this, we calculate

$$L_1 \cdot L_2 = (nA_{|S} - cC)_S^2 = \frac{5}{2}n^2 - nc - \frac{3}{2}c^2.$$

STEP C From the 2 previous steps, since $\mathcal{L}$ is mobile, we conclude that

$$4\Big(1 - \frac{c}{n}\Big)n^2 < (L_1 \cdot L_2)_P \le L_1 \cdot L_2 = \frac{5}{2}n^2 - nc - \frac{3}{2}c^2,$$

which gives $\frac{3}{2}(n-c)^2 < 0$, a contradiction. Q.E.D.

5.4 Determination of $\overline{\mathrm{NE}}\,Y$, and singular points with $B^3 \le 0$

In Sections 5.4–5.5, we show how to exclude *singular* points $P \in X$ as strong maximal centres. Recall from 3.4 that we write $f\colon (E \subset Y) \to (P \in X)$ for the unique extremal divisorial contraction to P, and $B = -K_Y$. We begin with a short summary of the contents, then, before getting down to business, we make a few philosophical remarks. There are 3 parts:

This section 5.4 introduces a new idea, the $Q = S \cap T$ method, that determines the Mori cone $\overline{\mathrm{NE}}\,Y$ in all cases, again modulo some bulky calculations that we relegate to 5.7. In 5.4.1, we use the exact shape of $\overline{\mathrm{NE}}\,Y = R + Q$ to construct test classes that exclude all the points with $B^3 \le 0$, with the single maverick exception $(**)$ of Lemma 5.4.3. In 5.5, we introduce the bad link method, and exclude points with $B^3 > 0$ and the maverick $(**)$. Together, the results give our main Theorems 5.1.3 and 5.1.4 and complete the proof of Theorem 1.3.

It is natural to try to use the test class method to exclude a centre whenever possible. We first observe that this can only work when $B^3 \le 0$.

Proposition 5.4.1 *Let $P \in X$ be a point centre and $f\colon (E \subset Y) \to (P \in X)$ an extremal divisorial contraction. If $B^3 > 0$, then $B^2 \in \operatorname{Int} \overline{\mathrm{NE}}\,Y$.* In other words, the test class method only has a fighting chance if $B^3 \le 0$.

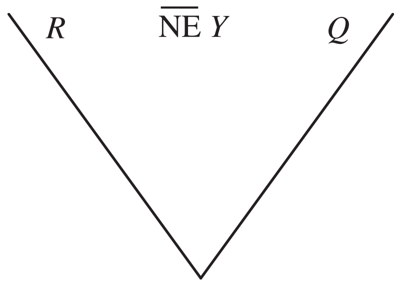

Figure 3: The Mori cone $\overline{\mathrm{NE}}\,Y$

Proof Indeed, let $R \subset \overline{\mathrm{NE}}\,Y$ be the extremal ray contracted by f, and Q the other ray (see Figure 3). If $B \cdot Q > 0$ then B is ample (this never actually happens) and we are done; thus we assume that $B \cdot Q \leq 0$.

Choosing $\Delta \in R$ and $\Gamma \in Q$, we write the 1-dimensional cycle B^2 as

$$B^2 = \alpha\Delta + \beta\Gamma;$$

we now show that both α and β are > 0. First

$$0 < A^3 = A \cdot B^2 = \alpha A \cdot \Delta + \beta A \cdot \Gamma = \beta A \cdot \Gamma,$$

and, since $A \cdot \Gamma > 0$, we get $\beta > 0$. Then

$$0 < B^3 = \alpha B \cdot \Delta + \beta B \cdot \Gamma \leq \alpha B \cdot \Delta,$$

and, since $B \cdot \Delta > 0$, also $\alpha > 0$. Q.E.D.

In fact, we use the test class method in its pure form (5.2) to exclude all singular points with $B^3 \leq 0$ except for the maverick $(**)$, for which $B^2 \in \operatorname{Int} \overline{\mathrm{NE}}\,Y$ (see Remark 5.4.9 below), so that the test class method cannot be used. In many cases, it is quite easy to find a P-isolating system on X, satisfying suitable numerical conditions, in the same way as for nonsingular points (and often easier). However, such linear systems quickly become harder and harder to detect. We even found in some examples that the most efficient way to write down the test class was to start by determining $\overline{\mathrm{NE}}\,Y$ precisely. Below, we calculate $\overline{\mathrm{NE}}\,Y$ in all cases. While, in a few instances (10 in all, listed in 5.7.2), it would have been easier just to write down a convenient isolating class, our presentation has the advantages that

(1) it provides a unified exposition of the material,

(2) it allows us to write down the best possible test class in all cases,

(3) this is precisely the sort of thing that we need in the bad link method, and therefore affords a unified treatment of points with $B^3 \leq 0$ and points with $B^3 > 0$, and

(4) it is the same calculation used to present the anticanonical ring of Y in the case when an untwisting exists, so the method also ties in with the untwisting of Chapter 4.

We do not really have a particularly good explanation as to why some points are excluded and some are not. We discuss this at more length in our concluding remarks 7.7.

Determination of $\overline{\mathrm{NE}}\, Y$

Notation 5.4.2 If $P \in X = X_d \subset \mathbb{P}(1, a_1, a_2, a_3, a_4)$ is a singular point on X and $f\colon (E \subset Y) \to (P \in X)$ the extremal divisorial contraction (3.4.2), we denote by S the surface

$$S = f_*^{-1}(x_0 = 0).$$

It is always the case that $S \in |B|$ and, if X is general (which we always assume), S is also irreducible. (See the discussion in 7.2.3.)

The main results here are the next two lemmas, whose proof consists of a lengthy but mechanical calculation which we postpone until 5.7 to avoid interrupting the flow of ideas.

Lemma 5.4.3 *Assume $B^3 \leq 0$. Then, with the sole exception of the following* maverick:

$$\text{No. 82: } X_{36} \subset \mathbb{P}(1,1,5,12,18), \text{ centre } P_2 = \frac{1}{5}(1,2,3), \tag{$**$}$$

there are integers b, c with $b > 0$ and $b/r \geq c \geq 0$ and a surface $T \in |bB + cE|$ such that

(1) The intersection $\Gamma = S \cap T$ is an irreducible curve (here and below, we say $S \cap T$ is irreducible *to mean that the scheme theoretic complete intersection is a reduced and irreducible curve); and*

(2) $T \cdot \Gamma \leq 0$.

The excluding column of the Big Table lists the class $T \sim bB + cE$ in each case.

Proof See 5.7. $\square$

Definition 5.4.4 A *special surface* is an irreducible surface $T \in |bB + cE|$, where $b > 0$ and $c < 0$ are integers.

Lemma 5.4.5 *As usual, let $P \in X$ be a singular point. In all 18 cases labelled "bad link" in the Big Table (this consists of those with $B^3 > 0$, in addition to the maverick (**) of Lemma 5.4.3), there is a special surface T with the properties*

(1) The scheme theoretic intersection $\Gamma = S \cap T$ is an irreducible curve and

(2) $B \cdot \Gamma \leq 0$.

The excluding column of the Big Table lists the class $T \sim bB + cE$ in each case.

Proof See 5.7. $\square$

We now discuss some of the implications.

Corollary 5.4.6 *For $P \in \operatorname{Sing} X$, let T and $\Gamma = S \cap T$ be as in the conclusion of Lemma 5.4.3 or 5.4.5. As usual, write R for the extremal ray of $\overline{\mathrm{NE}}\, Y$ contracted by $f\colon Y \to X$. Then $Q = \mathbb{R}_+[\Gamma]$ is the other ray of $\overline{\mathrm{NE}}\, Y$, so that $\overline{\mathrm{NE}}\, Y = R + Q$. (See Figure 3.)*

Proof Clearly $\Gamma \not\subset E$ because $\Gamma \notin R$, and hence $E \cdot \Gamma \geq 0$. Then from the stated inequalities, we read off at once

$$\begin{cases} 0 \geq T \cdot \Gamma \geq bB \cdot \Gamma & \text{in the conclusion of Lemma 5.4.3, and} \\ 0 \geq bB \cdot \Gamma > T \cdot \Gamma & \text{in the conclusion of Lemma 5.4.5.} \end{cases}$$

In the case of Lemma 5.4.3, if T is nef then $T \cdot \Gamma = 0$ and Γ is in the boundary of $\overline{\mathrm{NE}}$. Otherwise there exists a curve with $T \cdot C < 0$. Now $C \not\subset E$ (because $T \cdot R \geq 0$), so that necessarily also

$$B \cdot C = \frac{1}{b}(T - cE) \cdot C < 0,$$

and perforce $C \subset S \cap T$. But $S \cap T$ is Γ, so $C = \Gamma$. The case of Lemma 5.4.5 is similar and left to the reader. Q.E.D.

Remark 5.4.7 In fact, the following result also holds, although we don't need it in what follows. Denote by $\overline{\mathrm{NM}}^1\, Y \subset N^1 Y$ the closed cone of *effective* divisors. In the case of Lemma 5.4.3, we have

$$\overline{\mathrm{NM}}^1\, Y = \mathbb{R}_+[E] + \mathbb{R}_+[B],$$

and, in the case of Lemma 5.4.5,

$$\overline{\mathrm{NM}}^1\, Y = \mathbb{R}_+[E] + \mathbb{R}_+[T].$$

Proof Let $D \sim \alpha T + \beta E \geq 0$ be an effective divisor. Assuming that S is not contained in the support of D, we show that $\alpha, \beta \geq 0$; this obviously implies the statement. Restricting to S, we have

$$D_{|S} = \alpha T_{|S} + \beta E_{|S} = \alpha\Gamma + \beta\Delta \geq 0,$$

where we write $\Delta = E_{|S}$. Note that Γ and Δ are irreducible curves on S with $\Gamma^2 \leq 0$ (by Lemmas 5.4.3 and 5.4.5) and $\Delta^2 < 0$ (because Δ is contractible).

We now prove that if $C \sim \alpha\Gamma + \beta\Delta \geq 0$ is any effective divisor on S then $\alpha, \beta \geq 0$. We may assume that neither Γ nor Δ is contained in the support of C. Note that one of α or $\beta \geq 0$, since otherwise $-C > 0$.

If $\alpha \geq 0$ then

$$0 \leq C \cdot \Gamma = (\alpha\Gamma + \beta\Delta) \cdot \Gamma = \alpha\Gamma^2 + \beta\Delta \cdot \Gamma \leq \beta\Delta \cdot \Gamma.$$

Then since $\Delta \cdot \Gamma > 0$, necessarily $\beta \geq 0$. A similar argument shows that $\beta \geq 0$ implies $\alpha \geq 0$. Q.E.D.

5.4.1 Singular points with $B^3 \leq 0$

We now prove Theorem 5.1.3 with the one exception.

Theorem 5.4.8 *Assume that $B^3 \leq 0$. Then, with the sole exception of the maverick case (**) of Lemma 5.4.3, we have $B^2 \notin \operatorname{Int} \overline{\mathrm{NE}}\, Y$.*

In particular, P is not a strong maximal centre by Lemma 5.2.1.

We stress that the maverick is of course not an exception to $P \in X$ being excluded as a maximal centre (see Theorem 5.5.1 below).

Proof Let R, Q be as before, so that $\overline{\mathrm{NE}}\, Y = R + Q$ (see Figure 3). By Corollary 5.4.6,

$$Q = \mathbb{R}_+ \big[B \cdot (bB + cE)\big]$$

for some integers $b > 0$, $c \geq 0$. This shows that $B^2 \notin \operatorname{Int} \overline{\mathrm{NE}}\, Y$. Q.E.D.

Remark 5.4.9 The test class method cannot be used in the case of the maverick (**). Indeed, we know from Lemma 5.4.5 that a "special surface" $T = f_*^{-1}(x_1 = 0) \in |B - E|$ exists such that $T \cdot B^2 < 0$, which shows that $B^2 \in \operatorname{Int} \overline{\mathrm{NE}}\, Y$.

5.5 Bad links, and singular points with $B^3 > 0$

The next result completes the proof of Theorem 5.1.3 and proves 5.1.4.

Theorem 5.5.1 *Let $P \in X$ be a singular point and $f\colon (E \subset Y) \to (P \in X)$ the Kawamata blowup. If a special surface $T \in Y$ exists as in the conclusion of Lemma 5.4.5, P is not a strong maximal centre.*

Proof Assume by contradiction that P is a strong maximal centre. We start from the Kawamata blowup $f\colon (E \subset Y) \to (P \in X)$, the unique extremal divisorial extraction centred at P (Theorem 3.4.2), and play a 2-ray game on Y from the starting position given in Corollary 5.4.6: $\overline{\mathrm{NE}}\,Y = R + Q$, where R is the ray contracted by f, and $Q = \mathbb{R}_+[\Gamma]$. (Consult [Co2], Chapter 1 for the rules and general conventions of the 2-ray game.)

By the Sarkisov program [Co1], the 2-ray game must exist and terminate in a Sarkisov link; we show that, on the contrary, the game is always aborted, that is, the "link" goes "bad". The crux of the argument is that the 2-ray game must take place entirely in the Mori category: if contracting a ray leads at any stage to a canonical but nonterminal singularity, or to a fibre space other than a Mori fibre space, the rules of the game are violated. This is a contradiction, and proves that P is not a strong maximal centre.

The analysis breaks up in 2 cases: $B \cdot \Gamma = 0$ and $B \cdot \Gamma < 0$.

Case 1: $B \cdot \Gamma = 0$ (16 instances) Let $\varphi\colon Y \to W$ be the contraction of the extremal ray $Q = \mathbb{R}_+[\Gamma]$. First, note that φ cannot be a flopping contraction: if it were, Γ would be a contractible curve in T, which is impossible, since $\Gamma = S \cap T$ is a $\mathbb{Q}$-Cartier divisor on T with selfintersection

$$(\Gamma^2)_T = B^2 \cdot T = B \cdot \Gamma = 0.$$

It follows that φ is either a divisorial contraction, with the special surface T as exceptional divisor, or is a fibre space. In either case the 2-ray game is aborted. If φ is birational, then since $K_Y \cdot \Gamma = -B \cdot \Gamma = 0$, it is crepant, that is, the exceptional divisor T has discrepancy $a(T, K_W) = 0$ over W, so that W has a canonical but nonterminal singularity. Also $K_Y \cdot \Gamma = 0$ implies that φ is not a Mori fibre space. (If $\dim W < 3$, it is a fibre space of elliptic curves or K3 surfaces.) This concludes Case 1.

Case 2: $B \cdot \Gamma < 0$ Although we don't actually need this in what follows, we note first that there are only 3 examples, namely

- No. 47, $X_{21} \subset \mathbb{P}(1,1,5,7,8)$, centre $P = P_2 = \frac{1}{5}(1,2,3)$
- No. 62, $X_{26} \subset \mathbb{P}(1,1,5,7,13)$, centre $P = P_2 = \frac{1}{5}(1,2,3)$

- No. 82, $X_{36} \subset \mathbb{P}(1,1,5,12,18)$, centre $P = P_2 = \frac{1}{5}(1,2,3)$; this is the maverick case (**) of Lemma 5.4.3 with $B^3 = 0$, excluded from Theorem 5.4.8.

Write $\varphi\colon Y \to W$ for the contraction of the extremal ray $Q = \mathbb{R}_+[\Gamma]$. Then $-K_Y \cdot \Gamma = B \cdot \Gamma < 0$ means that φ is an *inverse flipping* contraction. Set $m_1 = -S\cdot\Gamma$, $m_2 = -T\cdot\Gamma$. Then $m_1, m_2 > 0$, and, locally around Γ, we obtain the inverse flip $t\colon Y \dashrightarrow Y'$ as the normalisation of the map $t\colon Y \dashrightarrow \mathbb{P}^1 \times W$ given by the linear system $|m_1T, m_2S|$; in particular, it exists. If S', T', etc. are the birational transforms, this implies in particular that $S' \cap T' = \emptyset$. We write $-K_{Y'} = B' \sim S'$.

