MA3A6 WEEK 10 ASSIGNMENT : DUE MONDAY 4PM WEEK 10

BILL HART

Note this assignment is the last *assessed* assignment. However there will be a final unassessed assignment to give you a feel for the remaining questions that could appear on the exam. I will post solutions on the web for this assignment and the final assignment.

1. Show that in a quadratic number field K a rational prime p either:

(i) Splits completely into a product of distinct prime ideals $(p) = \mathcal{P}_1\mathcal{P}_2$.

(ii) Ramifies, $(p) = \mathcal{P}_1^2$.

(iii) Remains inert, i.e. (p) is a prime ideal in $\mathcal{O}_K$.

You may use any theorems explicitly written out in lectures. However, note that the question is not trivial. One approach may be to prove that the degrees of the irreducible factors modulo p of the minimum polynomial $g(x)$ of an (algebraic integer) generator of K are the degrees of the *residue class field extensions* $\mathcal{O}_K/\mathcal{P}_i$ / $\mathbb{Z}/p\mathbb{Z}$, where $\mathcal{P}_i$ are the prime ideals dividing (p).

Note that norms are going to be important. These residue class fields $\mathcal{O}_K/\mathcal{P}_i$ are finite, so perhaps you can compute their orders and use that as a way of attacking the problem.

2. Let $S = \{2, 3, 5\}$. Find a number field K such that one of the primes in S ramifies in K, one of them splits completely and the other is inert in K.

3. Let $K = \mathbb{Q}(\alpha)$ be the number field with α a root of $x^5 + 7x^4 + 3x^2 - x + 1$.

(The function poldisc(f) in Pari will give you the discriminant of a general polynomial f, i.e. it will basically give you the discriminant of $\mathbb{Z}[\alpha]$ for α a root of f.)

Using Pari, compute discriminants of $\mathcal{O}_K$ and $\mathbb{Z}[\alpha]$ and by hand (without Pari), use this information, to factorise the rational prime 5 in $\mathcal{O}_K$.

4. Find out the Pari function for factoring rational primes into prime ideals in a number field. Use this function to write a short program to factor the first 10 rational primes into prime ideals in $\mathbb{Q}(\alpha)$ for α a root of $x^3 - 3x^2 + 3x + 1$. Examine the output Pari gives you carefully (and refer to the technical documentation describing the output) and use this output to determine which of these 10 rational primes ramify, which are inert and which split completely in K.

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