5. SCHLÄFLI MODULAR EQUATIONS FOR GENERALIZED WEBER FUNCTIONS

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Abstract. Various generalizations of the classical Weber functions are developed. Schläfli type modular equations are explicitly derived for particular ones of these functions which are associated with a congruence subgroup $\Gamma^0(N)$ which is of genus zero.

Introduction

The purpose of this paper is twofold. Firstly we demonstrate that various satisfactory analogues of the classical Weber functions

$$\mathfrak{f}(\tau) = \frac{e^{-\frac{\pi i}{24}}\,\eta\left(\frac{\tau+1}{2}\right)}{\eta(\tau)}, \quad \mathfrak{f}_1(\tau) = \frac{\eta\left(\frac{\tau}{2}\right)}{\eta(\tau)}, \quad \mathfrak{f}_2(\tau) = \sqrt{2}\,\frac{\eta\,(2\tau)}{\eta(\tau)} \tag{1}$$

exist and that they satisfy identities akin to those derived by Weber for his functions, in §34 of [6]. That is, we derive analogues of the identities

$$\mathfrak{f}(\tau+1) = e^{-\frac{\pi i}{24}}\,\mathfrak{f}_1(\tau), \quad \mathfrak{f}_1(\tau+1) = e^{-\frac{\pi i}{24}}\,\mathfrak{f}(\tau), \quad \mathfrak{f}_2(\tau+1) = e^{\frac{\pi i}{12}}\,\mathfrak{f}_2(\tau),$$
$$\mathfrak{f}(-1/\tau) = \mathfrak{f}(\tau), \quad \mathfrak{f}_1(-1/\tau) = \mathfrak{f}_2(\tau), \quad \mathfrak{f}_2(-1/\tau) = \mathfrak{f}_1(\tau),$$
$$\mathfrak{f}\,\mathfrak{f}_1\mathfrak{f}_2 = \sqrt{2}, \quad \mathfrak{f}^8 = \mathfrak{f}_1^8 + \mathfrak{f}_2^8.$$

In particular we call the classical Weber functions (1), *functions of signature* 2, after the $\tau/2$ in the definition of $\mathfrak{f}_1$, and describe generalizations of these for other prime signatures N. In fact, one starts with the function $\frac{\eta(\tau/N)}{\eta(\tau)}$ and applies modular transformations to obtain a family of related functions of signature N analogous to the other Weber functions. This does not quite give our functions; we need to modify these slightly in each case as explained at the start of section 1. In that section and the following one, we illustrate what can be done by obtaining sets of functions of signatures 3 and 5 respectively. Dealing with different signatures separately in this way preserves, for the most part, the elegance of the identities that they then obey, which compare well with those of Weber as listed above.

The second purpose of this paper is to define Schläfli type modular equations for the functions $u(\tau) = \frac{\eta(\tau/N)}{\eta(\tau)}$ as Weber does in §73 of [6] for his functions.

For a prime *degree* p we define

$$P(\tau) = (u(\tau)u(p\tau))^k \text{ and } Q(\tau) = \left(\frac{u(\tau)}{u(p\tau)}\right)^l$$

for some natural numbers k and l.

A Schläfli modular equation is then a polynomial relation between two composite functions of the form

$$A = P + c/P \text{ and } B = Q \pm 1/Q,$$

for some $c \in \mathbb{R}$.

We obtain explicit Schläfli modular equations for signatures $N = 3, 5, 7$ and 13, for various prime degrees p, in sections 5, 6, 7 and 8 respectively.

The method of obtaining such modular equations is via modular functions for the congruence subgroup

$$\Gamma^0(N) = \left\{ \begin{pmatrix} \delta & \gamma \\ \beta & \alpha \end{pmatrix} \in SL_2(\mathbb{Z}) : \gamma \equiv 0 \ (\mathrm{mod}\ N) \right\}.$$

The extended complex upper half plane modulo the action of this group can be thought of as a Riemann surface whose genus is zero for only a finite number of N (see section 3). Such functions for a given $\Gamma^0(N)$ can be thought of as meromorphic functions on this associated Riemann surface of genus zero. These functions form a rational function field. We note that a generator for this field which induces a bijection between the Riemann surface associated to $\Gamma^0(N)$ and the complex sphere is called a Hauptmodul (or principal modulus). This can be made unique by normalizing it. The usual normalization is to require it to be 0 at points corresponding to $\tau = 0$ and ∞ at points corresponding to $\tau = i\infty$.

To create our modular equation for a prime degree p we form a set of composite functions F_c which satisfy the following conditions:

(i) The F_c are meromorphic;

(ii) They are permuted by $\Gamma^0(N)$;

(iii) They have q-expansions of level N;

(iv) They have no poles on the complex upper half plane;

(v) They are permuted at least up to sign by a so called Fricke involution taking the cusp $\tau = 0$ to the 'other' (inequivalent) cusp of $\Gamma^0(N)$, $\tau = i\infty$;

(vi) The q-series of F_∞ vanishes up to and including the constant term; and,

(vii) The q-series of F_∞ and of each of the other F_c are related in such a way that the vanishing of one up to and including the constant term (or altogether) implies the vanishing of all the others up to and including the constant term (or altogether, respectively).

Note that (i), (ii) and (iii) imply that $G = \prod_c F_c$ is a modular function for $\Gamma^0(N)$. Now (v), (vi) and (vii) together imply that G^2, and therefore also G, has a zero at $\tau = 0$ and $\tau = i\infty$.

Combine these facts with (iv) and Liouville's theorem and we see that G is a constant, which in this case is clearly zero.

In fact, in the sequel, we will obtain all of the above by specifying sets of functions A_c and B_c, permuted by $\Gamma^0(N)$ and permuted up to sign by the Fricke involution, and by defining $F_c = F(A_c, B_c)$ for some polynomial $F(x, y)$ in two arguments.

We will pick A_∞ and B_∞ to be the functions A and B spoken of above. Supposing the conditions (i) to (vii) above hold, then G is identically zero and thus one of its factors F_c is then identically zero. Thus F_c has a q-series which vanishes altogether. But (vii) above then guarantees that the q-series of all the other F_c vanish identically, and in particular F_∞ is identically zero. This provides us with the modular equation we are after, since $F_\infty = 0$ is by definition a polynomial relationship between A_∞ and B_∞.

It should be noted that the author's interest in these modular equations arose from a study of the use of modular equations in evaluating singular values of quotients of the Dedekind eta function. The final section of this paper details a simple evaluation of this kind by making use of the modular equations derived earlier. Further details will be found in the author's thesis [4].