The structure morphism $\underline{\varphi'}\colon Y' \to W$ is the flipping contraction of an extremal ray $R' = \mathbb{R}_+[\Gamma']$ of $\overline{\mathrm{NE}}\, Y'$; let Q' be the other extremal ray. We show first that $T' \cdot Q' < 0$. The usual flip formalism together with the case assumption $B \cdot \Gamma < 0$ on Y gives $B' \cdot \Gamma' > 0$ on Y'; but B' is zero on any curves in T', so that $B' \cdot Q' \le 0$. In the same way, $E \cdot \Gamma > 0$ implies $E' \cdot \Gamma' < 0$, and hence (since an effective divisor is positive on some curves) $E' \cdot Q' > 0$. Therefore

$$T' \cdot Q' = (bB' + cE') \cdot Q' < 0$$

(recall that $c < 0$ in Lemma 5.4.5).

It follows that every curve $C \in Q'$ is contained in T'. Thus $S' \cdot Q' = 0$, and hence the contraction

$$\psi\colon Y' \to W'$$

of the ray Q' is given by $|nS'|$ for some $n \gg 0$ (anticanonical model). Because $S' \cap T' = \emptyset$, this contracts T' to a point, and does not contract any curves not in T'. Thus ψ is a divisorial contraction with exceptional divisor T', and is crepant over W', so that W' has nonterminal singularities. Q.E.D.

Remark 5.5.2 For all these bad links, we can view the 2-ray game entirely in terms of toric modifications of the ambient projective space. As such, they are basically similar to the quadratic involutions of Chapter 4, except that they lead to a contradiction rather than an untwisting. The first troop of 16 all fit into the pattern

$$\text{No. 21 } X_{14} \subset \mathbb{P}(1,1,2,4,7),\ \text{centre } P_3 = \frac{1}{4}(1,1,3)$$
$$\dashrightarrow Z_{18} \subset \mathbb{P}(1,1,2,6,9) \text{ given by } (x_0,x_1,y,z,t) \mapsto (x_0,x_1,y,yz,yt)$$

...

$$\text{No. 63 } X_{26} \subset \mathbb{P}(1,2,3,8,13),\ \text{centre } P_3 = \frac{1}{8}(1,3,5)$$
$$\dashrightarrow Z_{30} \subset \mathbb{P}(1,2,3,10,15) \text{ given by } (x,y,z,t,u) \mapsto (x,y,z,yt,yu)$$

etc. (we give a hint on how to do the calculation in 7.2), whereas the final eccentric trio involving inverse flips go thus:

No. 47, $X_{21} \subset \mathbb{P}(1,1,5,7,8)$, centre $P = P_2 = \frac{1}{5}(1,2,3)$

$\dashrightarrow Z_{24} \subset \mathbb{P}(1,1,6,8,9)$ given by $(x_0,x_1,y,z,t) \mapsto (x_0,x_1,x_1y,x_1z,x_1t)$

No. 62, $X_{26} \subset \mathbb{P}(1,1,5,7,13)$, centre $P = P_2 = \frac{1}{5}(1,2,3)$

$\dashrightarrow Z_{30} \subset \mathbb{P}(1,1,6,8,15)$ given by $(x_0,x_1,y,z,t) \mapsto (x_0,x_1,x_1y,x_1z,x_1^2t)$

No. 82, $X_{36} \subset \mathbb{P}(1,1,5,12,18)$, centre $P = P_2 = \frac{1}{5}(1,2,3)$

$\dashrightarrow Z_{42} \subset \mathbb{P}(1,1,6,14,21)$ given by $(x_0,x_1,y,z,t) \mapsto (x_0,x_1,x_1y,x_1^2z,x_1^3t)$

In each case, the final model Z has $-K_Z$ ample, but has a strictly canonical singularity, so is not a Fano 3-fold in the Mori category.

5.6 Proof of Theorem 5.3.1: construction of the test class for nonsingular points

Theorem 5.3.1 is trivial for the quartic $X_4 \subset \mathbb{P}^4$, and we exclude this case here. Fix a general Fano 3-fold hypersurface $X = X_d \subset \mathbb{P} = \mathbb{P}(1,a_1,a_2,a_3,a_4)$. This section discusses how to prove that a suitable class lA on X is P-isolating for a nonsingular point $P \in \operatorname{NonSing} X$.

Notation 5.6.1 We write $x_0,x_1,\dots,x_4$ (or $x,y,\dots$ etc.) for homogeneous coordinates on $\mathbb{P}$, and $P = (\xi_0,\xi_1,\xi_2,\xi_3,\xi_4)$ (or $P = (\xi,\eta,\dots)$) etc. for the coordinates of a point P. Let

$$\pi\colon X \dashrightarrow \mathbb{P}(1,a_1,a_2,a_3)$$

be the projection to the first 4 coordinates. Note that π is a morphism except possibly at P_4; it is generically finite, contracting at most a finite set of curves of X, that we write $\operatorname{Exc}\pi$.

Theorem 5.6.2 *With the exception of the renegade (*) of 5.3 (p. 208), we get P-isolating classes lA (Definition 5.2.4) on X as follows:*

(1) If $\xi_0 \neq 0$ then a_4A isolates P; if moreover $P \notin \operatorname{Exc}\pi$, then already a_3A isolates P.

(2) If $\xi_0 = 0$ and $\xi_1 \neq 0$ then a_1a_3A isolates P.

(3) If $\xi_0 = \xi_1 = 0$, set

$$l = \min\Bigl\{\{\mathrm{lcm}(a_i, a_j)\} \cup \{a_i + a_j \mid a_k \text{ divides } a_i + a_j\}\Bigr\},$$

(taken over all $\{i, j, k\} = \{2, 3, 4\}$); then lA isolates P.

Proof of Theorem 5.3.1 The conclusion

$$l \le 4/A^3.$$

is a straightforward corollary of this claim, together with a direct inspection of the list. Q.E.D.

We can write the equation of X in one of the forms[2]

$$F = x_4^3 + ax_4 + b \quad \text{or} \quad x_4^2 + b \quad \text{or} \tag{15}$$

$$= x_j x_4^2 + ax_4 + b \quad \text{with } j \ne 4, \tag{16}$$

where $a(x_0, \dots, x_3)$ and $b(x_0, \dots, x_3)$ are weighted homogeneous polynomials of appropriate degrees. In case (15), π is a finite morphism; in case (16), $\operatorname{Exc}\pi : (x_j = a = b = 0)$, which is a finite set of curves, by the argument given in Example 4.5.

Definition 5.6.3 Let $\{g_i\}$ be a finite set of homogeneous polynomials of degrees l_i in the variables $x_0, \dots, x_4$. We say that the set $\{g_i\}$ *isolates* P if P is a component of the set

$$X \cap \bigcap_i (g_i = 0).$$

Lemma 5.6.4 *If $\{g_i\}$ isolate P, then lA isolates P, where $l = \max\{l_i\}$.*

Proof This is the point at which we use $s > 1$ in Definition 5.2.4 of isolating: write $d = \mathrm{lcm}\{l_i\}$ for the least common multiple of the degrees of the g_i. By replacing each g_i by its power g_i^{d/l_i}, we arrange that they all have the same degree $d = sl$, and all belong to $\mathcal{I}_P^s(slA)$. Q.E.D.

We now show how to find a set of isolating polynomials of reasonably small degree. We work with the case division of Theorem 5.6.2.

[2] This can be proved by inspecting the list, or directly by a Newton polygon argument. Namely, by [Fl], Section 8, for each i, quasismoothness implies that one of $x_i^{n_i}$ or $x_i^{n_i}x_j$ appears in F, and so $\sum_{i=1}^4 \frac{1}{n_i} \ge 1$ by the adjunction formula $d = a_1 + a_2 + a_3 + a_4$ for a K3 $S_d \subset \mathbb{P}(a_1, \dots, a_4)$. Therefore $\frac{1}{n_4} \ge \frac{1}{4}$, and $1/4$ is only possible for $X_4 \in \mathbb{P}^4$. In all other cases, the x_4 of biggest degree appears in a quadratic or cubic monomial.

CASE 1: $\xi_0 \neq 0$ The following is obviously an isolating set of polynomials:

$$\{x_1', x_2', x_3', x_4'\}, \quad \text{where } x_i' = x_i - \frac{\xi_i}{\xi_0^{a_i}} x_0^{a_i}.$$

We deduce that a_4A isolates P. Moreover, if $P \notin \operatorname{Exc} \pi$, the set

$$\{x_1', x_2', x_3'\},$$

isolates P, and hence so does a_3A. This concludes Case 1.

CASE 2: $\xi_0 = 0$ and $\xi_1 \neq 0$ With some exceptions to be discussed shortly, the following set of polynomials isolates P:

$$\Big\{x_0,\ \xi_1^{a_2}x_2^{a_1} - \xi_2^{a_1}x_1^{a_2},\ \xi_1^{a_3}x_3^{a_1} - \xi_3^{a_1}x_1^{a_3}\Big\}, \tag{$*$}$$

which gives Theorem 5.6.2, (2). Since $(*)$ gives us x_0, x_2 and x_3 as algebraic functions of x_1, it only remains to restrict x_4, and this comes from F unless we are in case (16) and $P \in \operatorname{Exc}\pi$. Thus we are home if $(x_0 = 0) \cap \operatorname{Exc}\pi = \emptyset$. This is equivalent to saying that a, b restricted to the affine line $x_0 = x_j = x_4 = 0$, $x_1 \neq 0$ do not have common zeros. This holds in almost all cases, provided that X is general. Consider for example No. 31, $X_{16} \subset \mathbb{P}(1,1,4,5,6)$, with equation $F = yt^2 + a_{10}t + b_{16}$; after setting $x_0 = y = t = 0$, we have the x_1z line, and a, b have no common zeros in general because we could take $a = z^5, b = x_1^{16}$.

However, there is a bizarre quartet of exceptions to this, namely:

- No. 69, $X = X_{28} \subset \mathbb{P}(1,4,6,7,11)$
- No. 74, $X = X_{30} \subset \mathbb{P}(1,3,4,10,13)$
- No. 76, $X = X_{30} \subset \mathbb{P}(1,5,6,8,11)$
- No. 79, $X = X_{33} \subset \mathbb{P}(1,3,5,11,14)$.

(There are altogether about 40 cases to check, exactly the cases in the Big Table where P_4 is associated with a quadratic involution.)

We work out the first of these in detail to show what goes wrong, and to explain how to fix it. In this case the equation is $zu^2 + a_{17}u + b_{28}$, and $\operatorname{Exc}\pi$ is given by

$$z = a_{17}(x, y, z, t) = b_{28}(x, y, z, t) = 0.$$

The peculiar thing here is that $a_{17}(0, y, z, t) = yzt$ has z as a factor, so that a vanishes identically on the affine yt line, and $b_{28}(0, y, 0, t) = 0$ gives a common zero (say, the point $y^7 = t^4$); in other words, $(*)$ fails to isolate P. If $P = (0, \eta, 0, \tau, \upsilon)$, however, we obtain an isolating set by simply adding the polynomial

$$\eta\tau u - \upsilon yt$$

of degree $11 \le a_1a_3 = 28$, so it is still true that a_1a_3A isolates P. The other cases can be cured similarly.

Case 3: $P = (0, 0, \xi_2, \xi_3, \xi_4)$ We isolate P using x_0, x_1, and one further polynomial, which we choose to be of smallest degree γ among the following

$$\xi_j^{a_i'} x_i^{a_j'} - \xi_i^{a_j'} x_j^{a_i'}, \quad \xi_i\xi_j x_k^{c_k} - \xi_k^{c_k} x_i x_j,$$

where $\operatorname{lcm}(a_i, a_j) = a_i' a_j$ and $a_i + a_j = a_k c_k$ when $a_k \mid a_i + a_j$. For instance, consider No. 13, $P = (0, 0, \eta, \zeta, \tau) \in X_{11} \subset \mathbb{P}(1, 1, 2, 3, 5)$. Then $\gamma = 5$ and the following is a P-isolating set

$$\{x_1, x_2, \eta\zeta t - \tau yz\}.$$

In saying this, we use that $\Gamma : (x_0 = x_1 = 0) \cap X$ is an irreducible curve for general X, and so does not contain the curve $(x_0 = x_1 = \eta\zeta t - \tau yz = 0)$. Q.E.D.

5.7 Singular points: construction of T

We prove Lemmas 5.4.3 and 5.4.5. There are 2 subsections

5.7.1 Hyperplane sections and easy cases

5.7.2 Harder cases

Most of the calculations are fairly trivial, and we treat these in 5.7.1. However, there are 10 harder cases, which we collect and work out in voluptuous detail in 5.7.2; one of these is a real champion that deserves, and gets, an extraordinarily florid treatment.

5.7.1 Hyperplane sections and easy cases

In almost all of the cases (to be precise, all but 10, several hundred in all), we obtain $T \in Y$ simply as the birational transform of a suitably chosen hyperplane section: after slightly adjusting the coordinate system if necessary, we can take it to be $(x_i = 0) \subset Y$. In other words, we have

$$T = f_*^{-1}(x_i = 0) \quad \text{for some } i \in \{1, 2, 3, 4\}.$$

All the special surfaces of Lemma 5.4.5 arise thus. Here the calculations are quick and entirely straightforward, but (apart from the special surfaces, which occur predictably, see below) it seems difficult to discern a common pattern. We give several examples illustrating that any coordinate x_i can appear, and in a variety of combinations. For the convenience of the reader and ease of reference, we separate the cases with $B^3 \le 0$ from the special surfaces.

$B^3 \le 0$

Example 5.7.1 No. 9, $X_9 \subset \mathbb{P}(1,1,2,3,3)$, centre $P = P_2 = \frac{1}{2}(1,1,1)$

Recall (7.2.3) that $S : (x_0 = 0)$. Take $T : (x_1 = 0) \in |B|$, so that $(b,c) = (1,0)$ in the third column of the Big Table. The curve Γ is the birational transform of $C_9 \subset \mathbb{P}(2,3,3)$, typically given by $y^3 z_1 + z_1^3 + z_2^3$. This is irreducible and has degree $\frac{1}{2}$ in $\mathbb{P}$, but passes through P, and $T \cdot \Gamma$ drops by $\frac{1}{2}$ on blowing up. Alternatively, we calculate $T \cdot \Gamma = T^2 B = B^3 = (A - \frac{1}{2}E)^3 = \frac{1}{2} - \frac{1}{2} = 0$

Example 5.7.2 No. 18, $X_{12} \subset \mathbb{P}(1,2,2,3,5)$, centre $P \in P_1P_2 = 6 \times \frac{1}{2}(1,1,1)$

We first change coordinates so that $P = P_1$, the point at which $y_0 = 1$, all the other coordinates $= 0$. We may write the equation of X as

$$F = y_0^5 y_1 + O(\tfrac{2}{2}E),$$

where $O\left(\frac{2}{2}E\right)$ means a function that vanishes to order at least $\frac{2}{2}$ along E, in the same sense as in Proposition 3.4.6, (2). We then take $T : (y_1 = 0) \in |2A - E| = 2B$.

Example 5.7.3 No. 43, $X_{20} \subset \mathbb{P}(1,2,4,5,9)$, centre $P \in P_1P_2 = 5 \times \frac{1}{2}(1,1,1)$

As before, we change coordinates so that $P = P_1$, and write the equation of X as

$$F = y^8 z + O(\tfrac{2}{2}E).$$

Then set $T : (z = 0) \in |4A - E| = |4B + E|$. A small calculation shows

$$(4B + E)^2 B = (4A - E)^2 (A - \frac{1}{2}E) = \frac{16}{18} - 2 < 0.$$

Example 5.7.4 No. 23, $X_{14} \subset \mathbb{P}(1,2,3,4,5)$, centre $P \in P_1P_3 = 3 \times \frac{1}{2}(1,1,1)$

We may assume that $P = P_1$, and write the equation of X as

$$F = y^5 t + O(\tfrac{2}{2}E),$$

set $T : (t = 0) \in |4B + E|$, and check that $(4B + E)^2 \cdot B \le 0$.

