MA3D5 Galois theory

Assignment E

E.1. Find a splitting field L for $x^4 - 3$ over $\mathbf{Q}$. Determine the Galois group of $L/\mathbf{Q}$, list its subgroups and describe all the fields between L and $\mathbf{Q}$, saying which are normal over $\mathbf{Q}$. [You should assume the Main Theorem.]

E.2. Let $L = k(t)$ be the function field in an indeterminate t over a field k. Let G be the group of k-automorphisms of L generated by

$$\sigma\colon t \mapsto 1 - t \quad \text{and} \quad \tau\colon t \mapsto 1/t;$$

prove that $\sigma^2 = \tau^2 = \mathrm{id}$ and $(\sigma\tau)^3 = \mathrm{id}$, and deduce that $G \cong S_3$. Find the subfields of L fixed by σ and by τ respectively. Prove that $L^{\langle\sigma\tau\rangle} = k(y)$, where $y = \frac{t^3-3t+1}{t(t-1)}$; show that $y+\sigma(y) = 3$, and conclude that $K = L^G = k(z)$, where $z = y\sigma(y)$. Use Main Theorem 4.18 to find all fields F intermediate between K and L.

E.3. Exam style question

(i) Define a normal field extension. Show that if $k \subset K$ is a finite normal field extension, then K is a splitting field over k of some polynomial $f \in k[x]$. Deduce that if K is a subfield of some larger field L then $\sigma(K) = K$ for every k-automorphism σ over L.

(ii) Let p be a prime, and let K be a subfield of $\mathbb{C}$ containing $\varepsilon = \exp\frac{2\pi i}{p}$. Let $\alpha \in K$, and let β be a compex root of $x^p - \alpha$. Show that $k \subset k(\beta)$ is normal, and that $[k(\beta) : k] = 1$ or p. [Hint: Consider k-automorphisms of $k(\beta)$.]

E.4. Exam style question

(i) Define the Galois group $G = \mathrm{Gal}(K/k)$ of a finite field extension $k \subset K$. Prove that the order of G satisfies $|G| \leq [K : k]$, and state without proof the conditions under which equality holds.

(ii) For a subset $H \subset G$, we define $H^\dagger = K^H$ to be the set of elements of K fixed by every automorphism in H. Explain why $H^\dagger$ is an intermediate field $k \subset H^\dagger \subset K$ and prove that

$$H_1 \subset H_2 \implies H_1^\dagger \supset H_2^\dagger$$

State without proof the Main Theorem of Galois theory.

(iii) Find a splitting field K for the polynomial $x^3 - 5 \in \mathbf{Q}[x]$. Determine the Galois group of $K/\mathbf{Q}$, list its subgroups and describe all the fields between K and $\mathbf{Q}$.

Deadline: Thu 22nd Apr 2004 at 11:00.