1. Signature Three Functions

Referring to the three Weber functions $\mathfrak{f}, \mathfrak{f}_1, \mathfrak{f}_2$ as being of signature two, our first generalization is to a set of four new functions which we refer to as being of signature three. Denoting, as we shall when it is convenient, the n-th root of unity $e^{2\pi i/n}$ by ζ_n, we take the following

Definition 1.0.1. *Define the four functions*

$$\mathfrak{g}(\tau) = \sqrt{3}\,\frac{\eta(3\tau)}{\eta(\tau)}, \quad \mathfrak{g}_1(\tau) = \frac{\eta(\frac{\tau}{3})}{\eta(\tau)}, \quad \mathfrak{g}_2(\tau) = \frac{\eta(\frac{\tau+4}{3})}{\eta(\tau)}, \quad \mathfrak{g}_3(\tau) = \zeta_{12}^{-1}\frac{\eta(\frac{\tau+8}{3})}{\eta(\tau)}.$$

Various functions of this type already appear in the literature (see for example [3] or even [6] §72), however they appear there without the root of unity that we have added. This is included to simplify the modular transformation laws that these functions obey. With our new definitions for signature three, we have

Theorem 1.0.2.

$$\begin{pmatrix} \mathfrak{g} \\ \mathfrak{g}_1 \\ \mathfrak{g}_2 \\ \mathfrak{g}_3 \end{pmatrix} \circ S = \begin{pmatrix} \zeta_{12}\,\mathfrak{g} \\ \zeta_{12}^{-1}\,\mathfrak{g}_2 \\ \mathfrak{g}_3 \\ \mathfrak{g}_1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \mathfrak{g} \\ \mathfrak{g}_1 \\ \mathfrak{g}_2 \\ \mathfrak{g}_3 \end{pmatrix} \circ T = \begin{pmatrix} \mathfrak{g}_1 \\ \mathfrak{g} \\ \mathfrak{g}_3 \\ \mathfrak{g}_2 \end{pmatrix}$$

where S stands for the transformation $\tau \to \tau + 1$ and T for $\tau \to -1/\tau$.

Note: It will also be convenient to let S and T denote matrices associated to these fractional linear transformations. To this end all matrices and congruence subgroups in this paper will be thought of as belonging to the inhomogeneous modular group $\Gamma = SL_2(\mathbb{Z})/\{\pm I\}$ where I is the 2×2 identity matrix. (For more details on this see [5] I §2.)

Proof: We can obtain the first four of these relations by simple application of the transformation formula for the Dedekind eta function for $\tau \to \tau + 1$.

Since T is an involution, two of the final four identities follow from the other two. Thus it remains to prove only the first and third, say, of these identities.

The first follows easily from the transformation law of the eta function for the transformation $\tau \to -\frac{1}{\tau}$.

The other identity requires more effort however. Firstly we note that

$$\begin{aligned}
\mathfrak{g}_2\left(-\frac{1}{\tau+1}\right) &= \zeta_{24}\,\frac{\eta\left(\frac{\tau}{3\tau+3}\right)}{\eta\left(-\frac{1}{\tau+1}\right)} = \zeta_{24}\,\frac{\eta\left(\frac{1}{3+\frac{3}{\tau}}\right)}{\eta\left(-\frac{1}{\tau+1}\right)} \\
&= \zeta_{24}\sqrt{-i\left(-3-3/\tau\right)}\,\frac{\eta\left(-3-\frac{3}{\tau}\right)}{\eta\left(-\frac{1}{\tau+1}\right)} = \zeta_{12}^{-1}\sqrt{3i(1+1/\tau)}\,\frac{\eta\left(-\frac{3}{\tau}\right)}{\eta\left(-\frac{1}{\tau+1}\right)} \\
&= \zeta_{12}^{-1}\sqrt{3i(1+1/\tau)}\sqrt{-i\tau/3}\,\frac{\eta\left(\frac{\tau}{3}\right)}{\eta\left(-\frac{1}{\tau+1}\right)} = \zeta_{12}^{-1}\sqrt{\tau+1}\,\frac{\eta\left(\frac{\tau}{3}\right)}{\eta\left(-\frac{1}{\tau+1}\right)}.
\end{aligned}$$

Thus we see that

$$\mathfrak{g}_2\left(-\frac{1}{\tau}\right) = \zeta_{12}^{-1}\sqrt{\tau}\,\frac{\eta\left(\frac{\tau-1}{3}\right)}{\eta\left(-\frac{1}{\tau}\right)} = \zeta_{24}^{-5}\,\frac{\sqrt{\tau}}{\sqrt{-i\tau}}\,\frac{\eta\left(\frac{\tau+8}{3}\right)}{\eta\left(\tau\right)} = \mathfrak{g}_3(\tau).$$

□

We can now easily prove the following

Theorem 1.0.3. *The product of the four functions $\mathfrak{g}_i$ is a constant on the complex upper half plane:*

$$\mathfrak{g}(\tau)\,\mathfrak{g}_1(\tau)\,\mathfrak{g}_2(\tau)\,\mathfrak{g}_3(\tau) = \zeta_{12}\sqrt{3}.$$

Proof: From the previous theorem, it is clear that the product is a modular function invariant under all transformations in the full modular group. The product is thus an element of the modular function field $\mathbb{C}(j(\tau))$ where $j(\tau)$ is the absolute modular invariant.

Now, since the eta function is not zero on the upper half plane, it is clear from the definitions of the $\mathfrak{g}_i$ that this product has no poles or zeroes on the upper half plane. However, since $j(\tau)$ takes on every complex value in the upper half plane, this implies that our product is actually a constant.

We compute the leading term of the q-series for each of our four functions:

$$\mathfrak{g} = \sqrt{3}\,q^{\frac{1}{12}} + \ldots, \quad \mathfrak{g}_1 = q^{-\frac{1}{36}} - \ldots, \quad \mathfrak{g}_2 = \zeta_{36}^{2}\,q^{-\frac{1}{36}} - \ldots, \quad \mathfrak{g}_3 = \zeta_{36}\,q^{-\frac{1}{36}} - \ldots$$

The result then follows from multiplying these expressions. □

Theorem 1.0.4. *We have*

$$\mathfrak{g}(\tau)^6 - \mathfrak{g}_1(\tau)^6 - \mathfrak{g}_2(\tau)^6 + \mathfrak{g}_3(\tau)^6 = 0.$$

Proof: Once again we clearly have a modular function in $\mathbb{C}(j)$. However there are again no poles, so this function must be in $\mathbb{C}[j]$. However it is easy to check that the q-series has no negative powers of q as j does, and thus it is a constant. That constant is zero as follows from the q-series. □

2. Signature Five Functions

We take a further example of generalized Weber functions, this time of signature five.

Definition 2.0.5. *Define the six functions*

$$\mathfrak{h}(\tau) = \sqrt{5}\,\frac{\eta(5\tau)}{\eta(\tau)}, \quad \mathfrak{h}_1(\tau) = \frac{\eta(\frac{\tau}{5})}{\eta(\tau)}, \quad \mathfrak{h}_2(\tau) = \zeta_{12}\,\frac{\eta(\frac{\tau+6}{5})}{\eta(\tau)},$$

$$\mathfrak{h}_3(\tau) = \frac{\eta(\frac{\tau+12}{5})}{\eta(\tau)}, \quad \mathfrak{h}_4(\tau) = \zeta_{12}^{-1}\,\frac{\eta(\frac{\tau+18}{5})}{\eta(\tau)}, \quad \mathfrak{h}_5(\tau) = \zeta_{12}^{-2}\,\frac{\eta(\frac{\tau+24}{5})}{\eta(\tau)}.$$

We now prove

Theorem 2.0.6.

$$\begin{pmatrix} \mathfrak{h} \\ \mathfrak{h}_1 \\ \mathfrak{h}_2 \\ \mathfrak{h}_3 \\ \mathfrak{h}_4 \\ \mathfrak{h}_5 \end{pmatrix} \circ S = \begin{pmatrix} \zeta_6\,\mathfrak{h} \\ \zeta_6^{-1}\,\mathfrak{h}_2 \\ \mathfrak{h}_3 \\ \mathfrak{h}_4 \\ \mathfrak{h}_5 \\ \mathfrak{h}_1 \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} \mathfrak{h} \\ \mathfrak{h}_1 \\ \mathfrak{h}_2 \\ \mathfrak{h}_3 \\ \mathfrak{h}_4 \\ \mathfrak{h}_5 \end{pmatrix} \circ T = \begin{pmatrix} \mathfrak{h}_1 \\ \mathfrak{h} \\ \mathfrak{h}_5 \\ \mathfrak{h}_3 \\ \mathfrak{h}_4 \\ \mathfrak{h}_2 \end{pmatrix}$$