Example 5.7.5 No. 33, $X_{17} \subset \mathbb{P}(1,2,3,5,7)$, centre $P = P_1 = \frac{1}{2}(1,1,1)$

We assume by generality that the monomial y^5u appears in F. Then all terms divisible by y^5 in F can be collected together as y^5u' by setting $u' = u + yt + y^2t + y^3x$. Write the equation of X as

$$F = y^5u' + O(\tfrac{3}{2}E).$$

Then set $T : (u' = 0) \in |7A - \frac{3}{2}E| = |7B + 2E|$. We have

$$(7B + 2E)^2B = (7A - \frac{3}{2}E)^2(A - \frac{1}{2}E) = 49 \times \frac{17}{210} - \frac{9}{2} < 0.$$

Special surfaces (Lemma 5.4.5)

There are 18 special surfaces in all. We give $3 + 1$ examples, then briefly comment on the general shape of special surfaces.

Example 5.7.6 No. 10, $X_{10} \subset \mathbb{P}(1,1,1,3,5)$, centre $P = P_3 = \frac{1}{3}(1,1,2)$

Choosing suitable coordinates x_0, x_1, x_2, y, z on $\mathbb{P}$, the equation of X is

$$F = y^3(x_0 + x_1 + x_2) + O(\tfrac{4}{3}E).$$

The special surface is then given by

$$T : (x_1 = 0) \in |A - \frac{4}{3}E| = |B - E|;$$

this vanishes along E to unexpectedly high order. We easily check that

$$T \cdot B^2 = (A - \frac{4}{3}E)(A - \frac{1}{3}E)^2 = \frac{2}{3} - \frac{4}{6} = 0.$$

Example 5.7.7 No. 21, $X_{14} \subset \mathbb{P}(1,1,2,4,7)$, centre $P = P_3 = \frac{1}{4}(1,1,3)$

Again, in suitable coordinates, we may write the equation of X as

$$z^3y + O(\tfrac{6}{4}E),$$

and $T : (y = 0) \in |2B - E|$. We easily calculate

$$T \cdot B^2 = (2A - \frac{6}{4}E)(A - \frac{1}{4}E)^2 = \frac{2}{4} - \frac{6}{12} = 0.$$

Example 5.7.8 No. 55, $X_{24} \subset \mathbb{P}(1,2,3,7,12)$, centre $P = P_3 = \frac{1}{7}(1,2,5)$

We choose coordinates so that the equation of X is

$$F = t^3z + O(\tfrac{10}{7}E),$$

set $T : (z = 0) \in |3B - E|$, and calculate $T \cdot B^2 = 0$.

Example 5.7.9 No. 82, $X_{36} \subset \mathbb{P}(1,1,5,12,18)$, centre $P = P_2 = \frac{1}{5}(1,2,3)$

This is slightly different from the above. Choose coordinates so that the equation of X is

$$F = y^7 x_1 + O(\tfrac{6}{5}E),$$

set $T : (x_1 = 0) \in |B - E|$, calculate

$$T \cdot B^2 = (A - \frac{6}{5}E)(A - \frac{1}{5}E)^2 = \frac{1}{30} - \frac{6}{30} < 0.$$

Remark 5.7.10 Special surfaces T follow a nice general pattern, namely,

$$\begin{cases} \text{either} \quad P = P_3, \quad T : (z = 0), \quad T \cdot B^2 = 0; \\ \text{or} \qquad P = P_2, \quad T : (y = 0), \quad T \cdot B^2 < 0. \end{cases}$$

5.7.2 Harder cases

There remain 10 cases to discuss, honoured with a star in the Big Table in recognition of their distinguished difficulty. In each of these harder cases, as we shall soon see, it is easy to locate a P-isolating linear system $\mathcal{L}$ on X that gives a test class M on Y with $M \cdot B^2 \leq 0$, which shows, of course, that $B^2 \notin \overline{\mathrm{NE}}\, Y$ and immediately excludes $P \in X$ as a strong maximal centre, just as in 5.2. Thus, for the proof of Theorem 1.3, we do not actually need to do all this work to determine $\overline{\mathrm{NE}}\, Y$ precisely, and the impatient reader may fast forward through this section.

Taking into account affinities of behaviour, these 10 cases divide naturally into three groups, containing 6, 3, and 1 examples.

First group

Example 5.7.11 No. 19, $X_{12} \subset \mathbb{P}(1,2,3,3,4)$, centre $P \in P_1P_4 = 3 \times \frac{1}{2}(1,1,1)$

We prove that $M = 3B + E$ is a test class, and $T \in |6B + E|$.

Changing coordinates, we may assume that $P = P_1$. We write everything in the coordinate patch $(y = 1)$ of the weighted projective space, which is an affine quotient of type $\frac{1}{2}(1,1,1,0)$. Abusing notation slightly, we still denote the coordinate functions by x, z_0, z_1, t, of weights $1,1,1,2$ locally around P.

Note first that $\{x, z_0, z_1, t\}$ is a P-isolating set of polynomials, giving rise to the test class

$$M = 3A - \frac{1}{2}E = 3B + E$$

on Y. A small calculation gives

$$M \cdot B^2 = (3A - \frac{1}{2}E)(A - \frac{1}{2}E)^2 = \frac{3}{6} - \frac{1}{2} = 0,$$

which shows that $B^2 \notin \operatorname{Int} \overline{\mathrm{NE}}\, Y$, and excludes P as a maximal centre. The reader only interested in the proof of the rigidity theorem may stop reading now and move on to the next example, while we proceed to construct the surface T.

We expand the equation F of X as a sum of polynomials which are weighted homogeneous with respect to the local weighting

$$\begin{aligned} F &= F_2 + F_4 + \cdots \\ &= (t + z^2 + zx + x^2) + F_4 + \cdots , \end{aligned}$$

where, for brevity, z^2 stands for an arbitrary quadratic polynomial in z_0, z_1, etc., and we simply omit coefficients in front of monomials. We only write out the first term of the expansion, which is all that we need. Indeed, it is clear that the function

$$(t + z^2 + zx + x^2)_{|X}$$

vanishes to order at least $\frac{4}{2}$ along E, in other words, reinstating y,

$$u = z^2 + y(t + zx) + y^2x^2 \in |6A - \frac{4}{2}E| = |6B + E|.$$

Let $T : (u = 0)$. Using Bertini, it is easy to check that, if X is general, the curve $\Gamma = T \cap S$ is irreducible, and a small calculation gives

$$T \cdot \Gamma = (6A - \frac{4}{2}E)^2 \cdot (A - \frac{1}{2}E) = \frac{36}{6} - \frac{16}{2} < 0.$$

Remark 5.7.12 There are 4 other cases that behave almost exactly the same. The list, including the case just discussed, is:

- No. 19, $X_{12} \subset \mathbb{P}(1, 2, 3, 3, 4)$, centre $P \in P_1P_4 = 3 \times \frac{1}{2}(1, 1, 1)$
- No. 38, $X_{18} \subset \mathbb{P}(1, 2, 5, 5, 8)$, centre $P \in P_1P_4 = 2 \times \frac{1}{2}(1, 1, 1)$
- No. 66, $X_{27} \subset \mathbb{P}(1, 5, 6, 7, 9)$, centre $P = P_1 = \frac{1}{5}(1, 1, 4)$
- No. 75, $X_{30} \subset \mathbb{P}(1, 4, 5, 6, 15)$, centre $P = P_1 = \frac{1}{4}(1, 1, 3)$
- No. 77, $X_{32} \subset \mathbb{P}(1, 2, 5, 9, 16)$, centre $P \in P_1P_4 = 2 \times \frac{1}{2}(1, 1, 1)$

In all cases, after a coordinate change, $P = P_1$, $\{x, z, t, u\}$ is a P-isolating set of polynomials producing a test class M with $M \cdot B^2 \le 0$, and

$$T = \text{either } \{z^2 + y(t+?)\} \text{ or } \{t^2 + y(z+?)\},$$

(depending on the example). The calculations are straightforward and left to the reader.

The final example in this group differs only slightly from those just discussed.

Example 5.7.13 No. 86, $X_{38} \subset \mathbb{P}(1,5,6,8,19)$, centre $P = P_1 = \frac{1}{5}(1,1,4)$

We will show that $M = 6B + E$ is a test class, and $T \in |18B + 2E|$.

First, the P-isolating set $\{x, z, t, u\}$ gives the test class $M = 6A - \frac{1}{5}E$, and one checks that $M \cdot B^2 \le 0$, which excludes P as a maximal centre. The reader only interested in the proof of the rigidity theorem may pass on to the next case.

We write the equation of X as follows

$$F = y^4\big(z^3 + yz^2x + y^2(zx^2 + t) + y^3x^3\big) + O(\tfrac{8}{5}E),$$

where $O\big(\frac{8}{5}E\big)$ denotes a function which vanishes $\frac{8}{5}$ times along E. Because y is a unit at P_1, this says that

$$v = z^3 + yz^2x + y^2(zx^2 + t) + y^3x^3 \in |18A - \frac{8}{5}E| = |18B + 2E|.$$

This is all very nice, since

$$(18B + 2E)^2B = 4(9A - \frac{4}{5}E)^2(A - \frac{1}{5}E) = 4\Big(\frac{81}{120} - \frac{16}{20}\Big) < 0.$$

As before we set $T = (v = 0)$, and it is easy to check that $\Gamma = S \cap T$ is irreducible.

Second group

Example 5.7.14 No. 81, $X_{34} \subset \mathbb{P}(1,4,6,7,17)$, centre $P \in P_1P_2 = 2 \times \frac{1}{2}(1,1,1)$

We will show that $M = 17B + 8E$ is a test class, and $T \in |12B + E|$.

Indeed, $\{x, t, u\}$ is a P-isolating set, the corresponding test class is $M = 17A - \frac{1}{2}E = 17B + 8E$ and

$$M \cdot B^2 = (17A - \frac{1}{2}E)(A - \frac{1}{2}E)^2 = \frac{17}{84} - \frac{1}{2} < 0;$$

hence, P is not a maximal centre.

It is also easy to determine T. Indeed

$$F(0,y,z,0,0) = yz(y^6 + y^3z^2 + z^4) = yz(z^2 - \alpha^2y^3)(z^2 - \beta^2y^3),$$

where we may assume that $P = (0,1,\alpha,0,0)$. We simply need to observe that

$$v = z^2 - \alpha^2y^3$$

vanishes along E, that is, $v \in |12A - E| = |12B + 5E|$, and

$$(12B+5E)^2(A - \frac{1}{2}E) = 4(6A - \frac{1}{2}E)^2(A - \frac{1}{2}E) = 4\Big(\frac{36}{84} - \frac{1}{2}\Big) < 0.$$

Then set $T = (v = 0)$, etc.

The next example is similar but harder

Example 5.7.15 No. 60, $X_{24} \subset \mathbb{P}(1,4,5,6,9)$, centre $P \in P_1P_3 = 2 \times \frac{1}{2}(1,1,1)$

We will show that $M = 9B + 4E$ is a test class, and $T \in |18B + 7E|$.

$\{x,z,u\}$ isolates P, the corresponding test class is $M = 9A - \frac{1}{2}E = 9B + 4E$ and $M \cdot B^2 = \frac{9}{45} - \frac{1}{2} < 0$, so P is not a maximal centre.

To find T, we first try to proceed as before

$$F(0,y,0,t,0) = (t^2 - \alpha^2y^3)(t^2 - \beta^2y^3)$$

and, assuming that $P = (0,1,0,\alpha,0)$,

$$v = t^2 - \alpha^2y^3 \in |12A - E| = |12B + 5E|.$$

Unfortunately

$$(12B+5E)^2B = 4(6A - \frac{1}{2}E)^2(A - \frac{1}{2}E) = 4\Big(\frac{36}{45} - \frac{1}{2}\Big) > 0.$$

Since this doesn't work, there must be a function vanishing to higher order along E. Let us write the equation of X as

$$\begin{aligned} F(x,y,z,t,u) = (t^2 - \alpha^2y^3)(t^2 - \beta^2y^3) + \\ &+ t\big(y(t^2+y^3)x^2 + y^2z^2 + u^2 + (t^2+y^3)xz + y^2xu + yzu\big) \\ &+ O_{x,z,u}(4). \end{aligned}$$

Now simply take

$$\begin{aligned} v = \left(\frac{\beta^2 - \alpha^2}{\alpha^2}\right) t(t^2 - \alpha^2y^3) + \\ & y(t^2+y^3)x^2 + y^2z^2 + u^2 + (t^2+y^3)xz + y^2xu + yzu, \end{aligned}$$

so that $F = tv + \frac{\beta^2}{\alpha^2}(t^2 - \alpha^2y^3)^2 + O_{x,z,u}(4)$, and v vanishes along $\frac{4}{2}E$, hence $T = (v = 0) \in |18A - 2E| = |18B + 7E|$, etc.

Remark 5.7.16 The final example in this group is entirely analogous to that just shown, and we will not do it in detail:

$$\text{No. 69, } X_{28} \subset \mathbb{P}(1,4,6,7,11), \text{ centre } P \in P_1P_2 = 2 \times \frac{1}{2}(1,1,1).$$

Here $M = 11B + 5E$ is a test class, and $T \in |22B + 9E|$.

5.7.3 The amazing example

The final case, because of its crazy complexity, deserves a place of its own. We speak of

$$\text{No. 39, } X_{18} \subset \mathbb{P}(1,3,4,5,6), \text{ centre } P \in P_1P_4 = 3 \times \frac{1}{3}(1,1,2).$$

We will show that $M = 5B + E$ is a test class, and $T \in |20B + 3E|$.

To begin with, note that $u \in |6A - \frac{3}{3}E| = |6B + E|$ (unfortunately, you can check that $(6B+E)^2 \cdot B > 0$, so we are not done yet) and $t \in |5A - \frac{2}{3}E| = |5B + E|$, so that $\{x, t, u\}$ is a P-isolating set of polynomials, giving rise to the test class $M = 5B + E$. A small calculation yields

$$M \cdot B^2 = (5B + E)B^2 = (5A - \frac{2}{3}E)(A - \frac{1}{3}E)^2 = \frac{5}{20} - \frac{2}{6} < 0,$$

which shows that $B^2 \notin \overline{\mathrm{NE}}\, Y$ and rules out P as a maximal centre.

We now construct T and prove that it satisfies Lemma 5.4.3. The argument takes us 4 intricate steps, so you might again prefer to press fast forward (or maybe better still, just go out for a drink).

Before we plunge into the details of this calculation, we point out that it is not difficult to guess that we should be looking for a curve $\Gamma \in |D|$, with $D = 20B + 3E_{|S}$ (for a more systematic discussion, compare 7.2). Indeed, the Riemann–Roch formula of [YPG] on S immediately gives

$$h^0(D) - h^1(D) = 2 + \frac{1}{2}D^2 - \frac{1}{2}\Big(\frac{1}{2} + 2 \times \frac{2}{3}\Big) = 1,$$

where we used that $D^2 = -\frac{1}{6}$. The problem is, we have no way to guarantee that, *for X general*, the section $\Gamma \in |D|$ is *irreducible*. In fact, Γ is almost certain to break up as X specialises. This is why we still need to find an explicit equation for Γ and show that it is irreducible.

Step 1 To simplify the notation, we ignore the x-coordinate, in effect working with the K3 surface

$$S = S_{18} \subset \mathbb{P}(3,4,5,6)$$

with coordinates y, z, t, u of weights $3, 4, 5, 6$. With the single allowed coordinate change, we may arrange that $P = P_1 = \frac{1}{3}(1, 2) \in S$.

We write the equation of S as follows

$$\begin{aligned} F = y^2z^3 + a_1y^3zt + a_2y^4u + \\ + (a_3z^3u + a_4z^2t^2) + y(a_5ztu + a_6t^3) + a_7y^2u^2 + \\ + a_8u^3, \end{aligned} \tag{1}$$

where terms on the first, second and third line, vanish along E to order $\frac{3}{3}$, $\frac{6}{3}$, $\frac{9}{3}$.

