MA3A6 WEEK 9 ASSIGNMENT : DUE MONDAY 4PM WEEK 9

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1. Compute an aribtrary $\mathbb{Q}$-basis for $K = \mathbb{Q}(\sqrt{2}, \sqrt{3})$ consisting of algebraic integers and compute the discriminant of that basis. Use this to bound the discriminant of K. Write out a finite list of possible values that the discriminant could be.

We note $K = \mathbb{Q}(\sqrt{2}, \sqrt{3})$ is a degree 4 number field. We can take $\{1, \sqrt{2}, \sqrt{3}, \sqrt{6}\}$ as a Q-basis, as each of these elements is linearly dependent of the others over $\mathbb{Q}$. Note that the elements of this basis are algebraic integers of K.

As we have seen before, there are four embeddings of K into $\mathbb{C}$. The first is the identity monomorphism. The second takes $\sqrt{2}$ to $-\sqrt{2}$, the third takes $\sqrt{3}$ to $-\sqrt{3}$ and the fourth takes $\sqrt{2}$ to $-\sqrt{2}$ and $\sqrt{3}$ to $-\sqrt{3}$.

(One can verify this by finding a single generator of K and writing $\sqrt{2}$ and $\sqrt{3}$ in terms of this generator and seeing what happens when you replace the generator with each of the roots of its minimum polynomial, or one may simply note that each of the above is a distinct monomorphism of K into $\mathbb{C}$ described fully by its action on a basis for K and that there are four of them.)

Now we compute the discriminant of this basis.

$$\Delta(1, \sqrt{2}, \sqrt{3}, \sqrt{6}) = \begin{vmatrix} 1 & \sqrt{2} & \sqrt{3} & \sqrt{6} \\ 1 & -\sqrt{2} & \sqrt{3} & -\sqrt{6} \\ 1 & \sqrt{2} & -\sqrt{3} & -\sqrt{6} \\ 1 & -\sqrt{2} & -\sqrt{3} & \sqrt{6} \end{vmatrix}^2,$$

the rows of which I obtain by applying each of the monomorphisms of K into $\mathbb{C}$ to the basis in turn noting that $\sqrt{6} = \sqrt{2}\sqrt{3}$.

I note that the second column has a common factor of $\sqrt{2}$, the third column a factor of $\sqrt{3}$, etc. Pulling these out and evaluating the resulting 4×4 determinant, I find that the discriminant is $\Delta = 2^{10} \times 3^2$.

We note that the discriminant of K will be this value divided by some square factor of this value. It may therefore have any of the values 1, 2^2, 2^4, 2^6, 2^8, 2^{10}, 3^2, $2^2 3^2$, $2^4 3^2$, $2^6 3^2$, $2^8 3^2$, $2^{10} 3^2$.

2. Now use the algorithm demonstrated in class to determine the discriminant of K. Check your answer with Pari.

We apply the algorithm from class to find the discriminant of K. We note first that the only primes whose squares divide Δ are $p = 2, 3$. Thus we must look for algebraic integers of the form

$$\lambda_1 = \frac{1}{2}(a_1 + a_2\sqrt{2} + a_3\sqrt{3} + a_4\sqrt{6})$$

for $a_i = 0, 1$ not all zero, and

$$\lambda_2 = \frac{1}{3}(a_1 + a_2\sqrt{2} + a_3\sqrt{3} + a_4\sqrt{6})$$

for $a_i = 0, 1, 2$ not all zero.

In this case it is easy to note that $(1+\sqrt{3})\sqrt{2}/2$ when multiplied by $(1-\sqrt{3})\sqrt{2}/2$ gives -1 and the sum of the values is $\sqrt{2}$. Thus they are roots of $x^2 - \sqrt{2}x - 1$, i.e. roots of $x^4 - 4x^2 + 1$. But the first of these values is $(\sqrt{2}+\sqrt{6})/2$ which is one of the values we are after.

Adding this to our original $\mathbb{Q}$-basis and reducing to a four element basis again, we get the $\mathbb{Q}$-basis $\{1, \sqrt{2}, \sqrt{3}, (\sqrt{2}+\sqrt{6})/2\}$. The discriminant of this must be $\Delta/2^2 = 2^8 3^2$.

We must now look for algebraic integers of the form

$$\lambda_1 = \frac{1}{2}(a_1 + a_2\sqrt{2} + a_3\sqrt{3} + a_4(\sqrt{2}+\sqrt{6})/2)$$

for $a_i = 0, 1$ not all zero, and

$$\lambda_2 = \frac{1}{3}(a_1 + a_2\sqrt{2} + a_3\sqrt{3} + a_4(\sqrt{2}+\sqrt{6})/2)$$

for $a_i = 0, 1, 2$ not all zero.