Proof: Once again, the first eight of these identities follow easily from the appropriate transformation laws of the eta function. Clearly the last of the identities on the right follows from the third, thus it only remains to prove the three remaining identities. We have

$$\mathfrak{h}_2\left(-\frac{1}{\tau+1}\right)=\zeta_{12}\,\frac{\eta\left(\frac{\tau}{5(\tau+1)}+1\right)}{\eta\left(-\frac{1}{\tau+1}\right)}=\zeta_{24}^3\,\frac{\sqrt{-i\left(-5(\tau+1)/\tau\right)}\eta\left(\frac{-5(\tau+1)}{\tau}\right)}{\sqrt{-i(\tau+1)}\eta\left(\tau+1\right)}$$
$$=\zeta_{12}^{-1}\,\frac{\sqrt{-5/\tau}\eta\left(\frac{-5}{\tau}\right)}{\eta\left(\tau+1\right)}=\zeta_{24}\,\frac{\eta\left(\frac{\tau}{5}\right)}{\eta\left(\tau+1\right)}.$$

Thus we have

$$\mathfrak{h}_2\left(-\frac{1}{\tau}\right)=\zeta_{24}\,\frac{\eta\left(\frac{\tau-1}{5}\right)}{\eta\left(\tau\right)}=\zeta_{24}\,\frac{\eta\left(\frac{\tau+24}{5}-5\right)}{\eta\left(\tau\right)}=\zeta_{12}^{-2}\frac{\eta\left(\frac{\tau+24}{5}\right)}{\eta\left(\tau\right)}=\mathfrak{h}_5(\tau).$$

Similarly we have

$$\mathfrak{h}_3\left(-\frac{1}{\tau+3}\right)=\frac{\eta\left(\frac{2\tau+5}{5(\tau+3)}+2\right)}{\eta\left(-\frac{1}{\tau+3}\right)}=\zeta_{12}\,\frac{\sqrt{-i\left(\frac{-5(\tau+3)}{2\tau+5}\right)}\eta\left(\frac{-5(\tau+3)}{2\tau+5}\right)}{\eta\left(-\frac{1}{\tau+3}\right)}$$
$$=\frac{\sqrt{-i\left(\frac{-5(\tau+3)}{2\tau+5}\right)}\eta\left(\frac{-\tau-5}{2\tau+5}\right)}{\eta\left(-\frac{1}{\tau+3}\right)}=\frac{\sqrt{\left(\frac{5(\tau+3)}{\tau+5}\right)}\eta\left(\frac{2\tau+5}{\tau+5}\right)}{\eta\left(-\frac{1}{\tau+3}\right)}$$
$$=\zeta_{12}\,\frac{\sqrt{\left(\frac{5(\tau+3)}{\tau+5}\right)}\eta\left(\frac{-5}{\tau+5}\right)}{\eta\left(-\frac{1}{\tau+3}\right)}=\zeta_{12}\,\frac{\eta\left(\frac{\tau+5}{5}\right)}{\eta\left(\tau+3\right)}.$$

Then we have

$$\mathfrak{h}_3\left(-\frac{1}{\tau}\right)=\zeta_{12}\,\frac{\eta\left(\frac{\tau+2}{5}\right)}{\eta\left(\tau\right)}=\frac{\eta\left(\frac{\tau+12}{5}\right)}{\eta\left(\tau\right)}=\mathfrak{h}_3(\tau)$$

Finally we have

$$\mathfrak{h}_4\left(-\frac{1}{\tau+2}\right)=\zeta_{12}^{-1}\,\frac{\eta\left(\frac{3\tau+5}{5(\tau+2)}+3\right)}{\eta\left(-\frac{1}{\tau+2}\right)}=\zeta_{24}\,\frac{\sqrt{-i\left(\frac{-5(\tau+2)}{3\tau+5}\right)}\eta\left(\frac{-5(\tau+2)}{3\tau+5}\right)}{\eta\left(-\frac{1}{\tau+2}\right)}$$
$$=\zeta_{24}^{-1}\,\frac{\sqrt{-i\left(\frac{-5(\tau+2)}{3\tau+5}\right)}\eta\left(\frac{\tau}{3\tau+5}\right)}{\eta\left(-\frac{1}{\tau+2}\right)}=\zeta_{24}^{-1}\,\frac{\sqrt{-5(\tau+2)/\tau}\,\eta\left(\frac{-(3\tau+5)}{\tau}\right)}{\eta\left(-\frac{1}{\tau+2}\right)}$$
$$=\zeta_{12}^{-2}\,\frac{\sqrt{-5(\tau+2)/\tau}\,\eta\left(\frac{-5}{\tau}\right)}{\eta\left(-\frac{1}{\tau+2}\right)}=\zeta_{12}\,\frac{\eta\left(\frac{\tau}{5}\right)}{\eta\left(\tau+2\right)},$$

so that we have

$$\mathfrak{h}_4\left(-\frac{1}{\tau}\right)=\zeta_{12}\,\frac{\eta\left(\frac{\tau-2}{5}\right)}{\eta\left(\tau\right)}=\zeta_{12}^{-1}\,\frac{\eta\left(\frac{\tau+18}{5}\right)}{\eta\left(\tau\right)}=\mathfrak{h}_4(\tau)$$

□

Now we can prove

Theorem 2.0.7. *The product of the six functions $\mathfrak{h}_i$ is a constant on the complex upper half plane:*

$$\mathfrak{h}(\tau)\,\mathfrak{h}_1(\tau)\,\mathfrak{h}_2(\tau)\,\mathfrak{h}_3(\tau)\,\mathfrak{h}_4(\tau)\,\mathfrak{h}_5(\tau) = \zeta_{12}\,\sqrt{-5}.$$

Proof: The previous theorem shows the product is in $\mathbb{C}(j)$. Since there are no poles or zeroes, it is a constant which we determine from the q-series.

$$\mathfrak{h} = \sqrt{5}\,q^{\frac{1}{6}} + \ldots, \quad \mathfrak{h}_1 = q^{-\frac{1}{30}} - \ldots, \quad \mathfrak{h}_2 = \zeta_{15}^2\,q^{-\frac{1}{30}} - \ldots,$$
$$\mathfrak{h}_3 = \zeta_{10}\,q^{-\frac{1}{30}} - \ldots, \quad \mathfrak{h}_4 = \zeta_{15}\,q^{-\frac{1}{30}} - \ldots, \quad \mathfrak{h}_5 = \zeta_{30}\,q^{-\frac{1}{30}} - \ldots$$

Multiplying these expressions gives the stated result. □

Theorem 2.0.8. *We have*

$$\mathfrak{h}(\tau)^6 + \mathfrak{h}_1(\tau)^6 + \mathfrak{h}_2(\tau)^6 + \mathfrak{h}_3(\tau)^6 + \mathfrak{h}_4(\tau)^6 + \mathfrak{h}_5(\tau)^6 = -30.$$

Proof: Again the left hand side of the identity has no poles in the upper half plane and so is in $\mathbb{C}[j]$. Examining the q series tells us that it is a constant, and moreover that this constant is -30. □

3. Hauptmoduln for $\Gamma^0(N)$

We note that from the theorems in the previous two sections, $\mathfrak{g}_1^{12}$ is invariant under the transformations STS and S^3. These generate the group $\Gamma^0(3)$. Likewise $\mathfrak{h}_1^6$ is invariant under S^5, STS, S^2TS^{-2} and S^3TS^{-3} which generate $\Gamma^0(5)$. We will make use of the well known fact that for $p = 2, 3, 5, 7, 13$ (the primes p for which $\Gamma^0(p)$ is genus zero - see below)

$$h_p(\tau) = \left(\frac{\eta(\tau/p)}{\eta(\tau)}\right)^{\frac{24}{p-1}}$$

is a Hauptmodul for $\Gamma^0(p)$ (see for example [1] §7.2.1 for a conjugate statement).