To make life simpler now and later, it is convenient to normalise t and u so that $a_5 = a_1a_3 + 1$ and $a_6 = a_1a_4 + 1$, and we assume that this has already been done, although it will not come up for quite some time. As a first step, it is easy to find a function

$$v = z^3 + a_1yzt + a_2y^2u \in |12A - \tfrac{6}{3}E|. \tag{2}$$

We will often use the formula

$$z^3 = -a_1yzt - a_2y^2u + v. \tag{3}$$

Step 2 The main problem, which makes it hard to write down functions with high order of vanishing at P, is the term $a_3z^3u + a_4z^2t^2$ in F. In this step we use equation (3) to get rid of this term. This allows us to write down a function $w \in |19A - \frac{10}{3}E| = |19B + 3E|$. Unfortunately, you can check that $(19B + 3E)^2 > 0$, so we need to work even harder.

First, substitute (3) in the expression (1) for F, to obtain

$$\begin{aligned} F = a_4z^2t^2 + y\big((a_5 - a_1a_3)ztu + a_6t^3\big) + y^2\big((a_7 - a_2a_3)u^2 + v\big) + \\ + a_8u^3 + a_3uv, \end{aligned} \tag{4}$$

where the first and second lines contain terms that vanish to order $\frac{6}{3}$ and $\frac{9}{3}$ along E. This eliminates the unwanted term a_3z^3u. To eliminate the remaining $a_4z^2t^2$, we first multiply (4) by z:

$$\begin{aligned} zF = a_4z^3t^2 + y\big((a_5 - a_1a_3)z^2tu + a_6zt^3\big) + y^2\big((a_7 - a_2a_3)zu^2 + zv\big) + \\ + a_8zu^3 + a_3zuv \end{aligned}$$

and then again substitute for z^3 using (3)

$$\begin{aligned} zF = y\big((a_5 - a_1a_3)z^2tu + (a_6 - a_1a_4)zt^3\big) + \\ + y^2\big(-a_2a_4t^2u + (a_7 - a_2a_3)zu^2 + zv\big) + \\ + a_8zu^3 + a_3zuv + a_4t^2v. \end{aligned} \tag{5}$$

Grouping together the multiples of y, we can rewrite (5) as

$$zF = yw + (a_8zu^3 + a_3zuv + a_4t^2v),$$

where

$$\begin{aligned} w &= (a_5 - a_1a_3)z^2tu + (a_6 - a_1a_4)zt^3 + y\big(-a_2a_4t^2u + (a_7 - a_2a_3)zu^2 + zv\big) \\ &= z^2tu + zt^3 + y\big(-a_2a_4t^2u + (a_7 - a_2a_3)zu^2 + zv\big) \\ &\quad \in |19A - \tfrac{10}{3}E| = |19B + 3E|, \end{aligned} \tag{6}$$

where we made use of our initial choice $a_5 = a_1a_3 + 1$, $a_6 = a_1a_4 + 1$. We will use (6) in the form

$$zt^3 = -z^2tu - y\big(-a_2a_4t^2u + (a_7 - a_2a_3)zu^2 + zv\big) + w. \tag{7}$$

STEP 3 In this step, we write down our final element $\omega \in |20A - \frac{11}{3}E| = |20B + 3E|$. We begin multiplying F, as given in (4), by t

$$\begin{aligned} tF = a_4z^2t^3 + y\big(zt^2u + a_6t^4\big) + y^2\big((a_7 - a_2a_3)tu^2 + tv\big) + \\ + a_8tu^3 + a_3tuv. \end{aligned} \tag{8}$$

The plan is to eliminate from (8) the unwanted monomial $a_4z^2t^3$. We use (7) and then (3) again to get

$$\begin{aligned} z^2t^3 &= -z^3tu + y\big(a_2a_4zt^2u + (a_2a_3 - a_7)z^2u^2 - z^2v\big) + zw = \\ &= y\big((a_1 + a_2a_4)zt^2u + (a_2a_3 - a_7)z^2u^2 - z^2v\big) + a_2y^2tu^2 - \\ &\quad - tuv + zw, \end{aligned} \tag{9}$$

where we isolated in the last line terms which vanish $\frac{11}{3}$ times along E. Finally, we substitute (9) into (8) to get

$$\begin{aligned} tF = y\big((1 + a_1a_4 + a_2a_4^2)zt^2u + a_4(a_2a_3 - a_7)z^2u^2 + a_6t^4 - a_4z^2v\big) + \\ + y^2\big((a_7 - a_2a_3 + a_2a_4)tu^2 + tv\big) + \\ + a_8tu^3 + a_3tuv - a_4tuv + a_4zw, \end{aligned} \tag{10}$$

where the last line contains terms vanishing $\frac{11}{3}$ times along E. We can now rewrite (10) as

$$tF = y\omega + a_8tu^3 + a_3tuv - a_4tuv + a_4zw, \tag{11}$$

where

$$\begin{aligned} \omega = (1 + a_1 + a_2a_4)zt^2u + a_4(a_2a_3 - a_7)z^2u^2 + a_6t^4 - a_4z^2v + \\ + y\big((a_7 - a_2a_3 + a_2a_4)tu^2 + tv\big) \in |20A - \tfrac{11}{3}E| = |20B + 3E|. \end{aligned} \tag{12}$$

STEP 4 We leave to the reader the entirely straightforward task to use Bertini's theorem and the explicit equations (1) and (12) to verify that the curve $\Gamma = (F = \omega = 0)$ is irreducible (it is not hard).

This completes the proof of all the currently outstanding steps, lemmas and theorems. Q.E.D.

6 The Big Table

We tabulate the singular points of our hypersurfaces

$$X = X_d \subset \mathbb{P} = \mathbb{P}(1, a_1, a_2, a_3, a_4),$$

and, as promised in Theorem 3.2, we indicate for each point either how to exclude it as a maximal centre, or how to untwist it by a quadratic or elliptic involution.

The famous 95 are ordered as in Fletcher [Fl]. We choose coordinates x, y, z, t, u or x, y_1, y_2, z, t etc. in order of ascending degree; for example, in No. 12, $X_{10} \subset \mathbb{P}(1,1,2,3,4)$, the coordinates are x_0, x_1, y, z, t. In all cases x or x_0, x_1 etc. are of degree 1, and $(x_0 = 0) : S \subset X$ is the general elephant, which we assume to be a quasismooth K3 surface, as discussed in 7.2.3.

The first column tabulates the types of singularities. For example, in No. 12, $X_{10} \subset \mathbb{P}(1,1,2,3,4)$, the entry $P_3 = \frac{1}{3}(1,1,2)$ means that the only nonzero coordinate z at $P_3 = (0,0,0,1,0)$ has weight 3, and the local eigencoordinates at P_3 are x_0, x_1, y (compare Proposition 3.4.6).

The second column of the table documents the argument we use for exclusion. As in Notation 3.4.5, we write $A = -K_X$ and $B = -K_Y = A - \frac{1}{r}E$, where $Y \to X$ is the Kawamata blowup. Each entry first distinguishes $B^3 \leq 0$ versus $B^3 > 0$, as discussed in Remark 4 at the start of Chapter 5. Next, we indicate the class $T = bB + cE$ used in either the numerical exclusion of 5.4 or the bad link method 5.5. As we explained in Remark 5.4.7, this information completely determines $\overline{\mathrm{NE}}\,Y$, as well as the cone of effective divisors $\overline{\mathrm{NM}}^1\,Y$. Finally, (†) indicates the 19 bad links of 5.5 and $(*)$ the 10 hard cases of the computation of $\overline{\mathrm{NE}}\,Y$ of 5.7.2.

The right-hand column tells us how to untwist the centre. The quadratic involutions are treated in 4.4 and Theorem 4.9. For example, in No. 65, $X_{27} \subset \mathbb{P}(1,2,5,9,11)$, centre $P_4 = \frac{1}{11}(1,2,9)$, the entry Q.I. $zu^2, 16, 27$ says $F = zu^2 + a_{16}(x,y,z,t)u + b_{27}$; the degrees of a and b are trivial to figure out. The star in front of a monomial such as $*ut^2$ indicates an extra generality assumption (compare 7.4.2).

Refer to 4.10 and Theorem 4.13 for the elliptic involutions. In Example 4.11, No. 61, $X_{25} \subset \mathbb{P}(1,4,5,7,9)$, centre $P_3 = \frac{1}{7}(1,5,2)$, the table says E.I., $tu^2 - yt^3, 16, 25$. This means that the equation of X is $F =$

$tu^2 + ua_{16}(x,y,z) - yt^3 - \cdots + d_{25}(x,y,z,t)$. A star again indicates an extra generality assumption.

No. 1: $X_4 \subset \mathbb{P}(1,1,1,1,1)$, degree $A^3 = 4$

smooth	N/A	none

No. 2: $X_5 \subset \mathbb{P}(1,1,1,1,2)$, degree $A^3 = 5/2$

$P_4 = \frac{1}{2}(1,1,1)$		Q.I. $x_1z^2, 3, 5$

No. 3: $X_6 \subset \mathbb{P}(1,1,1,1,3)$, degree $A^3 = 2$

smooth	N/A	none

No. 4: $X_6 \subset \mathbb{P}(1,1,1,2,2)$, degree $A^3 = 3/2$

$P_3P_4 = 3 \times \frac{1}{2}(1,1,1)$		Q.I. $y_1y_2^2, 4, 6$

No. 5: $X_7 \subset \mathbb{P}(1,1,1,2,3)$, degree $A^3 = 7/6$

$P_4 = \frac{1}{3}(1,1,2)$		Q.I. $x_1z^2, 4, 7$
$P_3 = \frac{1}{2}(1,1,1)$		Q.I. $*zy^2, 5, 7$

No. 6: $X_8 \subset \mathbb{P}(1,1,1,2,4)$, degree $A^3 = 1$

$P_3P_4 = 2 \times \frac{1}{2}(1,1,1)$		Q.I. $zy^2, 6, 8$

No. 7: $X_8 \subset \mathbb{P}(1,1,2,2,3)$, degree $A^3 = 2/3$

$P_4 = \frac{1}{3}(1,1,2)$		Q.I. $y_2z^2, 6, 8$
$P_2P_3 = 4 \times \frac{1}{2}(1,1,1)$		E.I. $*y_2z^2 - y_1y_2^3, 5, 8$

No. 8: $X_9 \subset \mathbb{P}(1,1,1,3,4)$, degree $A^3 = 3/4$

$P_4 = \frac{1}{4}(1,1,3)$		Q.I. $x_1z^2, 5, 9$

No. 9: $X_9 \subset \mathbb{P}(1,1,2,3,3)$, degree $A^3 = 1/2$

$P_2 = \frac{1}{2}(1,1,1)$	$B^3 = 0, B$	
$P_3P_4 = 3 \times (1,1,2)$		Q.I. $z_1z_2^2, 6, 9$

No. 10: $X_{10} \subset \mathbb{P}(1,1,1,3,5)$, degree $A^3 = 2/3$

$P_3 = \frac{1}{3}(1,1,2)$	$B^3 > 0, B - E$ (†)	none

No. 11: $X_{10} \subset \mathbb{P}(1,1,2,2,5)$, degree $A^3 = 1/2$

$P_2P_3 = 5 \times \frac{1}{2}(1,1,1)$	$B^3 = 0, B$	none

No. 12: $X_{10} \subset \mathbb{P}(1,1,2,3,4)$, degree $A^3 = 5/12$

$P_4 = \frac{1}{4}(1,1,3)$		Q.I. $yt^2, 6, 10$
$P_3 = \frac{1}{3}(1,1,2)$		Q.I. $*z^2t, 7, 10$
$P_2P_4 = 2 \times \frac{1}{2}(1,1,1)$	$B^3 < 0, B$	

No. 13: $X_{11} \subset \mathbb{P}(1,1,2,3,5)$, degree $A^3 = 11/30$

$P_4 = \frac{1}{5}(1,2,3)$		Q.I. $x_1t^2, 6, 11$
$P_3 = \frac{1}{3}(1,1,2)$		Q.I. $*z^2t, 8, 11$
$P_2 = \frac{1}{2}(1,1,1)$	$B^3 < 0, B$	

No. 14: $X_{12} \subset \mathbb{P}(1,1,1,4,6)$, degree $A^3 = 1/2$

$P_3P_4 = 1 \times \frac{1}{2}(1,1,1)$	$B^3 = 0, B$	none

No. 15: $X_{12} \subset \mathbb{P}(1,1,2,3,6)$, degree $A^3 = 1/3$

$P_3P_4 = 2 \times \frac{1}{3}(1,1,2)$		Q.I. $tz^2, 9, 12$
$P_2P_4 = 2 \times \frac{1}{2}(1,1,1)$	$B^3 < 0, B$	

No. 16: $X_{12} \subset \mathbb{P}(1,1,2,4,5)$, degree $A^3 = 3/10$

$P_4 = \frac{1}{5}(1,1,4)$		Q.I. $yt^2, 8, 12$
$P_2P_3 = 3 \times \frac{1}{2}(1,1,1)$	$B^3 < 0, B$	

No. 17: $X_{12} \subset \mathbb{P}(1,1,3,4,4)$, degree $A^3 = 1/4$

$P_3P_4 = 3 \times \frac{1}{4}(1,1,3)$		Q.I. $z_1z_2^2, 8, 12$

No. 18: $X_{12} \subset \mathbb{P}(1,2,2,3,5)$, degree $A^3 = 1/5$

$P_4 = \frac{1}{5}(1,2,3)$		Q.I. $y_1t^2, 7, 12$
$P_1P_2 = 6 \times \frac{1}{2}(1,1,1)$	$B^3 < 0, 2B$	

No. 19: $X_{12} \subset \mathbb{P}(1,2,3,3,4)$, degree $A^3 = 1/6$

$P_1P_4 = 3 \times \frac{1}{2}(1,1,1)$	$B^3 < 0, 6B + E \ \ (*)$	none
$P_2P_3 = 4 \times \frac{1}{3}(1,2,1)$	$B^3 = 0, 2B$	

No. 20: $X_{13} \subset \mathbb{P}(1,1,3,4,5)$, degree $A^3 = 13/60$

$P_4 = \frac{1}{5}(1,1,4)$		Q.I. $yt^2, 8, 13$
$P_3 = \frac{1}{4}(1,1,3)$		Q.I. $*z^2t, 9, 13$
$P_2 = \frac{1}{3}(1,1,2)$		E.I. $yt^2 - *zy^3, 8, 13$

No. 21: $X_{14} \subset \mathbb{P}(1,1,2,4,7)$, degree $A^3 = 1/4$

$P_3 = \frac{1}{4}(1,1,3)$	$B^3 > 0, 2B - E \quad (\dagger)$	none
$P_2P_3 = 3 \times \frac{1}{2}(1,1,1)$	$B^3 < 0, B$	

No. 22: $X_{14} \subset \mathbb{P}(1,2,2,3,7)$, degree $A^3 = 1/6$

$P_3 = \frac{1}{3}(1,2,1)$	$B^3 = 0, 2B$	none
$P_1P_2 = 7 \times \frac{1}{2}(1,1,1)$	$B^3 < 0, 2B$	

No. 23: $X_{14} \subset \mathbb{P}(1,2,3,4,5)$, degree $A^3 = 7/60$

$P_4 = \frac{1}{5}(1,2,3)$		Q.I. $tu^2, 9, 14$
$P_3 = \frac{1}{4}(1,3,1)$		E.I. $tu^2 - yt^3, 9, 14$
$P_2 = \frac{1}{3}(1,2,1)$	$B^3 < 0, 2B$	
$P_1P_3 = 3 \times \frac{1}{2}(1,1,1)$	$B^3 < 0, 4B + E$	

No. 24: $X_{15} \subset \mathbb{P}(1,1,2,5,7)$, degree $A^3 = 3/14$

$P_4 = \frac{1}{7}(1,2,5)$		Q.I. $x_1t^2, 8, 15$
$P_2 = \frac{1}{2}(1,1,1)$	$B^3 < 0, B$	

No. 25: $X_{15} \subset \mathbb{P}(1,1,3,4,7)$, degree $A^3 = 5/28$

$P_4 = \frac{1}{7}(1,3,4)$		Q.I. $x_1t^2, 8, 15$
$P_3 = \frac{1}{4}(1,1,3)$		Q.I. $*tz^2, 11, 15$