Applying each of the monomorphismsand adding gives us the trace, which is $2a_2$ in the case of λ_1 and $4a_1/3$ in the case of λ_2. In the first case the trace being an integer for λ_1 and algebraic integer, gives us no information. But in the case of λ_2 it gives us that $a_1/3$ is an integer, which we can then subtract from λ_2. We thus only need to look for algebraic integers of the form

$$\lambda_2 = \frac{1}{3}(a_2\sqrt{2} + a_3\sqrt{3} + a_4(\sqrt{2}+\sqrt{6})/2)$$

for $a_i = 0, 1, 2$ not all zero, in the second case.

We now compute the norm of λ_1 which is $\omega/16$ where (after a painful calculation - I used Pari to do the arithmetic), $\omega = a_1^4 + (-4a_2^2 - 4a_4a_2 + (-6a_3^2 - 4a_4^2))a_1^2 + (24a_4a_3a_2 + 12a_4^2a_3)a_1 + (4a_2^4 + 8a_4a_2^3 - 12a_3^2a_2^2 + (-12a_4a_3^2 - 4a_4^3)a_2 + (9a_3^4 - 12a_4^2a_3^2 + a_4^4))$. Plugging in the fifteen possibilities, we find none is zero modulo 16 (again I used Pari to do the arithmetic).

Similarly the norm of λ_2 is $\omega/27$ where $\omega = 4a_2^4 + 8a_4a_2^3 - 12a_3^2a_2^2 + (-12a_4a_3^2 - 4a_4^3)a_2 + (9a_3^4 - 12a_4^2a_3^2 + a_4^4)$. Plugging in each of the 26 possibilities we find none that are zero modulo 27.

Thus there are no algebraic integers of either of the two forms and this $\mathbb{Q}$-basis must have dsicriminant equal to the discriminant of K. Thus the discriminant of K is in fact $2^8 3^2$, a fact which Pari easily verifies.

Sadly there is no way around the boring calculations in this algorithm. Obviously it is unlikely I would ask you to do such a computation in full on an exam, however it is important to understand every step of the technique, since I may ask you to complete any part of such a computation on the exam.

3. Let $\mathcal{P} = (2, \sqrt{-5})$ and $\mathcal{Q} = (2, 1+\sqrt{-5})$ be ideals in the ring $\mathbb{Z}[\sqrt{-5}]$. Compute $\mathcal{P}\mathcal{Q}$ and $\mathcal{P}+\mathcal{Q}$. (Give both the sum and product ideals in terms of one or two generators.)

We have that $\mathcal{P}+\mathcal{Q} = (2, \sqrt{-5}, 1+\sqrt{-5}) = (1, \sqrt{-5}) = \mathbb{Z}[\sqrt{-5}]$.

This is not a suprise since $\sqrt{-5}\sqrt{-5} = -5 \in \mathcal{P}$. Thus $-5 + 3 \times 2 \in \mathcal{P}$, i.e. $\mathcal{P} = (1, \sqrt{-5}) = \mathbb{Z}[\sqrt{-5}]$.

We also have $\mathcal{P}\mathcal{Q} = \mathbb{Z}[\sqrt{-5}]\mathcal{Q} = \mathcal{Q}$. This can also be seen from $\mathcal{P}\mathcal{Q} = (4, 2 + 2\sqrt{-5}, 2\sqrt{-5}, -5 + \sqrt{-5}) = (2, 2\sqrt{-5}, 1 + \sqrt{-5}) = (2, 1 + \sqrt{-5}) = \mathcal{Q}$.

4. Compute the number of cosets of the ideal $\mathcal{P} = (2, 1 + \sqrt{-5})$ in $R = \mathbb{Z}[\sqrt{-5}]$, i.e. compute the order of R/P and show that $\mathcal{P}$ is a maximal ideal of R. Is it prime?

Elements of $\mathcal{P} = (2, 1+\sqrt{-5})$ are of the form $2(a+b\sqrt{-5})+(1+\sqrt{-5})(c+d\sqrt{-5}) = (2a + c - 5d) + (2b + c + d)\sqrt{-5} = r + s\sqrt{-5}$ for $a, b, c, d, \in \mathbb{Z}$. Thus r and s can be anything integers of the same parity.

Adding any element $m + n\sqrt{-5}$ with m, n of opposite parity to (the generators) of $\mathcal{P}$ yields the whole of $\mathbb{Z}[\sqrt{-5}]$, thus $\mathcal{P}$ is maximal. It is therefore prime, since all maximal ideals are prime.

There are clearly two cosets of $\mathcal{P}$ in R, namely $0 + \mathcal{P}$ and $1 + \mathcal{P}$, since every element of R is of the form $0 + x$ or $1 + x$ for some $x \in \mathcal{P}$. Thus the order of R/P is two, i.e. the norm of $\mathcal{P}$ is 2.

This could also be found by noting that $\mathcal{P}^2 = (2)$ and $\mathcal{N}((2)) = |\mathcal{N}(2)| = 4$.

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