We recall some facts about the groups $\Gamma^0(N)$. According to [5] IV. §5.5, $\Gamma^0(p)$, for p an odd prime, is generated by S^p and all the elements of the form $S^vTS^{-v'}$, where $vv' + 1 \equiv 0 \pmod{p}$.

Let $\mathcal{F}$ be the usual fundamental region for the full modular group, then a fundamental region for $\Gamma^0(p)$ is given by

$$\mathcal{F}_p = \left[\bigcup_{-\frac{p-1}{2} \le i \le \frac{p-1}{2}} S^i(\mathcal{F})\right] \cup T(\mathcal{F}).$$

The group $\Gamma^0(p)$ therefore has index $p+1$ in the modular group Γ.

The group $\Gamma^0(N)$ is not normal in Γ and is conjugate to $\Gamma_0(N)$.

The genus of $\Gamma^0(N)$ is zero for precisely the values $N = 1, \ldots, 10, 12, 13, 16, 18, 25$.

We finish this section by noting that the Hauptmoduln $\mathfrak{g}_1^{12}$ and $\mathfrak{h}_1^6$ satisfy polynomial equations over $\mathbb{C}(j)$ of degree equal to the index of $\Gamma^0(3)$ or $\Gamma^0(5)$ in the modular group.

For $\mathfrak{g}_1^{12}$ note that Weber has a similar function x_0. In fact in his notation we have $\mathfrak{g}_1^{12} = x_0^2$. But he shows that x_0 is a root of

$$x^4 + 18x^2 + \gamma_3\,x - 27 = 0.$$

That is

$$(x^4 + 18x^2 - 27)^2 = \gamma_3^2 x^2.$$

But since $\gamma_3^2 = j - 1728$ then this is a quartic equation in x^2 over $\mathbb{C}(j)$. This is precisely what is required, since the index of $\Gamma^0(3)$ in Γ is four.

Also $\mathfrak{h}_1^6 = x_0^3$ in Weber's notation. But by rearranging what Weber has, x_0 satisfies

$$(x^6 + 10x^3 + 5)^3 = \gamma_2^3 x^3.$$

But since $\gamma_2^3 = j$ then this is a sextic in x^3 over $\mathbb{C}(j)$, and six is the index of $\Gamma^0(5)$ in Γ.

4. Functions Permuted by $\Gamma^0(N)$

In order to construct Schläfli modular equations for signature N we will need to construct functions invariant under $\Gamma^0(N)$. Our first step is to construct functions which are permuted up to sign by $\Gamma^0(N)$.

In §38 of [6], Weber calculates a formula for a linear transformation of the eta function. He has the following

Definition 4.0.9. *Define a function E by*

$$\eta\left(\frac{\delta\tau+\gamma}{\beta\tau+\alpha}\right) = E\left(\begin{pmatrix} \delta & \gamma \\ \beta & \alpha \end{pmatrix}; \tau\right) \eta(\tau)$$

for each

$$\begin{pmatrix} \delta & \gamma \\ \beta & \alpha \end{pmatrix} \in SL_2(\mathbb{Z}).$$

He then gives the formula

Lemma 4.0.10.

$$E\left(\begin{pmatrix} \delta & \gamma \\ \beta & \alpha \end{pmatrix}; \tau\right) = \begin{cases} \left(\frac{\beta}{\alpha}\right) i^{\frac{\alpha-1}{2}} e^{\frac{\pi i}{12}[\alpha(\gamma-\beta)-(\alpha^2-1)\beta\delta]}\sqrt{\alpha+\beta\tau}, & \textit{if } \alpha > 0 \textit{ is odd} \\ \left(\frac{\alpha}{\beta}\right) i^{\frac{1-\beta}{2}} e^{\frac{\pi i}{12}[\beta(\alpha+\delta)-(\beta^2-1)\alpha\gamma]}\sqrt{-i(\alpha+\beta\tau)}, & \textit{if } \beta > 0 \textit{ is odd} \end{cases}$$

involving Jacobi symbols.

For $N \in \mathbb{N}$ define

$$g_N(\tau) = \frac{\eta(\tau/N)}{\eta(\tau)}.$$

Theorem 4.0.11. *Let $A = \begin{pmatrix} \delta & \gamma \\ \beta & \alpha \end{pmatrix} \in \Gamma^0(N)$ be such that α is odd, then*

$$g_N(A\tau) = \left(\frac{N}{|\alpha|}\right) e^{-\frac{\pi i}{12}(N-1)[\alpha(\gamma/N+\beta)+(\alpha^2-1)\beta\delta]} g_N(\tau) \tag{2}$$

Proof: Note that

$$\frac{1}{N}\frac{\delta\tau+\gamma}{\beta\tau+\alpha}=\frac{(\delta/N)(N\tau)+\gamma}{\beta(N\tau)+(N\alpha)}=\frac{\delta(\tau/N)+(\gamma/N)}{(N\beta)(\tau/N)+\alpha} \tag{3}$$

Now suppose that α is positive. Since $N|\gamma$, then from the second of the expressions in (3) and Lemma (4.0.10) we have

$$\begin{aligned} g_N(A\tau) &= \frac{E\left(\left(\begin{array}{cc}\delta & \gamma/N\\ N\beta & \alpha\end{array}\right);\tau/N\right)}{E\left(\left(\begin{array}{cc}\delta & \gamma\\ \beta & \alpha\end{array}\right);\tau\right)}\frac{\eta(\tau/N)}{\eta(\tau)}\\ &= \left(\frac{N}{\alpha}\right)e^{-\frac{\pi i}{12}(N-1)[\alpha(\gamma/N+\beta)+(\alpha^2-1)\beta\delta]}\,g_N(\tau)\end{aligned}$$

Now if α is negative, we can multiply $\alpha,\beta,\gamma,\delta$ by -1. Note A then represents the same fractional linear transformation, but α is now positive. This observation leads to the stated result. $\square$

For notational convenience we will write this theorem as follows

$$g_N(A\tau)=\left(\frac{N}{|\alpha|}\right)\nu_N(\alpha,\beta,\gamma,\delta)\,g_N(\tau).$$

Note that ν_N is a root of unity, and if

$$s_N=\frac{24}{\gcd(24,N-1)} \tag{4}$$

then in fact, it is an s_N^{th} root of unity. Note that in particular if N is odd then $s_N|12$.