No. 26: $X_{15} \subset \mathbb{P}(1,1,3,5,6)$, degree $A^3 = 1/6$

$P_4 = \frac{1}{6}(1,1,5)$		Q.I. $yt^2, 9, 15$
$P_3P_4 = 2 \times \frac{1}{3}(1,1,2)$	$B^3 = 0, B$	

No. 27: $X_{15} \subset \mathbb{P}(1,2,3,5,5)$, degree $A^3 = 1/10$

$P_3P_4 = 3 \times \frac{1}{5}(1,2,3)$		Q.I. $t_1t_2^2, 10, 15$
$P_1 = \frac{1}{2}(1,1,1)$	$B^3 < 0, 5B + E$	

No. 28: $X_{15} \subset \mathbb{P}(1,3,3,4,5)$, degree $A^3 = 1/12$

$P_3 = \frac{1}{4}(1,3,1)$	$B^3 = 0, 3B$	none
$P_1P_2 = 5 \times \frac{1}{3}(1,1,2)$	$B^3 < 0, 3B$	

No. 29: $X_{16} \subset \mathbb{P}(1,1,2,5,8)$, degree $A^3 = 1/5$

$P_3 = \frac{1}{5}(1,2,3)$	$B^3 > 0, B - E \quad (\dagger)$	none
$P_2P_4 = 2 \times \frac{1}{2}(1,1,1)$	$B^3 < 0, B$	

No. 30: $X_{16} \subset \mathbb{P}(1,1,3,4,8)$, degree $A^3 = 1/6$

$P_3P_4 = 2 \times \frac{1}{4}(1,1,3)$		Q.I. $tz^2, 12, 16$
$P_2 = \frac{1}{3}(1,1,2)$	$B^3 = 0, B$	

No. 31: $X_{16} \subset \mathbb{P}(1,1,4,5,6)$, degree $A^3 = 2/15$

$P_4 = \frac{1}{6}(1,1,5)$		Q.I. $yt^2, 10, 16$
$P_3 = \frac{1}{5}(1,1,4)$		Q.I. $*tz^2, 11, 16$
$P_2P_4 = 1 \times \frac{1}{2}(1,1,1)$	$B^3 < 0, B$	

No. 32: $X_{16} \subset \mathbb{P}(1,2,3,4,7)$, degree $A^3 = 2/21$

$P_4 = \frac{1}{7}(1,3,4)$		Q.I. $yu^2, 9, 16$
$P_2 = \frac{1}{3}(1,2,1)$	$B^3 < 0, 2B$	
$P_1P_3 = 4 \times \frac{1}{2}(1,1,1)$	$B^3 < 0, 4B + E$	

No. 33: $X_{17} \subset \mathbb{P}(1,2,3,5,7)$, degree $A^3 = 17/210$

$P_4 = \frac{1}{7}(1,2,5)$		Q.I. $zu^2, 10, 17$
$P_3 = \frac{1}{5}(1,2,3)$		Q.I. $*ut^2, 12, 17$
$P_2 = \frac{1}{3}(1,2,1)$	$B^3 < 0, 2B$	
$P_1 = \frac{1}{2}(1,1,1)$	$B^3 < 0, 7B + 2E$	

No. 34: $X_{18} \subset \mathbb{P}(1,1,2,6,9)$, degree $A^3 = 1/6$

$P_3P_4 = 1 \times \frac{1}{3}(1,1,2)$	$B^3 = 0, B$	none
$P_2P_3 = 3 \times \frac{1}{2}(1,1,1)$	$B^3 < 0, B$	

No. 35: $X_{18} \subset \mathbb{P}(1,1,3,5,9)$, degree $A^3 = 2/15$

$P_3 = \frac{1}{5}(1,1,4)$	$B^3 > 0, 3B - E \quad (\dagger)$	none
$P_2P_4 = 2 \times \frac{1}{3}(1,1,2)$	$B^3 < 0, B$	

No. 36: $X_{18} \subset \mathbb{P}(1,1,4,6,7)$, degree $A^3 = 3/28$

$P_4 = \frac{1}{7}(1,1,6)$		Q.I. $yt^2, 11, 18$
$P_2 = \frac{1}{4}(1,1,3)$		E.I. $yt^2 - zy^3, 11, 18$
$P_2P_3 = 1 \times \frac{1}{2}(1,1,1)$	$B^3 < 0, B$	

No. 37: $X_{18} \subset \mathbb{P}(1,2,3,4,9)$, degree $A^3 = 1/12$

$P_3 = \frac{1}{4}(1,3,1)$	$B^3 = 0, 2B$	none
$P_2P_4 = 2 \times \frac{1}{3}(1,2,1)$	$B^3 < 0, 2B$	
$P_1P_3 = 4 \times \frac{1}{2}(1,1,1)$	$B^3 < 0, 4B + E$	

No. 38: $X_{18} \subset \mathbb{P}(1,2,3,5,8)$, degree $A^3 = 3/40$

$P_4 = \frac{1}{8}(1,3,5)$		Q.I. $yu^2, 10, 18$
$P_3 = \frac{1}{5}(1,2,3)$		Q.I. $*ut^2, 13, 18$
$P_1P_4 = 2 \times \frac{1}{2}(1,1,1)$	$B^3 < 0, 10B + 3E \ (*)$	

No. 39: $X_{18} \subset \mathbb{P}(1,3,4,5,6)$, degree $A^3 = 1/20$

$P_3 = \frac{1}{5}(1,4,1)$	$B^3 = 0, 3B$	none
$P_2 = \frac{1}{4}(1,3,1)$	$B^3 < 0, 3B$	
$P_2P_4 = \frac{1}{2}(1,1,1)$	$B^3 < 0, 3B + E$	
$P_1P_4 = 3 \times \frac{1}{3}(1,1,2)$	$B^3 < 0, 20B + 3E \ (*)$	

No. 40: $X_{19} \subset \mathbb{P}(1,3,4,5,7)$, degree $A^3 = 19/420$

$P_4 = \frac{1}{7}(1,3,4)$		Q.I. $tu^2, 12, 19$
$P_3 = \frac{1}{5}(1,3,2)$		E.I. $tu^2 - zt^3, 12, 19$
$P_2 = \frac{1}{4}(1,3,1)$	$B^3 < 0, 3B$	
$P_1 = \frac{1}{3}(1,1,2)$	$B^3 < 0, 7B + E$	

No. 41: $X_{20} \subset \mathbb{P}(1,1,4,5,10)$, degree $A^3 = 1/10$

$P_3P_4 = 2 \times \frac{1}{5}(1,1,4)$		Q.I. $tz^2, 15, 20$
$P_2P_4 = \frac{1}{2}(1,1,1)$	$B^3 < 0, B$	

No. 42: $X_{20} \subset \mathbb{P}(1,2,3,5,10)$, degree $A^3 = 1/15$

$P_2 = \frac{1}{3}(1,2,1)$	$B^3 < 0, 2B$	
$P_3P_4 = 2 \times \frac{1}{5}(1,2,3)$		Q.I. $tz^2, 15, 20$
$P_1P_4 = 2 \times \frac{1}{2}(1,1,1)$	$B^3 < 0, 10B + 3E$	

No. 43: $X_{20} \subset \mathbb{P}(1,2,4,5,9)$, degree $A^3 = 1/18$

$P_4 = \frac{1}{9}(1,4,5)$		Q.I. $yu^2, 11, 20$
$P_1P_2 = 5 \times \frac{1}{2}(1,1,1)$	$B^3 < 0, 4B + E$	

No. 44: $X_{20} \subset \mathbb{P}(1,2,5,6,7)$, degree $A^3 = 1/21$

$P_4 = \frac{1}{7}(1,2,5)$		Q.I. $tu^2, 13, 20$
$P_3 = \frac{1}{6}(1,5,1)$		E.I. $tu^2 - yt^3, 13, 20$
$P_1P_3 = 3 \times \frac{1}{2}(1,1,1)$	$B^3 < 0, 6B + 2E$	

No. 45: $X_{20} \subset \mathbb{P}(1,3,4,5,8)$, degree $A^3 = 1/24$

$P_4 = \frac{1}{8}(1,3,5)$		Q.I. $zu^2, 12, 20$
$P_1 = \frac{1}{3}(1,1,2)$	$B^3 < 0, 8B + E$	
$P_2P_4 = 2 \times \frac{1}{4}(1,3,1)$	$B^3 < 0, 3B$	

No. 46: $X_{21} \subset \mathbb{P}(1,1,3,7,10)$, degree $A^3 = 1/10$

$P_4 = \frac{1}{10}(1,3,7)$		Q.I. $xt^2, 11, 21$

No. 47: $X_{21} \subset \mathbb{P}(1,1,5,7,8)$, degree $A^3 = 3/40$

$P_4 = \frac{1}{8}(1,1,7)$		Q.I. $yt^2, 13, 21$
$P_2 = \frac{1}{5}(1,2,3)$	$B^3 > 0, B - E \quad (\dagger)$	

No. 48: $X_{21} \subset \mathbb{P}(1,2,3,7,9)$, degree $A^3 = 1/18$

$P_4 = \frac{1}{9}(1,2,7)$		Q.I. $zu^2, 12, 21$
$P_1 = \frac{1}{2}(1,1,1)$	$B^3 < 0, 3B + E$	
$P_2P_4 = 2 \times \frac{1}{3}(1,2,1)$	$B^3 < 0, 2B$	

No. 49: $X_{21} \subset \mathbb{P}(1,3,5,6,7)$, degree $A^3 = 1/30$

$P_3 = \frac{1}{6}(1,5,1)$	$B^3 = 0, 3B$	none
$P_2 = \frac{1}{5}(1,3,2)$	$B^3 = 0, 3B$	
$P_1P_3 = 3 \times \frac{1}{3}(1,2,1)$	$B^3 < 0, 7B + E$	

No. 50: $X_{22} \subset \mathbb{P}(1,1,3,7,11)$, degree $A^3 = 2/21$

$P_3 = \frac{1}{7}(1,3,4)$	$B^3 > 0, B - E \quad (\dagger)$	none
$P_2 = \frac{1}{3}(1,1,2)$	$B^3 < 0, B$	

No. 51: $X_{22} \subset \mathbb{P}(1,1,4,6,11)$, degree $A^3 = 1/12$

$P_3 = \frac{1}{6}(1,1,5)$	$B^3 > 0, 4B - E \quad (\dagger)$	none
$P_2 = \frac{1}{4}(1,1,3)$	$B^3 = 0, B$	
$P_2P_3 = \frac{1}{2}(1,1,1)$	$B^3 < 0, B$	

No. 52: $X_{22} \subset \mathbb{P}(1,2,4,5,11)$, degree $A^3 = 1/20$

$P_3 = \frac{1}{5}(1,4,1)$	$B^3 = 0, 2B$	none
$P_2 = \frac{1}{4}(1,1,3)$	$B^3 < 0, 2B$	
$P_1P_2 = 5 \times \frac{1}{2}(1,1,1)$	$B^3 < 0, 4B + E$	

No. 53: $X_{24} \subset \mathbb{P}(1,1,3,8,12)$, degree $A^3 = 1/12$

$P_3P_4 = 1 \times \frac{1}{4}(1,1,3)$	$B^3 = 0, B$	none
$P_2P_4 = 2 \times \frac{1}{3}(1,1,2)$	$B^3 < 0, B$	

No. 54: $X_{24} \subset \mathbb{P}(1,1,6,8,9)$, degree $A^3 = 1/18$

$P_4 = \frac{1}{9}(1,1,8)$		Q.I. $yt^2, 15, 24$
$P_2P_4 = 1 \times \frac{1}{3}(1,1,2)$	$B^3 < 0, B$	
$P_2P_3 = \frac{1}{2}(1,1,1)$	$B^3 < 0, B$	

No. 55: $X_{24} \subset \mathbb{P}(1,2,3,7,12)$, degree $A^3 = 1/21$

$P_3 = \frac{1}{7}(1,2,5)$	$B^3 > 0, 3B - E \quad (\dagger)$	none
$P_2P_4 = 2 \times \frac{1}{3}(1,2,1)$	$B^3 < 0, 2B$	
$P_1P_4 = 2 \times \frac{1}{2}(1,1,1)$	$B^3 < 0, 3B + E$	

No. 56: $X_{24} \subset \mathbb{P}(1,2,3,8,11)$, degree $A^3 = 1/22$

$P_4 = \frac{1}{11}(1,3,8)$		Q.I. $yu^2, 13, 24$
$P_1P_3 = 3 \times \frac{1}{2}(1,1,1)$	$B^3 < 0, 3B + E$	

No. 57: $X_{24} \subset \mathbb{P}(1,3,4,5,12)$, degree $A^3 = 1/30$

$P_3 = \frac{1}{5}(1,3,2)$	$B^3 = 0, 3B$	none
$P_2P_4 = 2 \times \frac{1}{4}(1,3,1)$	$B^3 < 0, 3B$	
$P_1P_4 = 2 \times \frac{1}{3}(1,1,2)$	$B^3 < 0, 12B + 2E$	

No. 58: $X_{24} \subset \mathbb{P}(1,3,4,7,10)$, degree $A^3 = 1/35$

$P_4 = \frac{1}{10}(1,3,7)$		Q.I. $zu^2, 14, 24$
$P_3 = \frac{1}{7}(1,3,4)$		Q.I. $*ut^2, 17, 24$
$P_2P_4 = \frac{1}{2}(1,1,1)$	$B^3 < 0, 3B + E$	

No. 59: $X_{24} \subset \mathbb{P}(1,3,6,7,8)$, degree $A^3 = 1/42$

$P_3 = \frac{1}{7}(1,6,1)$	$B^3 = 0, 3B$	none
$P_2P_4 = \frac{1}{2}(1,1,1)$	$B^3 < 0, 3B + E$	
$P_1P_2 = 4 \times \frac{1}{3}(1,1,2)$	$B^3 < 0, 6B + E$	

No. 60: $X_{24} \subset \mathbb{P}(1,4,5,6,9)$, degree $A^3 = 1/45$

$P_4 = \frac{1}{9}(1,4,5)$		Q.I. $tu^2, 15, 24$
$P_2 = \frac{1}{5}(1,4,1)$	$B^3 < 0, 4B$	
$P_3P_4 = 1 \times \frac{1}{3}(1,1,2)$	$B^3 < 0, 5B + E$	
$P_1P_3 = 2 \times \frac{1}{2}(1,1,1)$	$B^3 < 0, 18B + 7E$ $(*)$	

No. 61: $X_{25} \subset \mathbb{P}(1,4,5,7,9)$, degree $A^3 = 5/252$

$P_4 = \frac{1}{9}(1,4,5)$		Q.I. $tu^2, 16, 25$
$P_3 = \frac{1}{7}(1,5,2)$		E.I. $tu^2 - yt^3, 16, 25$
$P_1 = \frac{1}{4}(1,3,1)$	$B^3 < 0, 9B + E$	

No. 62: $X_{26} \subset \mathbb{P}(1,1,5,7,13)$, degree $A^3 = 2/35$

$P_3 = \frac{1}{7}(1,1,6)$	$B^3 > 0, 5B - E$ $(\dagger)$	none
$P_2 = \frac{1}{5}(1,2,3)$	$B^3 > 0, B - E$ $(\dagger)$	

No. 63: $X_{26} \subset \mathbb{P}(1,2,3,8,13)$, degree $A^3 = 1/24$

$P_3 = \frac{1}{8}(1,3,5)$	$B^3 > 0, 2B - E$ $(\dagger)$	none
$P_2 = \frac{1}{3}(1,2,1)$	$B^3 < 0, 2B$	
$P_1P_3 = 3 \times \frac{1}{2}(1,1,1)$	$B^3 < 0, 3B + E$	

No. 64: $X_{26} \subset \mathbb{P}(1,2,5,6,13)$, degree $A^3 = 1/30$

$P_3 = \frac{1}{6}(1,5,1)$	$B^3 = 0, 2B$	none
$P_2 = \frac{1}{5}(1,2,3)$	$B^3 = 0, 2B$	
$P_1P_3 = 4 \times \frac{1}{2}(1,1,1)$	$B^3 < 0, 6B + 2E$	

