# MA3D5 Galois theory

## Assignment D

**D.1.** Working from first principles, factorise $x^7 - 1$ over $\mathbb{F}_2$. [Hint: first take out the obvious factor $1 - x$; after which it factors as product of two cubic polynomials.] Show that either irreducible cubic factor gives rise to the same field extension $\mathbb{F}_2 \subset \mathbb{F}_8$. (Compare Example 3.28.) Deduce that $\mathbb{F}_2[x]$ contains exactly two irreducible cubic polynomials.

**D.2.** Consider the field extension $\mathbb{F}_3 \subset \mathbb{F}_9$. The minimal polynomial over $\mathbb{F}_3$ of any $\alpha \in \mathbb{F}_9 \setminus \mathbb{F}_3$ is an irreducible quadratic polynomial. Use Theorem 3.30 to prove that there are exactly 3 monic irreducible quadratic polynomials $f_i(x) \in \mathbb{F}_3[x]$, and that $x^8 - 1$ factors over $\mathbb{F}_3$ as a product

$$x^8 - 1 = (x - 1) \cdot (x - 2) \cdot \prod f_i.$$

**D.3.** Let $k \subset K$ be a finite extension; prove that the number of homomorphisms $\varphi\colon K \to L$ to any "sufficiently large" field $L$ equals $[K : k]$ if and only if $k \subset K$ is separable. [Hint: Rerun the proof of Theorem 4.3. The point is to reduce to a primitive extension $k(\alpha)$ with minimal polynomial $f$, at which point you are down to counting the *distinct* roots in $L$ of $f$.]

**D.4.** Exam style question

(i) Let $k$ be a field and $f \in k[x]$; explain the notion of a splitting field $k \subset K$ for $f$ over $k$. Prove that a splitting field for $f$ over $k$ exists. Define a normal extension and explain without proof the relation between these two notions.

(ii) Suppose that $f = x^4 - 2ax^2 + b \in k[x]$ is irreducible, and let $K$ be a splitting field for $f$ over $K$; prove that the degree $[K : k]$ is either 4 or 8.

(iii) Let $p$ be a prime number. Prove that for any field $k$ and any $a \in k$, the polynomial $f(x) = x^p - a$ is either irreducible, or has a root. [Hint: If $f = f_1 f_2$, factorise $f_1, f_2$ into linear factors over a bigger field, and consider their constant terms.]

Deadline: Thu 4th Mar at 11:00.