Theorem 4.0.12. *Let $g(\tau)$ be a function, periodic with even period $P\in\mathbb{N}$ such that $N|P$. Suppose for $A=\left(\begin{array}{cc}\delta & \gamma\\ \beta & \alpha\end{array}\right)\in\Gamma^0(N)$ and α odd that*

$$g(A\tau)=\left(\frac{N}{|\alpha|}\right)\nu(\alpha,\beta,\gamma,\delta)\,g(\tau) \tag{5}$$

then if p is an odd prime with $(p,P)=1$, then if $p|\beta$,

$$g(pA\tau)=\left(\frac{N}{|\alpha|}\right)\nu(\alpha',\beta',\gamma',\delta')\,g(p\tau) \tag{6}$$

where $\alpha'=\alpha$, $\beta'=\beta/p$, $\gamma'=p\gamma$, $\delta'=\delta$; and if $p\not|\,\beta$,

$$g(pA\tau)=\left(\frac{N}{|\alpha'|}\right)\nu(\alpha',\beta',\gamma',\delta')\,g(\tau'/p) \tag{7}$$

where $\alpha'=\frac{\alpha+Pmc'\beta}{p}$, $\beta'=\beta$, $\gamma'=\gamma+Pmc'\delta$, $\delta'=p\delta$ and $\tau'=\tau-Pmc'$, where $c'\equiv-\alpha/\beta$ (mod p) and Pm is the smallest positive multiple of P congruent to 1 (mod p).

Proof: Note that

$$p\,\frac{\delta\tau+\gamma}{\beta\tau+\alpha}=\frac{\delta(p\tau)+(p\gamma)}{(\beta/p)(p\tau)+\alpha}=\frac{(p\delta)(\tau/p)+\gamma}{\beta(\tau/p)+(\alpha/p)} \tag{8}$$

If $p|\beta$, then using the first expression in (8) and the given transformation law for g, we obtain (6).

If $p \not| \beta$, then we may use the second expression, except that now it may be that $p \not| \alpha$. But since $p \not| \beta$, we can set $c' \equiv -\alpha/\beta \pmod p$. Then setting $\tau' = \tau - Pmc'$, the second expression of (8) becomes

$$\frac{(p\delta)(\tau'/p) + (\gamma + Pmc'\delta)}{\beta(\tau'/p) + \left(\frac{\alpha + Pmc'\beta}{p}\right)}.$$

Note that $N|P$ and hence $N|\gamma'$, the determinant of this transformation with respect to (τ'/p) is still 1, and α' is still odd, and so by the transformation formula for g, our result follows. □

Similarly we have

Theorem 4.0.13. *With hypotheses as per the previous theorem, if* $p|(\delta + Pmc\beta)$ *then*

$$g((A\tau + Pmc)/p) = \left(\frac{N}{|\alpha|}\right) \nu(\alpha', \beta', \gamma', \delta')\, g(\tau/p) \tag{9}$$

where $\alpha' = \alpha$, $\beta' = p\beta$, $\gamma' = (\gamma + Pmc\alpha)/p$ *and* $\delta' = \delta + Pmc\beta$*; and otherwise, if* $p \not| (\delta + Pmc\beta)$ *then*

$$g((A\tau + Pmc)/p) = \left(\frac{N}{|\alpha|}\right) \nu(\alpha', \beta', \gamma', \delta')\, g(\tau'/p) \tag{10}$$

where $\alpha' = \alpha$, $\beta' = p\beta$, $\gamma' = (\gamma + Pmc'\alpha)/p$, $\delta' = \delta + Pmc'\beta$, $\tau' = \tau - Pmc'$ *and* $c' \equiv -\frac{\gamma + c\alpha}{\delta + c\beta}$ *(mod p).*

Proof: Note that

$$\frac{1}{p}\left(\frac{\delta\tau + \gamma}{\beta\tau + \alpha} + Pmc\right) = \frac{1}{p}\frac{(\delta + Pmc\beta)\tau + (\gamma + Pmc\alpha)}{\beta\tau + \alpha} \tag{11}$$

the determinant of which, with respect to τ, is 1. Also, since we have $N|P$, then $N|(\gamma + Pmc\alpha)$.

We can use (3) (replace N by p throughout) to rewrite the right hand side of (11). This results in two new expressions for (11). If $p|(\delta + Pmc\beta)$ we can use the first of these expressions, otherwise, use the second. In the first case, the result (9) follows immediately. In the second case it may be that $p \not| (\gamma + Pmc\alpha)$. But since $p \not| (\delta + Pmc\beta)$ we can write

$$c' \equiv -\frac{\gamma + Pmc\alpha}{\delta + Pmc\beta} \equiv -\frac{\gamma + c\alpha}{\delta + c\beta} \pmod p,$$

and setting $\tau' = \tau - Pmc'$ we obtain the given result (10). □

Theorem 4.0.14. *Let N be odd and P, the period of* g_N*, be even. Let p be a prime with* $(p, 6P) = 1$*. Let* p' *be the smallest positive residue of p (mod* S_N*) with* S_N *defined as per (4), then*

$$v_{N,c} = \frac{g_N((\tau + Pmc)/p)}{g_N(\tau)^{p'}} \quad \text{and} \quad v_{N,\infty} = \frac{g_N(p\tau)}{g_N(\tau)^{p'}} \quad \text{for } -(p-1)/2 \le c \le (p-1)/2$$

are permuted up to sign by $\tau \to A\tau$ *where* $A = \begin{pmatrix} \delta & \gamma \\ \beta & \alpha \end{pmatrix} \in \Gamma^0(N)$ *and* α *is odd. The sign is given by* $\left(\frac{N}{|\alpha'|}\right)\left(\frac{N}{|\alpha|}\right)$ *where* α' *is the appropriate value as given by the appropriate theorem (4.0.12) or (4.0.13).*

Proof: We first prove if N is odd then $N|P$, and in fact $24|(N-1)P/N$. Since $\eta(\tau+1) = \zeta_{24}\,\eta(\tau)$, it is clear that a 24-th root of unity is induced by taking $\tau \to \tau + N$ in $\eta(\tau/N)$. Thus $g_N(\tau)$ is periodic with minimal period $P = tN$ for t as small as possible, where $tN \equiv t \pmod{24}$. In particular it is always true that $N|P$ and $24|t(N-1) = (N-1)P/N$, hence our claim.

To prove our theorem we merely need to show that

$$\nu_N(\alpha', \beta', \gamma', \delta') = \nu_N(\alpha, \beta, \gamma, \delta)^{p'}$$

where $\alpha', \beta', \gamma', \delta'$ are given by the various expression of the last two theorems.

Firstly as

$$\nu_N(\alpha', \beta', \gamma', \delta') = e^{-\frac{\pi i}{12}(N-1)[\alpha'(\gamma'/N+\beta')+(\alpha'^2-1)\beta'\delta']}$$

then since $N-1$ is even, we are only interested in the values of $\alpha', \beta', \gamma', \delta' \pmod{12}$. Also since $(p, 6) = 1$ then $p \equiv 1, 5, 7$ or $11 \pmod{12}$ and so $p \equiv 1/p \pmod{12}$.

Also, since $(N, 2) = 1$ and $24|(N-1)P/N$ we have that

$$(N-1)\frac{\gamma'}{N} = (N-1)\frac{(\gamma + Pmc'\delta)}{N} \equiv (N-1)\frac{\gamma}{N} \pmod{24}$$

Applying these results, $\nu_N(\alpha', \beta', \gamma', \delta')$ becomes $\nu_N(\alpha, p\beta, p\gamma, \delta)$ in (6), (9) and (10) and $\nu_N(p\alpha, \beta, \gamma, p\delta)$ in (7).

A simple calculation then shows that in all cases $\nu_N(\alpha', \beta', \gamma', \delta') = \nu_N(\alpha, \beta, \gamma, \delta)^{p'}$ as required, since $p \equiv p' \pmod{12}$.