No. 65: $X_{27} \subset \mathbb{P}(1,2,5,9,11)$, degree $A^3 = 3/110$

$P_4 = \frac{1}{11}(1,2,9)$		Q.I. $zu^2, 16, 27$
$P_2 = \frac{1}{5}(1,4,1)$	$B^3 < 0, 2B$	
$P_1 = \frac{1}{2}(1,1,1)$	$B^3 < 0, 11B + 4E$	

No. 66: $X_{27} \subset \mathbb{P}(1,5,6,7,9)$, degree $A^3 = 1/70$

$P_3 = \frac{1}{7}(1,5,2)$	$B^3 = 0, 5B$	none
$P_2 = \frac{1}{6}(1,5,1)$	$B^3 < 0, 5B$	
$P_1 = \frac{1}{5}(1,1,4)$	$B^3 < 0, 12B + E$ $(*)$	
$P_2P_4 = 1 \times \frac{1}{3}(1,2,1)$	$B^3 < 0, 5B + E$	

No. 67: $X_{28} \subset \mathbb{P}(1,1,4,9,14)$, degree $A^3 = 1/18$

$P_3 = \frac{1}{9}(1,4,5)$	$B^3 > 0, B - E$ $(\dagger)$	none
$P_2P_4 = \frac{1}{2}(1,1,1)$	$B^3 < 0, B$	

No. 68: $X_{28} \subset \mathbb{P}(1,3,4,7,14)$, degree $A^3 = 1/42$

$P_1 = \frac{1}{3}(1,1,2)$	$B^3 < 0, 7B + E$	
$P_3P_4 = 2 \times \frac{1}{7}(1,3,4)$		Q.I. $ut^2, 21, 28$
$P_2P_4 = \frac{1}{2}(1,1,1)$	$B^3 < 0, 3B + E$	

No. 69: $X_{28} \subset \mathbb{P}(1,4,6,7,11)$, degree $A^3 = 1/66$

$P_4 = \frac{1}{11}(1,4,7)$		Q.I. $zu^2, 17, 28$
$P_2 = \frac{1}{6}(1,1,5)$	$B^3 < 0, 4B$	
$P_1P_2 = 2 \times \frac{1}{2}(1,1,1)$	$B^3 < 0, 22B + 9E$ $(*)$	

No. 70: $X_{30} \subset \mathbb{P}(1,1,4,10,15)$, degree $A^3 = 1/20$

$P_2 = \frac{1}{4}(1,1,3)$	$B^3 < 0, B$	none
$P_3P_4 = \frac{1}{5}(1,1,4)$	$B^3 = 0, B$	
$P_2P_3 = 1 \times \frac{1}{2}(1,1,1)$	$B^3 < 0, B$	

No. 71: $X_{30} \subset \mathbb{P}(1,1,6,8,15)$, degree $A^3 = 1/24$

$P_3 = \frac{1}{8}(1,1,7)$	$B^3 > 0, 6B - E$ $(\dagger)$	none
$P_2P_4 = 1 \times \frac{1}{3}(1,1,2)$	$B^3 < 0, B$	
$P_2P_3 = 1 \times \frac{1}{2}(1,1,1)$	$B^3 < 0, B$	

No. 72: $X_{30} \subset \mathbb{P}(1,2,3,10,15)$, degree $A^3 = 1/30$

$P_3P_4 = 1 \times \frac{1}{5}(1,2,3)$	$B^3 = 0, 2B$	none
$P_2P_4 = 2 \times \frac{1}{3}(1,2,1)$	$B^3 < 0, 2B$	
$P_1P_3 = 3 \times \frac{1}{2}(1,1,1)$	$B^3 < 0, 3B + E$	

No. 73: $X_{30} \subset \mathbb{P}(1,2,6,7,15)$, degree $A^3 = 1/42$

$P_3 = \frac{1}{7}(1,6,1)$	$B^3 = 0, 2B$	none
$P_2P_4 = 1 \times \frac{1}{3}(1,2,1)$	$B^3 < 0, 2B$	
$P_1P_2 = 5 \times \frac{1}{2}(1,1,1)$	$B^3 < 0, 6B + 2E$	

No. 74: $X_{30} \subset \mathbb{P}(1,3,4,10,13)$, degree $A^3 = 1/52$

$P_4 = \frac{1}{13}(1,3,10)$		Q.I. $zu^2, 17, 30$
$P_2 = \frac{1}{4}(1,3,1)$	$B^3 < 0, 3B$	
$P_2P_3 = \frac{1}{2}(1,1,1)$	$B^3 < 0, 3B + E$	

No. 75: $X_{30} \subset \mathbb{P}(1,4,5,6,15)$, degree $A^3 = 1/60$

$P_1 = \frac{1}{4}(1,1,3)$	$B^3 < 0, 10B + E$ $(*)$	none
$P_3P_4 = 1 \times \frac{1}{3}(1,1,2)$	$B^3 < 0, 5B + E$	
$P_2P_4 = 2 \times \frac{1}{5}(1,4,1)$	$B^3 < 0, 4B$	
$P_1P_3 = 2 \times \frac{1}{2}(1,1,1)$	$B^3 < 0, 5B + 2E$	

No. 76: $X_{30} \subset \mathbb{P}(1,5,6,8,11)$, degree $A^3 = 1/88$

$P_4 = \frac{1}{11}(1,5,6)$		Q.I. $tu^2, 19, 30$
$P_3 = \frac{1}{8}(1,5,3)$		E.I. $tu^2 - zt^3, 22, 30$
$P_2P_3 = 1 \times \frac{1}{2}(1,1,1)$	$B^3 < 0, 5B + 2E$	

No. 77: $X_{32} \subset \mathbb{P}(1,2,5,9,16)$, degree $A^3 = 1/45$

$P_3 = \frac{1}{9}(1,2,7)$	$B^3 > 0, 5B - E \quad (\dagger)$	none
$P_2 = \frac{1}{5}(1,4,1)$	$B^3 < 0, 2B$	
$P_1P_4 = 2 \times \frac{1}{2}(1,1,1)$	$B^3 < 0, 18B + 7E \ (*)$	

No. 78: $X_{32} \subset \mathbb{P}(1,4,5,7,16)$, degree $A^3 = 1/70$

$P_3 = \frac{1}{7}(1,5,2)$	$B^3 = 0, 4B$	none
$P_2 = \frac{1}{5}(1,4,1)$	$B^3 < 0, 4B$	
$P_1P_4 = 2 \times \frac{1}{4}(1,1,3)$	$B^3 < 0, 7B + E$	

No. 79: $X_{33} \subset \mathbb{P}(1,3,5,11,14)$, degree $A^3 = 1/70$

$P_4 = \frac{1}{14}(1,3,11)$		Q.I. $zu^2, 19, 33$
$P_2 = \frac{1}{5}(1,1,4)$	$B^3 < 0, 3B$	

No. 80: $X_{34} \subset \mathbb{P}(1,3,4,10,17)$, degree $A^3 = 1/60$

$P_3 = \frac{1}{10}(1,3,7)$	$B^3 > 0, 4B - E \quad (\dagger)$	none
$P_2 = \frac{1}{4}(1,3,1)$	$B^3 < 0, 3B$	
$P_1 = \frac{1}{3}(1,1,2)$	$B^3 < 0, 10B + 2E$	
$P_2P_3 = 1 \times \frac{1}{2}(1,1,1)$	$B^3 < 0, 3B + E$	

No. 81: $X_{34} \subset \mathbb{P}(1,4,6,7,17)$, degree $A^3 = 1/84$

$P_3 = \frac{1}{7}(1,4,3)$	$B^3 = 0, 6B$	none
$P_2 = \frac{1}{6}(1,1,5)$	$B^3 < 0, 4B$	
$P_1 = \frac{1}{4}(1,3,1)$	$B^3 < 0, 7B + E$	
$P_1P_2 = 2 \times \frac{1}{2}(1,1,1)$	$B^3 < 0, 12B + 5E \ (*)$	

No. 82: $X_{36} \subset \mathbb{P}(1,1,5,12,18)$, degree $A^3 = 1/30$

$P_2 = \frac{1}{5}(1,2,3)$	$B^3 = 0, B - E \quad (\dagger)$	none
$P_3P_4 = 1 \times \frac{1}{6}(1,1,5)$	$B^3 = 0, B$	

No. 83: $X_{36} \subset \mathbb{P}(1,3,4,11,18)$, degree $A^3 = 1/66$

$P_3 = \frac{1}{11}(1,4,7)$	$B^3 > 0, 3B - E \quad (\dagger)$	none
$P_2P_4 = 1 \times \frac{1}{2}(1,1,1)$	$B^3 < 0, 3B + E$	
$P_1P_4 = 2 \times \frac{1}{3}(1,1,2)$	$B^3 < 0, 18E + 6B$	

No. 84: $X_{36} \subset \mathbb{P}(1,7,8,9,12)$, degree $A^3 = 1/168$

$P_2 = \frac{1}{8}(1,7,1)$	$B^3 < 0, 7B$	none
$P_1 = \frac{1}{7}(1,2,5)$	$B^3 < 0, 8B$	
$P_3P_4 = 1 \times \frac{1}{3}(1,1,2)$	$B^3 < 0, 8B + 2E$	
$P_2P_4 = 1 \times \frac{1}{4}(1,3,1)$	$B^3 < 0, 7B + E$	

No. 85: $X_{38} \subset \mathbb{P}(1,3,5,11,19)$, degree $A^3 = 2/165$

$P_3 = \frac{1}{11}(1,3,8)$	$B^3 > 0, 5B - E \quad (\dagger)$	none
$P_2 = \frac{1}{5}(1,1,4)$	$B^3 < 0, 3B$	
$P_1 = \frac{1}{3}(1,2,1)$	$B^3 < 0, 5B + E$	

No. 86: $X_{38} \subset \mathbb{P}(1,5,6,8,19)$, degree $A^3 = 1/120$

$P_3 = \frac{1}{8}(1,5,3)$	$B^3 = 0, 5B$	none
$P_2 = \frac{1}{6}(1,5,1)$	$B^3 < 0, 5B$	
$P_1 = \frac{1}{5}(1,1,4)$	$B^3 < 0, 18B + 2E \ (*)$	
$P_2P_3 = 1 \times \frac{1}{2}(1,1,1)$	$B^3 < 0, 5B + 2E$	

No. 87: $X_{40} \subset \mathbb{P}(1,5,7,8,20)$, degree $A^3 = 1/140$

$P_2 = \frac{1}{7}(1,1,6)$	$B^3 < 0, 5B$	none
$P_3P_4 = 1 \times \frac{1}{4}(1,1,3)$	$B^3 < 0, 7B + E$	
$P_1P_4 = 2 \times \frac{1}{5}(1,2,3)$	$B^3 < 0, 20B + 3E$	

No. 88: $X_{42} \subset \mathbb{P}(1,1,6,14,21)$, degree $A^3 = 1/42$

$P_3P_4 = 1 \times \frac{1}{7}(1,1,6)$	$B^3 = 0, B$	none
$P_2P_4 = 1 \times \frac{1}{3}(1,1,2)$	$B^3 < 0, B$	
$P_2P_3 = 1 \times \frac{1}{2}(1,1,1)$	$B^3 < 0, B$	

No. 89: $X_{42} \subset \mathbb{P}(1,2,5,14,21)$, degree $A^3 = 1/70$

$P_2 = \frac{1}{5}(1,4,1)$	$B^3 < 0, 2B$	none
$P_3P_4 = 1 \times \frac{1}{7}(1,2,5)$	$B^3 = 0, 2B$	
$P_1P_3 = 3 \times \frac{1}{2}(1,1,1)$	$B^3 < 0, 5B + 2E$	

No. 90: $X_{42} \subset \mathbb{P}(1,3,4,14,21)$, degree $A^3 = 1/84$

$P_2 = \frac{1}{4}(1,3,1)$	$B^3 < 0, 3B$	none
$P_3P_4 = \frac{1}{7}(1,3,4)$	$B^3 = 0, 3B$	
$P_1P_4 = 2 \times \frac{1}{3}(1,1,2)$	$B^3 < 0, 21B + 5E$	
$P_2P_3 = \frac{1}{2}(1,1,1)$	$B^3 < 0, 3B + E$	

No. 91: $X_{44} \subset \mathbb{P}(1,4,5,13,22)$, degree $A^3 = 1/130$

$P_3 = \frac{1}{13}(1,4,9)$	$B^3 > 0, 5B - E \quad (\dagger)$	none
$P_2 = \frac{1}{5}(1,3,2)$	$B^3 < 0, 4B$	
$P_1P_4 = 1 \times \frac{1}{2}(1,1,1)$	$B^3 < 0, 5B + 2E$	

No. 92: $X_{48} \subset \mathbb{P}(1,3,5,16,24)$, degree $A^3 = 1/120$

$P_2 = \frac{1}{5}(1,1,4)$	$B^3 < 0, 3B$	none
$P_3P_4 = 1 \times \frac{1}{8}(1,3,5)$	$B^3 = 0, 3B$	
$P_1P_4 = 2 \times \frac{1}{3}(1,2,1)$	$B^3 < 0, 5B + E$	

No. 93: $X_{50} \subset \mathbb{P}(1,7,8,10,25)$, degree $A^3 = 1/280$

$P_2 = \frac{1}{8}(1,7,1)$	$B^3 < 0, 7B$	none
$P_1 = \frac{1}{7}(1,3,4)$	$B^3 < 0, 8B$	
$P_3P_4 = 1 \times \frac{1}{5}(1,2,3)$	$B^3 < 0, 8B + E$	
$P_2P_3 = 1 \times \frac{1}{2}(1,1,1)$	$B^3 < 0, 7B + 3E$	

No. 94: $X_{54} \subset \mathbb{P}(1,4,5,18,27)$, degree $A^3 = 1/180$

$P_2 = \frac{1}{5}(1,3,2)$	$B^3 < 0, 4B$	none
$P_1 = \frac{1}{4}(1,1,3)$	$B^3 < 0, 18B + 3E$	
$P_3P_4 = 1 \times \frac{1}{9}(1,4,5)$	$B^3 = 0, 4B$	
$P_1P_3 = 1 \times \frac{1}{2}(1,1,1)$	$B^3 < 0, 5B + 2E$	

No. 95: $X_{66} \subset \mathbb{P}(1,5,6,22,33)$, degree $A^3 = 1/330$

$P_1 = \frac{1}{5}(1,2,3)$	$B^3 < 0, 6B$	none
$P_3P_4 = 1 \times \frac{1}{11}(1,5,6)$	$B^3 = 0, 5B$	
$P_2P_4 = \frac{1}{3}(1,2,1)$	$B^3 < 0, 5B + E$	
$P_2P_3 = 1 \times \frac{1}{2}(1,1,1)$	$B^3 < 0, 5B + 2E$	

7 Final remarks

Although our proof of Theorem 1.3 is now complete, many mysteries remain. Some properties that we observe to hold for our general hypersurfaces certainly fail for general Fano 3-folds. Some features of our proof, both methods of exclusion and constructions of untwisting, occur over a whole range of cases, including many not treated here, and can sometimes even be stated as general properties of all Fanos, although we have no conceptual a priori proofs. As we discuss in the final 7.9, the many individual cases of special behaviour that have kept us company throughout this long paper possibly indicate that no such simple logic is applicable.

7.1 Iskovskikh–Manin by our methods

The classical result of Iskovskikh–Manin on the nonsingular quartic 3-fold [IM] becomes quite straightforward by our methods. See 5.2 for the method for excluding curves as maximal centres, where we took their argument as a model. For excluding any point as a maximal centre, our method is very short and simple: it uses only the statement of the log surface method in Theorem 5.3.2 and the obvious test class A with $A^3 = 4$. Compare also [Pu1] and [Co2].

7.2 Numerics

This is a short tutorial on some nice methods of calculation that we have found useful on many occasions.

