MA3D5 Galois theory

Assignment C

C.1. $k \subset K$ is a field extension. If $\alpha \in K$ is algebraic with minimal polynomial of degree n over k, and β is algebraic with minimal polynomial of degree m, prove that $\alpha + \beta$ is algebraic with minimal polynomial of degree $\leq nm$.

C.2. $k \subset k(\alpha)$ is a primitive field extension. If $\alpha \in K$ is algebraic over k with minimal polynomial of odd degree, prove that $k(\alpha^2) = k(\alpha)$. Conversely, if $k(\alpha^2) = k(\alpha)$, does it follow that α is of odd degree?

C.3. Let $\varepsilon = \exp \frac{2\pi i}{7}$. The powers of 3 mod 7 run through all the elements of $\mathbf{Z}/7$:

n	0	1	2	3	4	5
3^n	1	3	2	6	4	5

(that is, 3 is a primitive root mod 7). Consider the two quantities

$$\alpha_1 = \varepsilon + \varepsilon^2 + \varepsilon^4 \quad \text{and} \quad \alpha_2 = \varepsilon^3 + \varepsilon^6 + \varepsilon^5$$

and prove that

(i) α_1 and α_2 are the two roots of a quadratic polynomial over $\mathbf{Q}$.

(ii) $\varepsilon, \varepsilon^2$ and ε^4 are the 3 roots of a cubic polynomial over $\mathbf{Q}(\alpha_1)$, and likewise for $\varepsilon^3, \varepsilon^6, \varepsilon^5$.

[Hint: Compare the end of Section 3.2. In each part, the easy way is to use elementary symmetric functions to find the polynomial explicitly.]

Continued ...

C.4. Exam-style question:

(A) Define a normal field extension $k \subset K$. For a polynomial $f \in k[x]$, define a splitting field of f over k. Explain briefly and without detailed proof the relation between these two notions.

(B) Let $f = x^6 + 3$ and $g = x^6 + 2$, and write $k_1 = \mathbf{Q}[x]/(f)$ and $k_2 = \mathbf{Q}[x]/(g)$. Prove the following statements:

(i) k_1 is the splitting field of f over $\mathbf{Q}$.

(ii) $\mathbf{Q} \subset k_2$ is not a normal extension, and the splitting field of g over $\mathbf{Q}$ is an extension of degree 12.

What is it about f and g that makes such a crucial difference? [Hint: The lecturer is obsessed with cube roots of 1.]

Deadline: Thu 20th Feb at 11:00.