It is clear that the signs that appear in the transformations of this theorem are as given. □

It appears that this theorem can be extended to even N by merely raising the functions $\upsilon_{N,c}$ and $\upsilon_{N,\infty}$ to the power of two to compensate for the fact that $(N-1)$ is no longer divisible by two.

We can subject the signs that appear in this theorem to further analysis and obtain a more convenient expression. For example

Theorem 4.0.15. *The signs appearing in the preceeding theorem are given by +1 in the case of equations* (6)*,* (9) *and* (10) *and* $\left(\frac{N}{p}\right)$ *in the case of equation* (7)*.*

Proof: In the case of the first three equations referenced we have that $\alpha' = \alpha$, which leads immediately to the given result.

In the case of equation (7) we note that as the transformations we are dealing with are in $\Gamma^0(N)$ then we must have that $\gcd(N, \alpha') = \gcd(N, \alpha) = 1$. We also know that α and α' are odd. Thus we can use the quadratic reciprocity law, generalized for Jacobi symbols, to determine that the required sign is given by

$$(-1)^{\frac{(N-1)(|\alpha'|+|\alpha|-2)}{4}}\left(\frac{|\alpha'|}{N}\right)\left(\frac{|\alpha|}{N}\right).$$

Now since $p|\alpha'| \equiv |\alpha| \pmod{N}$, from the expression for α' belonging to equation (7), then this sign becomes

$$(-1)^{\frac{(N-1)(|\alpha'|+|\alpha|-2)}{4}}\left(\frac{p}{N}\right).$$

It is clear that at worst, this expression depends on N, $|\alpha|$ and $|\alpha'|$ modulo 8. But since $24|(N-1)P/N$ as per the proof of the above theorem, we see from the expression for α' that $(N-1)|\alpha| \equiv (N-1)p|\alpha'| \pmod{8}$, so that the sign is given by

$$(-1)^{\frac{(N-1)((p+1)|\alpha'|-2)}{4}}\left(\frac{p}{N}\right).$$

Since α' is odd, this can be further simplified to

$$(-1)^{\frac{(N-1)(p-1)}{4}}\left(\frac{p}{N}\right)=\left(\frac{N}{p}\right),$$

as required. □

Let us denote the numerators of the functions $v_{N,c}$ and $v_{N,\infty}$ by v_c and v_∞ respectively, for an odd N which we now consider to be fixed.

If we change the definition of these functions v_c we can arrange that the sign in the previous theorem becomes $+1$ in all cases. We note that the case represented by equation (7) is only relevant when we have a permutation taking v_∞ to some v_c, with $c \neq \infty$, or vice versa. Thus we multiply the functions v_c, for $c \neq \infty$, by $\left(\frac{N}{p}\right)$ to obtain a sign of $+1$ in all cases. For convenience we record the definitions of these functions here

$$u(\tau)=g_N(\tau), \qquad v_\infty(\tau)=g_N(p\tau),$$

$$v_c(\tau)=\left(\frac{N}{p}\right)g_N\left(\frac{\tau+Pmc}{p}\right) \quad \text{for} \quad -(p-1)/2\le c\le (p-1)/2.$$

From now on we refer to the functions v_c and v_∞ collectively as the v_c. Of course we also adjust the numerators of the $v_{N,c}$ appropriately.

We will assume for now that $\Gamma^0(N)$ has a set of generators of the form $\begin{pmatrix}\delta & \gamma\\ \beta & \alpha\end{pmatrix}$ where α is odd. Again we check this for individual N of interest later. Thus we can speak of the functions $v_{N,c}$ as being permuted by $\Gamma^0(N)$.

We wish to construct functions of the form

$$P_c=(uv_c)^k \tag{12}$$

for some $k\in\mathbb{N}$, which are permuted by $\Gamma^0(N)$, and similarly for functions

$$Q_c=(v_c/u)^l \tag{13}$$

for some $l\in\mathbb{N}$.

In addition, we will want the P_c to be permuted exactly as the Q_c. These functions are obtained by multiplying or dividing the functions $v_{N,c}$, etc., by some power of u which is known to be invariant under $\Gamma^0(N)$. For suppose that u^m is known to be such a power, with $m\in\mathbb{N}$ as small as possible. We know that the $v_{N,c}=v_c/u^{p'}$ are permuted. Thus let

$$P_c=(v_c/u^{p'})^k\,u^{ma};\;\; a\in\mathbb{Z} \tag{14}$$

where $(p'+1)k=ma$. These satisfy the requirements for the P_c. Also choose

$$Q_c=(v_c/u^{p'})^l\,u^{mb};\;\; b\in\mathbb{Z} \tag{15}$$

where $(p'-1)l=mb$. These are also permuted, and in precisely the same way as the P_c. We note that these equations are soluble if we set

$$k=\frac{m}{\gcd(p'+1,m)} \quad \text{and} \quad l=\frac{m}{\gcd(p'-1,m)}. \tag{16}$$

Now we not only want functions which have the properties just mentioned but which are also invariant at least up to sign under a Fricke involution, $\tau\to -N/\tau$. To this end we prove

Lemma 4.0.16. *The transformation $\tau\to-\frac{N}{\tau}$ sends g_N to $\frac{\sqrt{N}}{g_N}$.*

Proof: From the definition,

$$g_N(N\tau) = \frac{\eta(\tau)}{\eta(N\tau)}.$$

Sending $\tau \to -1/\tau$ and applying the transformation formula of the Dedekind eta function to the right hand side, we obtain the required result. $\square$

We now know what the Fricke involution does to $u = g_N(\tau)$ and we are in a position to see what it does to the v_c.

Using the lemma, it is trivial to determine the Fricke involutory action on v_∞ and v_0.

$$v_\infty : \ \ v_\infty(-N/\tau) = g_N\left(-Np/\tau\right) = \frac{\sqrt{N}}{g_N\left(\tau/p\right)} = \left(\frac{N}{p}\right)\frac{\sqrt{N}}{v_0}$$

For the other v_c we solve

$$\begin{pmatrix} 1 & Pmc \\ 0 & p \end{pmatrix}\begin{pmatrix} 0 & -N \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \tau \\ 1 \end{pmatrix} = \begin{pmatrix} \delta & \gamma \\ \beta & \alpha \end{pmatrix}\begin{pmatrix} 0 & -N \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & A \\ 0 & p \end{pmatrix}\begin{pmatrix} \tau \\ 1 \end{pmatrix}$$

where $A = Pmk$ for some integer k, and where the first matrix on the right is in $SL_2(\mathbb{Z})$. Note in particular that since $N|P$ always, we have that A/N is an integer.

This matrix equation yields the four equations

$$Pmc = \gamma, \quad -N = -Np\delta + \gamma A, \quad p = \alpha, \quad 0 = -Np\beta + \alpha A$$

Clearly the last two of these yield $\beta = \frac{A}{N}$ and the first two yield

$$\delta = \frac{(P/N)mcA + 1}{p}. \tag{17}$$

Once again for $\begin{pmatrix} \delta & \gamma \\ \beta & \alpha \end{pmatrix} \in SL_2(\mathbb{Z})$ we check that the determinant is one and we see that demanding that $A = Pmk$ only leaves us to fulfil the requirement that $\delta \in \mathbb{Z}$, i.e.

$$(P/N)mcA + 1 \equiv 0 \ (\mathrm{mod}\ p). \tag{18}$$

Since $(p, N) = 1$ and $Pm \equiv 1$ (mod p), then so long as we do not have $c \equiv 0$ (mod p) this is equivalent to

$$k \equiv -\frac{N}{c} \ (\mathrm{mod}\ p)$$

as one might have expected.