7.2.1 Predicting the degrees

The invariants $d, a_1, \dots, a_4$ and $A^3 = (-K_X)^3$ of our anticanonical hypersurfaces $X_d \subset \mathbb{P}(1, a_1, \dots, a_4)$ satisfy the two relations

$$\sum a_i = d \quad \text{and} \quad A^3 = \frac{d}{\prod a_i}.$$

Note that by Proposition 3.4.6, (1), for a Kawamata blowup $Y \to X$

$$-K_Y = B = A - \frac{1}{r}E \quad \text{so that} \quad B^3 = A^3 - \frac{1}{ra(r-a)}.$$

Thus if we suspect that the anticanonical ring of Y is a hypersurface, we can usually predict the degrees of the generators and relations without effort; this is particularly effective when used in conjunction with the lists of Fletcher [Fl]. It applies to all the involutions constructed in Chapter 4: when $Y \to X$ leads to a quadratic involution, all but one of the generators of X survive, so that if the anticanonical model of Y is $Z_{d'} \subset \mathbb{P}(1, a_1, \dots, a'_i, \dots, a_4)$, the values of a'_i and d' are determined. (The calculation we left implicit for the first troop of 16 bad links in Remark 5.5.2 is exactly the same.) Similarly, if $Y \to X$ leads to an elliptic involution, we eliminate two of the generators x_i, x_j in favour of two new generators x'_i, x'_j that should appear in the Weierstrass equation (6) and thus have degrees $a'_i, a'_j = 2e, 3e$; for example, in Example 4.11,

$$(2e)(3e) = 9d \quad \text{and} \quad 2e + 3e = d - 9,$$

which has only one solution $e = 9$ and $d = 54$ in integers.

7.2.2 Hilbert function

A slightly more sophisticated form of this argument involves the Hilbert function (or Hilbert polynomial) of graded rings as in Fletcher [Fl], II.8, and developed more systematically in Altınok [A]. Write $P_n(X) = \chi(X, nA)$ and define the Hilbert polynomial by $P_X(t) = \sum_{n\geq 0} P_n t^n$; in our present case, since A is ample, $\chi(X, nA) = h^0(X, nA)$ for each n by vanishing, and the Hilbert polynomial equals the Hilbert function of $R(X, A)$. If $X_d \subset \mathbb{P}(1, a_1, \ldots, a_4)$ then obviously

$$P_X(t) = \frac{1 - t^d}{\prod_{i=0}^{4}(1 - t^{a_i})}\,. \tag{17}$$

We can define the relative Hilbert polynomial of a morphism $f\colon Y \to X$ by $P_f = P_X - P_Y$. Here we use this only for f the Kawamata blowup of a point of type $\frac{1}{r}(1, a, r-a)$; a calculation using the plurigenus formula of [YPG, §10] gives

$$P_f = P_X - P_Y = \frac{t^r}{(1-t)(1-t^a)(1-t^{r-a})(1-t^r)}\,. \tag{18}$$

It is clear that $\chi(Y, nB) = h^0(Y, nB)$ for all $n \geq 0$ if and only if we are in the "projectively normal" case when $R(Y, nB) \to R(E, nB_{|E})$ is surjective for all $n \geq 0$. It is a fun exercise to verify $P_f = P_X - P_Y$ in all the cases of Theorem 4.9, where we take P_X and P_Y to be given by (17) and P_f by the right hand side of (18).

The method can be modified to work when $R(Y, B) \to R(E, B_{|E})$ is not surjective, which happens in the elliptic involution cases, and even when inverse flips intervene as in the final eccentric trio of Remark 5.5.2.

7.2.3 Restriction from Fano to K3

By the generality assumption, we can assume that $S : (x_0 = 0)$ is a quasismooth K3 surface, and x_0 is the local coordinate of degree 1 at every point of type $\frac{1}{r}(1, a, r-a)$ in the sense of Proposition 3.4.6, (2), so that $S \subset X$ is the general elephant.

We write $A_S = A_{|S} = -K_{X|S}$. The restriction map $H^0(X, nA) \to H^0(S, nA_S)$ is surjective for all $n \geq 0$, so that $R(S, A_S) = R(X, A)/(x_0)$. For many purposes, there is not much difference in working with linear systems on X or their restrictions to S, other than technical convenience. The convenience, however, may sometimes be quite substantial, because linear systems on S can be treated in terms of $\mathbb{Q}$-divisor on the minimal resolution, and we have pretty good intuition for how linear systems on K3s behave. For example, let $Y \to X$ be the Kawamata blowup of a $\frac{1}{r}(1, a, r-a)$ point $P \in X$, and $T \subset Y$ the birational transform of S. Then $P \in S$ of type $\frac{1}{r}(a, r-a)$ is an A_{r-1} Du Val, and the $(a, r-a)$ blowup $T \to S$ extracts the ath curve of its resolution graph, leaving two cyclic singularities of index a and $r-a$.

We explain the calculation used in the amazing example of 5.7.3 in these terms. There $S_{18} \subset \mathbb{P}(3,4,5,6)$ is a K3 surface with $A^2 = \frac{1}{20}$, and we extract a curve E by making the $(1,2)$ blowup of the quotient singularity $\frac{1}{3}(1,2)$ at $P_1 = (1,0,0,0)$. Now $B = A - \frac{1}{3}E$, and $E^2 = -\frac{3}{2}$, and we look at the class $20B + 3E = 20A - \frac{11}{3}E$, obtaining $(20B+3E)^2 = -\frac{1}{6}$, and $\chi(20B+3E) = 1$. Write y, z, t, u for the coordinates of $\mathbb{P}(3,4,5,6)$ so that z, t are local coordinates at P_1; they vanish along E with multiplicity $\frac{1}{3}, \frac{2}{3}$ respectively. It is not hard to write out a monomial basis of the 11-dimensional space $H^0(20A)$. For an element to vanish $\frac{11}{3}$ times along E, we have to write it in terms of the local coordinates z, t, then set to zero the 10 coefficients of the monomials

$$\begin{array}{lllll} z^2 & t & & & \\ z^5 & z^3t & zt^2 & & \\ z^8 & z^t & z^4t^2 & z^2t^3 & t^4 \end{array}$$

Translating to the local coordinates z, t using the equation F_{18} is the tricky part, and what makes the calculations of 5.7.3 so amazingly hard.

7.3 Involutions and projections

Our construction of quadratic and elliptic involutions in Chapter 4 consisted of calculating the subring $R(Y,B) \subset R(X,A)$ obtained by imposing zeros of order $\frac{d}{r}$ along E as "antiadjunction" conditions on $H^0(X, \mathcal{O}_X(dA))$, which amounts algebraically to eliminating one or more of the variables. We explain briefly how to view this geometrically as a *projection*.

The case of quadratic involution is straightforward: in the notation of Theorem 4.9, we eliminate ξ and introduce $w = \xi\eta$ in its place. Geometrically, $X \dashrightarrow Z$ projects away from the ξ point, and the exceptional locus maps to a normal embedding of $\mathbb{P}(1, a, r-a)$ as the codimension 2 subvariety $(\eta = w + a = 0)$. The image Z has equation $g = w(w+a) + \eta b = 0$; the inverse map $Z \dashrightarrow X$ consists of setting $\xi = w/\eta = b/(w+a)$. This fits into a *generic* pattern: write $\Pi : (x = y = 0)$ for a codimension 2 complete intersection and $Z : (Ay - Bx = 0)$ for a hypersurface containing Π; here x, y, A, B could be independent indeterminates, or a regular sequence in a suitable ambient space. Setting $z = A/x = B/y$ defines a rational map (an inverse projection or *unprojection*) $Z \dashrightarrow X$ to the codimension 2 complete intersection $X : (zx = A, zy = B)$. The familiar unprojection of a cubic surface $S_3 \subset \mathbb{P}^3$ to $S_4 \subset \mathbb{P}^4$ fits into this generic case. Whereas in general, we expect the codimension to increase by 1 under this unprojection, in each case of Theorem 4.9, $A = w + a$ involved the variable w linearly, so that X happens to remain a hypersurface.

The elliptic involution is more subtle, and we only mention briefly the meaning of the curious equations around (12) of Example 4.11. If we treat a, d, v, w, y as independent indeterminates, it is not hard to see that 4.11, (14) defines the nonnormal toric variety Σ obtained as the image of the normalisation

$$\mathbb{C}^3 \to \Sigma \subset \mathbb{C}^5 \quad \text{given by} \quad (a, d, x) \mapsto (a, d, v = ax, w = dx, y = x^2),$$

which folds the x-axis in half, identifying $(0, 0, \pm x)$. Write Z for the generic hypersurface containing Σ, given by

$$A(vd - aw) + B(v^2 - a^2y) + C(vw - ady) + D(w^2 - dy^2) = 0.$$

Treating $a, d, v, w, y, A, B, C, D$ as independent indeterminates, Σ can still be "unprojected" by introducing rational functions u, t satisfying

$$\begin{pmatrix} a & v & aC - dD \\ d & w & aB \\ v & ay & wD + aA \\ w & dy & dA - vB + wC \end{pmatrix} \begin{pmatrix} u \\ t \\ 1 \end{pmatrix} = 0.$$

These equations are the first 4 Pfaffians of the 5×5 skewsymmetric matrix

$$\begin{pmatrix} a & d & v & w \\ & t & D & -u \\ & & u + C & B \\ -\mathrm{sym} & & & A + yt \end{pmatrix}.$$

(The fifth Pfaffian is not linear in u, t.) In the generic context, u, t define a birational map $Z \dashrightarrow X$ to the Gorenstein codimension 3 variety X defined by the five Pfaffians. Whereas in general this type of unprojection should increase the codimension by 2, in the cases of Theorem 4.13, $D = 1$, so that the equations (12) involve w and a linearly, and X happens to be a hypersurface.

There are many other types of unprojection: for example, let $\Gamma \subset \mathbb{P}^3$ be a nonnormal twisted quartic, and $S = S_4 \subset \mathbb{P}^3$ a nonsingular quartic surface containing it. Then Γ can be contracted to a node by a map $S \to \overline{S}$, and it is easy to check that $\overline{S}$ is a codimension 5 K3 surface $\overline{S} \subset \mathbb{P}(1^7, 2)$. This case is essentially the hyperplane section of the double projection of the genus 6 Fano 3-fold $X \subset \mathbb{P}^7$, which has image a quartic of $\mathbb{P}^4$ containing the nonsingular projection of the Veronese.

We confidently expect that the unprojection idea will in the fullness of time serve as the basis for a *classification of all Fano 3-folds*. In other words, for the Fano 3-folds whose anticanonical ring are of codimension ≥ 5, a first guess would be that the great majority of them have a singular point $\frac{1}{r}(1,a,r-a)$ for which three of the generators of the canonical ring have degrees equal to $1,a,r-a$. Then they project down to a graded ring of smaller codimension containing a projectively normal $\mathbb{P}(1,a,r-a)$. We have in mind here an analogy with Fano's study of nonsingular Fanos by projection and double projection and the experience of Selma Altınok's thesis [A] on Fanos with anticanonical ring of codimension 3 and 4.

7.4 Comments on the results

7.4.1 Mini-definition of rigidity

We could define rigidity via the *mini-definition*: every Sarkisov link $X \dashrightarrow Y$ has $Y \cong X$. In other words, we could take a statement like Theorem 3.2 as the definition of rigid, although, of course, the equivalence of the two definitions depends on knowing the Sarkisov program.

7.4.2 The key monomials

When a key monomial $*y^2z$ is absent (compare Example 4.6), we expect the point P_t to be excluded as a maximal centre by a bad link. We haven't studied all these cases systematically, but some of them are easy enough, and what happens supports Conjecture 1.3 in a rather interesting way. Consider No. 5, $X_7 \subset \mathbb{P}(1,1,1,2,3)$, centre $P = P_3 = \frac{1}{2}(1,1,1)$, Example 4.6. An interesting degeneration takes place here: in general, the projection $X \dashrightarrow \mathbb{P}^2$ given by $\{x_0 : x_1 : x_2\}$ has fibres that are nonsingular elliptic curves $E_7 \subset \mathbb{P}(1,2,3)$ passing through the singularities P_3 and P_4. If the monomial y^2z is not present, X contains the (y,z)-line $\Gamma = P_3P_4$, and these fibres E_7 degenerate to nonsingular elliptic curves $E'_6 \subset \mathbb{P}(1,2,3)$ plus Γ, where E'_6 meets Γ transversally at one point; Γ would be a -1-curve on a nonsingular model of $\mathbb{P}(1,2,3)$.

Assume that the tangent monomial at P_y is x_2y^3. If $Y \to X$ is the Kawamata blowup then $B \cdot \Gamma < 0$, and we get a bad link involving an inverse flip as in 5.5. In fact the anticanonical model of Y is $X \dashrightarrow Z_9 \subset \mathbb{P}(1,1,1,3,4)$, given by

$$(x_0,x_1,x_2,y,z) \mapsto (x_0,x_1,x_2,x_2y,x_2z),$$

exactly as at the end of 5.5.

7.4.3 Comments on $\operatorname{Bir} X$

(i) *Why on earth does it always have to be involutions?* For our Fano hypersurfaces X, we have proved, among other things, that $\operatorname{Bir} X$ is generated by birational involutions; this seems to hold also for every Fano variety for which rigidity has been established up to now. It may also be a general feature of badly nonrigid varieties, e.g., Max Noether's case of $\mathbb{P}^2$, which, although familiar, lacks any convincing explanation.

(ii) There are many cases when, even though $\operatorname{Bir} X$ is an infinite group, X still has an invariant elliptic fibration or an invariant pencil of K3s. Then $\operatorname{Bir} X$ can be interpreted as the group of biregular automorphisms of the generic fibre. For example, in the case just discussed, $X_7 \subset \mathbb{P}(1,1,1,2,3)$ has in general two involutions i_y and i_z, both of which leave invariant the projection to $\mathbb{P}^2$, and generate an infinite reflection group of biregular automorphisms of the generic fibre E_7; in fact i_z interchanges y and z, and i_y is the reflection in z. There seems to be an analogy here with results on automorphisms of K3s, and even with the currently popular finiteness theorems and conjectures for the Kähler cones of Calabi–Yau 3-folds and their automorphism groups.

(iii) When the coefficient of the term z^2t in a standard quadratic involution degenerates to zero as in 7.4.2, the birational involution of X remains present, but degenerates to an involution no longer untwisting P_z. Recall the example of the projection from an Eckardt point in 2.11, where a birational involution *degenerated to a biregular involution.* The example of 7.4.2 is more interesting: in a varying family, as $c \to 0$, the involutions i_y *degenerates to coincide with* i_z. (This is easily proved: the bad link argument given in 7.4.2 exclude P_y as a maximal centre, so that, together with the main argument of Chapter 5, it shows that the variety X_7 with $c = 0$ has only the single involution i_z. But i_y acted by -1 on the generic fibre E_7, and that can't change into $+1$.) Thus we see the group $\operatorname{Bir} X$ jumping down from an infinite reflection group to $\mathbb{Z}/2$. As far as the generic fibre E_7 is concerned, it just means that the points P_y and P_z come together.

7.4.4 Inverse flips

In the construction of a Sarkisov link, if flips and inverse flips occur, then we necessarily do inverse flips first, then flops, and flips last. This is just an automatic consequence of the way the 2-ray game works: on the extraction $Y \to X$, it is a MMP directed by $\mathcal{H}$.

If we do an extremal extraction $Y \to X$ then an inverse flip means that the maps $R(X, mA) \to R(E, mA_{|E})$ fails to be surjective, with the cokernel growing like m^3.

Inverse flips certainly exist in Sarkisov links on Fano 3-folds. Consider, for example, the case treated by Iskovskikh, and later by him and Pukhlikov, of the involution centred in a conic q on $X = X_{2,3} \subset \mathbb{P}^5$ which is *special* in the sense that its plane also contains a line l of X. This involution is a Sarkisov link of Type II (Definition 3.1.4) conveniently derived using the method of Chapter 4 based on the anticanonical ring. It is obtained by first making the blowup $Y \to X$ of q, followed by the inverse flip $Y \dashrightarrow Y'$ in the birational transform of l, which is a $(-1,-2)$-line on Y. Then Y' has anticanonical model $Z_{12} \subset \mathbb{P}(1,1,1,4,6)$ which has a biregular involution i_Z as explained in 4.1.

