MA3D5 Galois theory

Instructions: The assignments count for 15% of the credit for the course. There are 5 worksheets, whose purpose is to ensure students do the bare minimum work required to keep up with the course; they are not intended to be difficult, and you can discuss them in the support class. Solutions must be left in the filing cabinet by the front office by the

> **deadlines**: 10:00 am on Thu of Weeks 2, 4, 6, 8 plus Term 3, Week 1 (Thu 21st Apr 2005).

We'll arrive at the total marks by taking the first, the best 2 out of the next 3, plus the last. In other words, the first sheet covering prerequisites and the final sheet including past exam questions are intended as compulsory vacation work.

I would be grateful for comments on the worksheets or the course material.

Preliminary warm-up (not for credit)

P.1 See Section 1.4 of the notes for elementary symmetric functions. If $\sigma_1, \dots, \sigma_n$ are elementary symmetric functions of $\alpha_1, \dots, \alpha_n$, compute the elementary symmetric functions of $\alpha_1^2, \dots, \alpha_n^2$.

P.2 Suppose that $f(x) = x^2 + ax + b = 0$ has roots α, β. Find the quadratic polynomial with roots

$$\text{(a) } \alpha^2, \beta^2; \quad \text{(b) } \alpha^3, \beta^3; \quad \text{(c) } \alpha^7\beta^4, \alpha^4\beta^7.$$

P.3 Write $\varepsilon := \exp \frac{2\pi i}{5}$ for the natural primitive 5th root of 1 as in Section 1.5 of the notes; it is a root of the quartic $f(x) = x^4 + x^3 + x^2 + x + 1 = 0$. Find the quadratic equation whose two roots are $\varepsilon + \varepsilon^4$ and $\varepsilon^2 + \varepsilon^3$, and hence give radical formulas for $\cos \frac{2\pi}{5}$, $\cos \frac{4\pi}{5}$. Ditto for sin.

P.4 Consider the "reciprocal" or "palindromic" quartic polynomial

$$f(x) = ax^4 + bx^3 + cx^2 + bx + a.$$

If α is a root, prove that so is $1/\alpha$.

Show that $\frac{1}{x^2} f(x)$ can be written as a quadratic polynomial in $y = x + \frac{1}{x}$. Use this to find a formula for the roots of f involving two square root signs.

Assignment A
Deadline: Thu 13th Jan 2005 at 10:00

A.1 (i) If $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$ has roots $\alpha_1, \dots, \alpha_n$, use elementary symmetric functions to find the polynomial with roots $c\alpha_1, c\alpha_2, \dots, c\alpha_n$. Say why the result is not surprising.

(ii) Use elementary symmetric polynomials to find the polynomial whose roots are $1/\alpha_1, \dots, 1/\alpha_n$. Check against common sense.

A.2 Suppose that α, β, γ are the three roots of the cubic equation $f(x) = x^3 + 3px + 2q = 0$, and let ω be a primitive cube root of 1 as in Section 1.5 of the notes. Study the effect of permuting α, β, γ on the quantities

$$t_1 = \alpha + \omega\beta + \omega^2\gamma \quad \text{and} \quad t_2 = \alpha + \omega^2\beta + \omega\gamma.$$

Deduce that t_1^3, t_2^3 are invariant under the 3-cycle $(\alpha\beta\gamma)$ and are interchanged by the transposition $(\beta\gamma)$. Use elementary symmetric functions to find the quadratic polynomial with roots t_1^3, t_2^3.

A.3 Let $\varepsilon := \exp\frac{2\pi i}{5}$ be the natural primitive 5th root of 1.

Calculate from first principles the elementary symmetric functions of

(i) the 5 roots $\varepsilon^0 = 1, \varepsilon, \varepsilon^2, \varepsilon^3, \varepsilon^4$ of $x^5 - 1$; and

(ii) the 4 roots $\varepsilon, \varepsilon^2, \varepsilon^3, \varepsilon^4$ of $x^4 + x^3 + x^2 + x + 1$.

[Hint: The 5 roots are vertexes of a regular 5-gon centred at 0.]

A.4 Carry out in detail the calculation to prove formulas (1.12) of the notes for the roots of a quartic. Namely, if

$$\left.\begin{array}{rcl} \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 &=& 0, \\ \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_1\alpha_4 + \alpha_2\alpha_3 + \alpha_2\alpha_4 + \alpha_3\alpha_4 &=& r, \\ \alpha_1\alpha_2\alpha_3 + \alpha_1\alpha_2\alpha_4 + \alpha_1\alpha_3\alpha_4 + \alpha_2\alpha_3\alpha_4 &=& -s, \\ \alpha_1\alpha_2\alpha_3\alpha_4 &=& t, \end{array}\right\}$$

and u, v, w are defined by

$$\left.\begin{array}{rcl} 2\alpha_1 &=& u + v + w, \\ 2\alpha_2 &=& u - v - w, \\ 2\alpha_3 &=& -u + v - w, \\ 2\alpha_4 &=& -u - v + w. \end{array}\right\} \tag{1}$$

then prove that

$$\left.\begin{array}{rcl} u^2+v^2+w^2 &=& -2r, \\ uvw &=& -s, \\ u^2v^2+u^2w^2+v^2w^2 &=& r^2-4t. \end{array}\right\} \qquad (2)$$

How to use Maple

This is not part of the assignment sheet or of the course, but I recommend doing it. At a unix terminal, type maple; you get a clever logo, and the prompt >. For example, you can calculate $\sum \alpha_i^3$ in terms of elementary symmetric functions (*and get the right answer*!) by the following few lines:

```
> s1:=a+b+c; s2:=a*b+a*c+b*c; s3:=a*b*c;
                                s1 := a + b + c
                             s2 := a b + a c + b c
                                  s3 := a b c
> expand(a^3+b^3+c^3-s1^3);
        2         2          2                   2      2          2
   - 3 a  b - 3 a  c - 3 a b  - 6 a b c - 3 a c  - 3 b  c - 3 b c
> expand(a^3+b^3+c^3-s1^3+3*s1*s2);
                                    3 a b c
> evalb(expand(s1^3-3*s1*s2+3*a*b*c) = a^3+b^3+c^3);
                                      true
```

Mathematica is very similar.