We again use our formula (2), and the lemma (4.0.16) to deduce

$$v_c\left(-\frac{N}{\tau}\right) = \left(\frac{N}{|\delta|}\right)\frac{\sqrt{N}}{v_k}. \tag{19}$$

This expression can be simplified. Since γ is divisible by N, then $(\delta, N) = 1$. Clearly from (17) we have that δ is odd. Thus by the quadratic law for Jacobi symbols, the sign in (19) is $(-1)^{\frac{(N-1)(|\delta|-1)}{4}}\left(\frac{|\delta|}{N}\right)$. However $p|\delta| \equiv 1$ (mod N) and $|\delta| \equiv p$ (mod 4) and so (19) becomes

$$v_c\left(-\frac{N}{c}\right) = \left(\frac{N}{p}\right)\frac{\sqrt{N}}{v_k}.$$

We are now in a position to do calculations for individual N.

5. Schläfli-type Modular Equations for Signature Three

We now derive modular equations where the signature $N = 3$.

In section 3 we noted that $\Gamma^0(3)$ was generated by STS and S^3. Alternatively, by applying the well known identity $STSTST = I$ we see that TST and S^3 can be taken as generators.

These transformations are represented by matrices of the form $\begin{pmatrix} \delta & \gamma \\ \beta & \alpha \end{pmatrix}$ with α odd as required. In fact TST is represented by $\begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}$ and S^3 by $\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$.

As per the calculation in the proof of theorem (4.0.14) we note that the period of our function $g_3(\tau)$ of signature three (i.e. the function $\mathfrak{g}_1(\tau)$ of section 1) is $P = 36$. Note that the period is even as required.

The signs that are induced by the Fricke involution are given by $\left(\frac{3}{p}\right)$.

For $N = 3$ we have $S_N = 12$ as per (4). Thus $p' \equiv p \pmod{12}$ in equation (16). Since the Hauptmodul for $\Gamma^0(3)$ is u^{12} and is invariant under $\Gamma^0(3)$ we can set $m = 12$. Now we can use (16) to determine values for k and l for any particular prime p we are considering.

As we hinted in the introduction, we initially define the functions

$$A_c = P_c \pm 3^k/P_c \quad \text{and} \quad B_c = Q_c \pm 1/Q_c$$

with P_c and Q_c defined as in (12) and (13), and with the signs for A_c and B_c somehow dependent on the signs induced by the Fricke involution as it acts on P_c and Q_c respectively. It is always possible, for example, to choose these signs so that the Fricke involution permutes the A_c and B_c with no sign changes; thus all the A_c and B_c are permuted without sign changes by $\Gamma^0(N)$ and by the Fricke involution.

Now, as per condition (vi) of the introduction, we find a polynomial in two arguments, $F(x, y)$, such that the q-series of $F_\infty = F(A_\infty, B_\infty)$ vanishes up to and including the constant term. However, even in the cases that this can be done, we still need to satisfy condition (vii) of the introduction, i.e. we need to be able to show that by virtue of the vanishing of the q-series of F_∞ we also have the vanishing of the q-series of the other $F_c = F(A_c, B_c)$. Thus we need to show that the q-series of the various A_c are related in an appropriate way, and similarly for the B_c.

Consider the q-series $A_\infty(q_N)$ of the function A_∞, where $q_N = \exp\left(\frac{2\pi i\tau}{N}\right)$ for some $N \in \mathbb{N}$. The substitution

$$\tau \to \frac{\tau}{p} \tag{20}$$

takes $A_\infty(q_N)$ to $A_\infty(q_{pN})$. But as functions of τ this substitution takes $uv_\infty \to \left(\frac{N}{p}\right) v_0 u$. This will not always take A_∞ to A_0, depending of course on whether k is odd or even in the definition of P_c.

This same transformation will take B_∞ to $\pm B_0$ depending on the index l in the definition of Q_c and the sign that we choose in the definition of the B_c. (It is important to note for this case, that (u/v_∞) goes to (v_0/u).)

The problem here is that the components of F_0, namely A_0 and B_0, may not have q-series related to the components of F_∞ in the same way. Thus we may not be able to guarantee that the q series of F_∞ and F_0 both vanish at the same time.

For the other F_c, the substitution

$$\tau \to \tau + 36mc \tag{21}$$

takes q_{pN} in the q-series of A_0 to $\exp\left[\left(\frac{2\pi i}{pN}\right)36mc\right]q_{pN} = \zeta^c q_{pN}$, where ζ is some root of unity. This transformation always sends A_0 to A_c as functions. Likewise this substitution sends B_∞ to B_0. Thus in these cases it is clear that if the q-series of F_0 vanishes, then so do those of the other F_c.

This leaves us only to sort out the problem we mentioned above for F_∞. There are two different methods that we can use to solve this problem, depending on the signature N that we are working with. We will deal with each case as it arises.

All primes p which we can consider for $N = 3$ must not divide $6P = 216$. These are all congruent to 1, 5, 7 or 11 modulo 12. We will deal with each of these cases separately.

I. For $\mathbf{p \equiv 1 \pmod{12}}$

From equation (16) we have $k = 6$ and $l = 1$. Thus $P_c = (uv_c)^6$ and $Q_c = (v_c/u)$ are permuted in precisely the same manner by TST and S^3. The Fricke involution always induces the sign $+1$ in this case. Thus the sets of functions

$$A_c = (uv_c)^6 + 3^6/(uv_c)^6 \tag{22}$$

$$B_c = (v_c/u) + (u/v_c) \tag{23}$$

are permuted by $\Gamma^0(3)$ and the Fricke involution.

The q-series of F_∞ and F_0 are then always related in the appropriate way. We find appropriate polynomials $F(x, y)$ below.

II. For $\mathbf{p \equiv 5 \pmod{12}}$

In this case we have $k = 2$ and $l = 3$. The Fricke involution induces -1. Thus we choose the definitions

$$A_c = (uv_c)^2 + 3^2/(uv_c)^2 \tag{24}$$

$$B_c = (v_c/u)^3 - (u/v_c)^3. \tag{25}$$

The transformation (20), in this case, takes A_∞ to A_0 and B_∞ to B_0 as required (since $-(v_\infty/u)^3$ goes to $(u/v_0)^3$, etc.).

III. For $\mathbf{p \equiv 7 \pmod{12}}$

We have $k = 3$ and $l = 2$. The Fricke involution again induces -1. With $k = 3$ we initially try $A_c = (uv_c)^3 - 3^6/(uv_c)^3$, however (20) sends $(uv_\infty)^3$ to $-(v_0u)^3$ sending A_∞ to $-A_0$, but sends $(u/v_\infty)^2$ to $(v_0/u)^2$, i.e. B_∞ to B_0. This is the first case where we need to alter our functions A_c and B_c so that the q-series obey condition (vii) of the introduction.

The way that we avoid the problem is simply to raise P_c to the power of two. We define

$$A_c = (uv_c)^6 + 3^6/(uv_c)^6 \tag{26}$$

$$B_c = (v_c/u)^2 + (u/v_c)^2. \tag{27}$$

Now the A_c and B_c are permuted in the required manner, as are their q-series.

This is not the only possible solution to the problem, but it is the only solution which we have found which always appears to lead to a modular equation for signature 3.