Why do none of these appear on our 95 hypersurfaces? Equivalently, why is $B = -K_Y$ nef? Maybe we're just not looking deep enough into singular varieties.

7.5 The Sarkisov category and the rigid boundary

Our main theorem is really very surprising: a priori, it is really hard to imagine any particular reason why a Sarkisov link $X \dashrightarrow X'$ cannot exist between two Fano 3-folds X and X' in different families (in fact we know that this does occur in other contexts); or between X and X' in the same family but with different moduli. Of course, the intermediate Jacobian says something, but we don't expect it always to give a blanket contradiction.

The following simple example shows how terminal singularities (even quite mild ones) may drastically affect the boundary line between rigid and nonrigid varieties.

Example 7.5.1 A quartic 3-fold $X \subset \mathbb{P}^4$ may have a terminal singularity of local type $x^2 + y^2 + z^4 + t^4$, so that every surface in the pencil $\lambda x + \mu y = 0$ has an essential singularity. Then X is birational to a pencil of rational surfaces, and so is certainly not rigid.

For a long time, in the absence of examples, we entertained the conjecture that a Fano must either be rigid, or have a strict Mori fibre space structure. Now Corti and Mella have a few "birigid" examples having *exactly two* Fano minimal models and no Mori fibre space; Grinenko and Pukhlikov have proposed related examples. It seems there will many more phenomena of this nature, intermediate between rigid and wildly nonrigid cases. (In fact this area is developing rapidly, and what we write here may soon be out of date.)

At one point in our studies, we hoped that rigid versus nonrigid should be a firm dichotomy in 3-folds; examples such as Example 7.5.1 show that this is too optimistic. Exactly as with questions of rationality or unirationality, it seems that the boundary is mysterious and possibly chaotic. Probably a lot of things suddenly explode together at the boundary: we get links to completely different Mori fibre space models, so that untwisting by selfmaps becomes out of the question, the group $\operatorname{Bir} X$ becomes wildly infinite dimensional, etc.

7.6 The spectrum: all the arguments of the rainbow

Logically speaking, there cannot be any intersection between the cases when our methods of untwisting and of excluding apply, so it might seem to be a stroke of luck that they cover precisely every case. In fact, there really is a big overlap: our methods based on the anticanonical ring work more or less uniformly across a whole range of cases, sometimes giving an untwisting, and sometimes ending up in an anticanonical contraction other than a Mori fibre space, and thus a bad link. In this context, it is instructive to compare the calculations of Chapter 4 that lead to an untwisting with those of Chapter 5 that lead to a bad link (see for example Remark 5.5.2).

We can view the different excluding or untwisting arguments used in this paper as forming a *spectrum* ranging from entirely numerical arguments (obtaining a contradiction from an intersection calculation) to entirely anticanonical Mori theoretic arguments. This point was already mentioned in 2.11. The attentive reader will surely have noticed items of several different colours:

(i) Simple numerical arguments such as intersection number on a surface; see for example Lemma 2.5, p. 180, and Step 1 of the proof of Theorem 5.1.1 on p. 206.

(ii) The pure test class method 5.2; here we use intersection number on extremal extraction $Y \to X$, but still no extra clever choices.

(iii) Variations on the test class method 5.3; note that Theorems 5.3.2 and 5.3.3 make more sophisticated use of the minimal model program and log MMP to set up a more refined intersection number calculation.

(iv) The calculation of $\overline{\mathrm{NE}}$ in 5.4, which we first used in 5.4.1 to help find a test class, again feeding into a numerical calculation.

(v) The bad link method discussed in 2.11 and 5.5; this used the knowledge of $\overline{\mathrm{NE}}$, together with facts about the Sarkisov program.

(vi) Anticanonical model; when $|-K_Y|$ (or a multiple) is big on an extremal extraction $Y \to X$ we can in many practical cases follow through a 2-ray game all the way to an anticanonical model. This may lead to a contradiction, as we have seen in 5.5, or it may give a Sarkisov link that untwists our centre as in Chapter 4.

The *test class method* is at the numerical end of this spectrum; our version of it simplifies and strengthens the original approach used by Iskovskikh and Manin to treat the nonsingular quartic 3-fold. It requires little detailed knowledge of the geometry of X, and in several cases it implies that Γ is not even a weak maximal centre. At the biregular end we have the *bad link method* (see 5.5), which rests on the inner logic of the Sarkisov program and needs a finer understanding of some of the geometry of X. Our treatment has followed this organisation roughly.

It would be nice to consolidate our methods, and do everything by the bad link method: after making an extraction $Y \to X$, we have $\overline{\mathrm{NE}}\, Y = R + Q$, and for a link to exist, it must be possible to detect Q, and to contract it (or inverse flip it) in the Mori category. In almost all cases, Q can be thought of a curve (or limit of curves) on the K3 ($x_0 = 0$), usually rational. We know that when this fails, we get a contradiction of "bad link" type. However, in practice, this method is hard to use systematically.

As a simple illustration, suppose that we are trying to exclude curves on the quartic $C \subset X_4 \subset \mathbb{P}^4$ as part of the proof of the Manin–Iskovskikh theorem (and we tie our hands behind our backs, not allowing ourselves to use intersection numbers). If C is a line, we blow up $Y \to X$; the anticanonical system $|-K_Y|$ of Y gives a fibre space $\varphi\colon Y \to \mathbb{P}^2$ of plane cubics. Therefore $\overline{\mathrm{NE}}\, Y = R + Q$ where Q is the fibre of φ. Since this is not a Mori fibre space, the 2-ray game stops, and we have a contradiction to C a maximal centre.

If C is a conic, we blowup $Y \to X$, but not the residual conic C'; then $|-K_Y|$ is a pencil of K3s having C' as its fixed curve, but is not nef. If you want to continue the two ray game, the inverse flip $Y \dashrightarrow Y'$ of C' exists, and contracts C' to a node on every surface $S \in |-K_Y|$. Then Y' has a strictly canonical singularity.

If $C \subset X_4$ is a twisted cubic, a similar kind of argument works: look at the quadrics through C, getting a nef system $|B + E|$ that defines a fibration of Y which is not a Mori fibre space. But we have no idea how to extend this kind of thing to curves of higher degree. The paradox is that we only seem to be able to make the bad link method work for exclusion when there is a reasonably substantial anticanonical system, so that we can at least start out on a 2-ray game (in other words, if our centre is reasonably close to being a maximal singularity).

Note that, as we said in 5.2, we cannot use bad link or other biregular methods to exclude nonsingular points as maximal centres, because we don't have a classification of extremal extractions $Y \to X$; and even if we did, it might be too complicated to use. Thus the test class stuff and the clever

Theorems 5.3.2 and 5.3.3 are the only arguments we know. (Although it would be reasonable to hope that nonsingular points could be excluded for some very coarse reasons.)

7.7 Remarks relating to the proof of exclusion

The key question is whether there is a fundamental reason for some centres to be excluded and others not. Let $f\colon (E \subset Y) \to (\Gamma \subset X)$ be an extremal divisorial contraction.

In a vague sense, it is natural to expect that, if $B^3 \leq 0$, E is not a maximal singularity. Indeed, if the 2-ray game originating from f is to produce a Sarkisov link, B must at least be big (that is, $\kappa(X, B) = 3$, but not necessarily satisfying B nef or $B^3 > 0$). This explanation works in many cases, but one must not be deceived. Inverse flips increase B^3, so it is still possible for B to be big even if $B^3 \leq 0$ (this happens, for instance, in the maverick exception No. 82, $X_{36} \subset \mathbb{P}(1, 1, 5, 12, 18)$, centre $P_2 = \frac{1}{5}(1, 2, 3)$, but in many more cases – not touched upon in this paper – when X is not general but still quasismooth). Also, in all bad links, B is big but the point in question is excluded anyway. It should also be noted (see 7.4.4 above) that inverse flips do sometimes occur in links of Fano 3-folds, although never in this paper.

We find it interesting that, although the test class method and the bad link method are fundamentally different in spirit, both rest on the existence of a surface $T \in |bB + cE|$ satisfying suitable conditions. This seems to suggest that there should be a unified method of excluding. At the end of all our calculations, we can prove the following

Theorem 7.7.1 *Let X be general, and $f\colon (E \subset Y) \to (\Gamma \subset X)$ an arbitrary extremal divisorial contraction. Then E is a maximal singularity if and only if*

$$B^2 \in \operatorname{Int} \overline{\mathrm{NM}}_1 Y,$$

where $\overline{\mathrm{NM}}_1 Y \subset N_1 Y$ is the cone of nef curves.

Proof If P is a maximal centre, we know from the classification of links in Chapter 4 that B is nef and big. More than that, we know that the anticanonical map

$$Y \to \overline{Y} = \operatorname{Proj} \bigoplus_{n \geq 0} H^0(-nK_Y)$$

only contracts a finite number of curves on Y. Then $B^2 \cdot D > 0$ for all effective surfaces $D \subset Y$ and $B^2 \in \operatorname{Int} \overline{\mathrm{NM}}_1 Y$. Assume now that $\Gamma \subset X$

is not a maximal centre. If the reason for excluding it was the test class method, then for all extremal divisorial contractions $E \subset Y \to \Gamma \subset X$, $B^2 \notin \operatorname{Int} \overline{\mathrm{NE}}\, Y$ and, in particular, $B^2 \notin \operatorname{Int} \overline{\mathrm{NM}}_1\, Y$. In all other cases, by Lemma 5.4.5, there is a special surface T on Y with $B^2 \cdot T \leq 0$, in other words $B^2 \notin \operatorname{Int} \overline{\mathrm{NM}}_1\, Y$. Q.E.D.

Remark 7.7.2 (i) It is important to understand that we only know how to prove Theorem 7.7.1 *after* completing the whole analysis of untwisting and exclusion in this paper.

(ii) If a direct proof of Theorem 7.7.1 could be found, it would greatly simplify the task of classifying maximal centres. For instance, if $B^3 \leq 0$, it is immediate that $B^2 \notin \operatorname{Int} \overline{\mathrm{NM}}_1\, Y$. In the case of a singular point with $B^3 > 0$, the existence of a special surface T with $B^2 \cdot T \leq 0$ (even without the additional conditions of Lemma 5.4.5, which are needed in the bad link method) immediately implies that $B^2 \notin \operatorname{Int} \overline{\mathrm{NM}}_1\, Y$.

(iii) Theorem 7.7.1 should be contrasted with the backbone of the test class method 5.2: E maximal implies $B^2 \in \operatorname{Int} \overline{\mathrm{NE}}\, Y$. Because $\overline{\mathrm{NM}}_1 \subset \overline{\mathrm{NE}}$, as already observed, the statement contains the test class method as a special case.

(iv) We do not know any counterexamples to Theorem 7.7.1, even if X is not general, or not a hypersurface.

(v) In some sense, the test class method

$$E \text{ maximal} \implies B^2 \in \operatorname{Int} \overline{\mathrm{NE}}\, Y$$

is analogous to Pukhlikov's condition on K^2 in [Pu2]. Pukhlikov proves that if $X \to S$ is a 3-fold Mori fibration in del Pezzo surfaces (satisfying technical conditions including X nonsingular) of degree $d \leq 3$ and $K_X^2 \notin \operatorname{Int} \overline{\mathrm{NE}}\, X$, then $X \to S$ is birationally rigid – see [Pu2] and [Co2], Chapter 5 for treatments of this result. Although easily verified in many natural examples, the K^2 condition is not very well understood. Could it be that $K_X^2 \notin \operatorname{Int} \overline{\mathrm{NM}}_1\, X$ implies $X \to S$ birationally rigid?

7.8 Generality

Together with the Main Theorem 1.3, we stated Conjecture 1.3 that the statement holds for any quasismooth X; this still looks pretty good to us. We list here the places in the paper where generality assumptions on X beyond quasismoothness are involved.

(i) In 9 of the quadratic involutions and 2 of the elliptic involutions, we assumed the presence of a key monomial such as $*z^2t \in f$ as an extra generality assumption in order to construct the untwisting; as we discuss in 7.4.2, if the monomial has zero coefficient, we expect that the centre P is excluded by a bad link, and the involution degenerates to a birational map that no longer untwists P.

(ii) In the proof of Theorem 5.6.2, Case 2: to find convenient polynomials isolating P, we required that a, b have no common zeros on a certain affine line; with the bizarre quartet of exceptions discussed there, this holds by Bertini.

(iii) Similarly in Theorem 5.6.2, Case 3, we asked $(x_0 = x_1 = 0) \cap X$ to be an irreducible curve; this again holds by Bertini.

(iv) In the calculation of $\overline{\mathrm{NE}}\,Y$ in 5.7, we needed the curve $\Gamma = S \cap T$ to be irreducible. This is also OK by Bertini, but checking it requires a case-by-case inspection; the shape of T does not follow any specially regular pattern.

In summary, generality always mean "outside a Zariski closed set", but the extent to which the open condition can be made explicit varies with the requirement. For the involutions, we only require the presence of a specific monomial (and it is even easy enough to analyse what happens if it does not appear). In contrast, in the proof of Theorem 5.6.2, the condition that $\Gamma = S \cap T$ is irreducible is much less clear-cut: in many cases when Γ has fairly low degree, classifying the various exceptions is fun, but Γ occasionally has higher degree, for example in the "amazing example" treated in 5.7.3.

Many of these generality assumptions could be removed without much trouble, but in some instances this would be far from straightforward. Relaxing generality allows all sorts of curious and delightful monsters to crawl out from their underground hiding places. Although we have never found any that seriously threaten the truth of our conjecture, it is clear that doing away with generality assumptions altogether would require a volume of calculations that is quite beyond us.

7.9 Eccentricity

At a number of points in this paper there are statements that have exceptions or arguments that break down due to antisocial behaviour on the part of a small and unrepresentative group of disruptive individuals. At the top of our list of trouble-makers are

(i) The renegade $(*)$ of 5.3 (p. 208). This is an exception to Theorem 5.3.1: if $P \in X_5 \subset \mathbb{P}(1,1,1,1,2)$ lies on an exceptional line, even though the point is excluded, no P-isolating system with the required numerical properties exists.

(ii) The maverick exception

$$\text{No. 82: } X_{36} \subset \mathbb{P}(1,1,5,12,18), \text{ centre } P_2 = \frac{1}{5}(1,2,3)$$

to Lemma 5.4.3. This is the only point with $B^3 \leq 0$ for which no test class exists; the point is excluded using the bad link method.

(iii) The eccentric trio of Remark 5.5.2 (which includes in its ranks the maverick) Nos. 47, 62 and 82: the bad link starts with an inverse flip.

(iv) The amazing example of 5.7.3, where the calculation of $\overline{\mathrm{NE}}\,Y$ becomes fantastically intricate.

(v) The bizarre quartet in the proof of Theorem 5.6.2, Case 2, where an otherwise harmless genericity assumption becomes impossible.

These are just the most striking examples of an already idiosyncratic proof: the tricksy 10 cases of quadratic involutions, the 10 hard cases of calculation of $\overline{\mathrm{NE}}\,Y$ and the errant nature of the several hundred easy cases, the unpredictability of singular points P with $B^3 > 0$ – some of these are excluded, some are not – are similar expressions of an unruly population.

There is an amusing and always surprising interplay between generality and eccentricity; relaxing generality encourages eccentricity. For example:

- in 7.4.2, if its starred monomial $x_j x_i^2$ is absent, a quadratic involution degenerates to coincide with some different involution no longer untwisting the centre; or
- the bizarre quartet, stubbornly resisting a perfectly innocuous generality assumption; or
- the 18 special surfaces of Definition 5.4.4, whose very existence is eccentric, but which all line up in a neat predictable pattern.

In the face of this extravagant exuberance of behaviour, two extreme conclusions present themselves. One is that we haven't understood what's really going on; we have cobbled together some ad hoc techniques, but failed to uncover the more powerful methods that are capable of imposing order. The alternative point of view is that our arguments are necessarily complicated, tailored as they must be to encompass the special circumstances of a population of individualists resenting overenthusiastic attempts to legislate.

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