MA3A6 WEEK 5 ASSIGNMENT : DUE MONDAY 4PM WEEK 5

BILL HART

1. Determine all the field conjugates of $\beta = 1 + \sqrt{\frac{1+\sqrt{5}}{2}}$ for the number field $K = \mathbb{Q}(\alpha)$ where α is a root of $f(x) = x^4 - x^2 - 1$.

Let $g(x) = x^2 - x - 1$, then $f(x) = g(x^2)$. The roots of $g(x)$ are $\frac{1\pm\sqrt{5}}{2}$, thus the roots of $g(x)$ are $\pm\sqrt{\frac{1\pm\sqrt{5}}{2}}$.

Since we are after all the field conjugates of β, we should write β in terms of one of the roots of $f(x)$. Then all we have to do is replace this root of $f(x)$ with each of the other roots of $f(x)$ in turn to get the field conjugates of β.

Let's pick the root $\alpha = \sqrt{\frac{1+\sqrt{5}}{2}}$ of $f(x)$. We note that $\beta = \alpha + 1$. So replacing α with each of its four conjugates in turn, in the expression for β, we get $1 \pm \sqrt{\frac{1\pm\sqrt{5}}{2}}$. These four values are the field conjugates of β.

2. Let K be a number field and let $\sigma_1, \sigma_2, \ldots, \sigma_n$ be the monomorphisms from K into $\mathbb{C}$. The absolute norm of an element $\beta \in K$ is defined to be $\mathcal{N}(\beta) = \prod_i \beta_i$ where $\beta_i = \sigma_i(\beta)$.

Let $K = \mathbb{Q}(\alpha)$ where α is a root of $x^4 - x^2 - 1$. Determine $\mathcal{N}\left(\frac{3+\sqrt{5}}{2}\right)$ for the number field K. (Note the norm depends on the number field.)

Let α be as in the previous solution. We want the norm of $\gamma = \frac{3+\sqrt{5}}{2} = 1 + \alpha^2$. We replace α with each of its conjugates in this expression for γ to get the field conjugates of γ. We get $1 + \frac{1+\sqrt{5}}{2}, 1 + \frac{1+\sqrt{5}}{2}, 1 + \frac{1-\sqrt{5}}{2}, 1 + \frac{1-\sqrt{5}}{2}$ are the field conjugates of γ. The norm of γ in this field is the product of these field conjugates, i.e. $\left(1 - \left(\frac{3+\sqrt{5}}{2}\right)\right)^2 = 1^2 = 1$.

3. Prove that all quadratic number fields are Galois.

All quadratic number fields are of the form $K = \mathbb{Q}(\sqrt{d})$ for some $d \in \mathbb{Z}$. The roots of the minimum polynomial are $\sqrt{d}$ and $-\sqrt{d}$ which are both in K. Thus all roots of the minimum polynomial of the generator are in K and thus a quadratic field is Galois.

4. A number field is said to be totally real if each of the conjugate roots of the defining polynomial is real.

Consider the cyclotomic number field $\mathbb{Q}(\zeta_p)$ for a prime p, where $\zeta_p^p = 1$ and $\zeta_p \neq 1$.

Let $L = \mathbb{Q}(\beta)$ be the subfield of K generated by the element $\beta = \zeta_p + \zeta_p^{-1}$. Show that L is totally real.

Since p is prime, the minimum polynomial is $(x^n - 1)/(x - 1)$. The roots of this are the p-th roots of unity, excluding 1, i.e. they are ζ_p^i for $i = 1, \ldots, (p-1)$.

Since β is in K, the roots of its minimum polynomial are amongst the field conjugates of β. But the field conjugates of β are just $\beta_i = \zeta_p^i + \zeta_p^{-i}$. But ζ_p^i and ζ_p^{-i} are complex conjugates of each other, so their sum is real, i.e. each of the β_i are real. But if all the field conjugates of β are real, then all the conjugates of β are real. Thus the field $L = \mathbb{Q}(\beta)$ is totally real.

Is L Galois?

Yes. Each of the β_i can be written in terms of powers of β and is thus in $L = \mathbb{Q}(\beta)$.

To prove the last statement, consider $\beta_i = \zeta_p^i + \zeta_p^{-i}$. Start with $\beta^i = (\zeta_p + \zeta_p^{-1})^i$ and expand it out into an expression in terms of powers of ζ and ζ^{-1}. It starts $\beta^i = \zeta_p^i + \zeta_p^{-i} + p\zeta_p^{i-1} + p\zeta_p^{1-i} + \cdots$, i.e. $\beta_i = \beta^i + s(\zeta_p, \zeta_p^{-1})$, where $s(x, y) = s(y, x)$, i.e. the rest of the expression is the same if you swap ζ_p and ζ_p^{-1}, and where $s(x, y)$ has degree $i - 1$ in x and y.

But now we can write $s(\zeta_p, \zeta_p^{-1}) = p(\zeta_p + \zeta_p^{-1})^{i-1} + t(\zeta_p, \zeta_p^{-1})$ where $t(x, y) = t(y, x)$ and it has degree $i - 2$ in x and y, etc. We keep doing this and we eventually get that $\beta_i = \beta^i + p\beta^{i-1} + a_{i-2}\beta^{i-2} + \cdots + a_1\beta + a_0$ for some $a_i \in \mathbb{Z}$, i.e. $\beta_i \in L = \mathbb{Q}(\beta)$ for all i.

Thus L is Galois.

I didn't expect you to prove this, but well done if you did!

Those of you studying Galois theory may have had an easier time with this problem.

E-mail address: hart_wb@yahoo.com