IV. For $\mathbf{p \equiv 11 \pmod{12}}$

We have $k = 1$ and $l = 6$, the Fricke involution inducing $+1$. We easily check that the following functions and their q-series are permuted as required

$$A_c = (uv_c) + 3/(uv_c) \tag{28}$$

$$B_c = (v_c/u)^6 + (u/v_c)^6. \tag{29}$$

We now deal with some small primes p and derive the q-series for the corresponding A_∞ and B_∞ and determine a polynomial $F(x, y)$ such that the q-series of F_∞ vanishes up to and including the constant term, as required. These are our modular equations. It will be convenient to express the various q-series with $q = e^{2\pi i\tau}$ replaced by a higher power of q in order to avoid fractional exponents. This does not affect the resulting modular equations.

In addition, we denote our q-series by a leading power and a series of coefficients, e.g. $q^{-1} + 2 - 4q^2 + O(q^3)$ will be written $(q^{-1}; 1, 2, 0, -4, \ldots)$, to save space (and the eyes).

For $p = 5$:

$$A_\infty(q^3) = (q^{-1}; 1, -2, \ldots)$$
$$B_\infty(q^3) = (q^{-1}; 1, 3, \ldots)$$

For $p = 7$:

$$A_\infty(q^3) = (q^{-4}; 1, -6, 9, 16, -66, \ldots)$$
$$B_\infty(q^3) = (q^{-1}; 1, 2, 6, 6, 15, \ldots)$$

For $p = 11$:

$$A_\infty(q^3) = (q^{-1}; 1, -1, 2, 4, 5, 6, \ldots)$$
$$B_\infty(q^3) = (q^{-5}; 1, 6, 27, 92, 279, 756, \ldots)$$

For $p = 13$:

$$A_\infty(q^3) = (q^{-7}; 1, -6, 9, 16, -66, 54, 98, -300, \ldots)$$
$$B_\infty(q^3) = (q^{-1}; 1, 1, 3, 1, 3, 6, 6, 9, \ldots)$$

For $p = 17$:

$$A_\infty(q^3) = (q^{-3}; 1, -2, -1, 4, -3, 0, 16, 10, 42, 86, 130, 230, 422, \ldots)$$
$$B_\infty(q^3) = (q^{-4}; 1, 3, 9, 19, 42, 81, 155, 276, 485, 824, 1368, 2206, 3550, \ldots)$$

Thus we have the modular equations

$p = 5: \quad A = B - 5,$

$p = 7: \quad A = B^4 - 14B^3 + 45B^2 + 56B - 250,$

$p = 11: \quad B = A^5 + 11A^4 + 51A^3 + 121A^2 + 144A + 66,$

$p = 13: \quad A = B^7 - 13B^6 + 45B^5 + 52B^4 - 493B^3 + 351B^2 + 1215B - 1404,$

$p = 17: \quad A^4 = B^3 - 17AB^2 + 34A^2B + 34A^3 - 238B^2 - 442AB - 389A^2 + 1244B + 1428A - 1556.$

Note that the first three of these modular equations are equivalent to modular equations arising from Ramanujan's alternative cubic theory as recounted in equations (7.22), (7.27) and (7.33) of [2].

6. Schläfli-type Modular Equations for Signature Five

The argument is similar to the signature three case. Here $N = 5$ and we calculate that the Hauptmodul, which is of course invariant under $\Gamma^0(5)$, is u^6. Also we have $S_N = 6$ and $P = 30$.

The group $\Gamma^0(5)$ is generated by S^5, TST, $S^{-3}TS^3$ and S^3TS^{-3}. These are represented by matrices

$$\begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} -3 & -10 \\ 1 & 3 \end{pmatrix}, \begin{pmatrix} 3 & -10 \\ 1 & -3 \end{pmatrix}.$$

The signs induced by the Fricke involution are $\left(\frac{5}{p}\right) = \left(\frac{p}{5}\right)$.

These signs clearly depend only on p modulo 5, whilst p' will depend on p modulo 6. Thus we must break into cases modulo 30.

We must also check what the transformations

$$\tau \to \tau/p \quad \text{and} \quad \tau \to \tau + 30mc$$

do to the q-series of v_∞ and v_0, etc.

I. For $\mathbf{p \equiv 1, 19}$ (mod 30)

We have $k = 3$ and $l = 1$. The Fricke involution induces +1. Thus set

$$A_c = (uv_c)^3 + 5^3/(uv_c)^3$$
$$B_c = (v_c/u) + (u/v_c).$$

The q-series are permuted the appropriate way by the transformations above.

II. For $\mathbf{p \equiv 7, 13}$ (mod 30)

Here $k = 3$ and $l = 1$. The Fricke involution induces -1. But in this case, the first transformation above takes uv_∞ to $-v_0u$, thus taking A_∞ to $-A_0$, whilst at the same time it takes B_∞ to B_0. This is the same problem which we mentioned in the signature three case. However the same technique we used there does not lead to modular equations here (valid polynomials $F(x, y)$ making the q-series of F_∞ vanish do not always exist).

For this signature it appears that the correct technique is to adjust our signs so that the Fricke involution induces a -1 when applied to each of the A_c and B_c

$$A_c = (uv_c)^3 + 5^3/(uv_c)^3$$
$$B_c = (v_c/u) + (u/v_c).$$

We then only take terms in our polynomial $F(x, y)$ whose total orders are of the same parity. This ensures that each of the F_c induce -1 when the Fricke involution is applied to them.

Also the q-series of both A_∞ and B_∞ change sign together under the tranformations listed above. Thus again the q-series of F_∞ always vanishes with that of F_0.

III. For $\mathbf{p \equiv 11, 29}$ (mod 30)

Here $k = 1$ and $l = 3$. The Fricke involution induces +1. So we choose

$$A_c = (uv_c) + 5/(uv_c)$$
$$B_c = (v_c/u)^3 + (u/v_c)^3.$$

The q-series are permuted appropriately by the transformations.

IV. For $\mathbf{p \equiv 17, 23}$ (mod 30)

Here $k = 1$ and $l = 3$. The Fricke involution induces -1. Again uv_∞ becomes $-v_0u$ sending A_∞ to $-A_0$ whilst sending B_∞ to B_0. Thus we choose

$$A_c = (uv_c) + 5/(uv_c)$$
$$B_c = (v_c/u)^3 + (u/v_c)^3.$$

We are again able to give examples of such modular equations for small p.

For $p = 7$:

$$A_\infty(q^5) = (q^{-4}; 1,\ -3,\ 0,\ 5,\ 0,\ 3,\ -16,\ -3,\ 1,\ 49, \ldots)$$
$$B_\infty(q^5) = (q^{-1}; 1,\ 1,\ 3,\ 2,\ 4,\ 6,\ 10,\ 14,\ 17, \ldots)$$

For $p = 11$:

$A_\infty(q^5) = (q^{-2}; 1,\ -1,\ -1,\ 5,\ 7,\ 9,\ 15,\ 25,\ 30,\ 53, \ldots)$

$B_\infty(q^5) = (q^{-5}; 1,\ 3,\ 9,\ 22,\ 51,\ 105,\ 212,\ 402,\ 744,\ 1326,\ 2317, \ldots)$

For $p = 13$:

$A_\infty(q^5) = (q^{-7}; 1, -3,\ 0,\ 5,\ 0,\ 3, -16, 0,\ 15,\ 0,\ 18, -48,\ 0,\ 42,\ 134,\ 413,\ 981,$ $2750,\ 6476,\ 13173,\ 26599,\ 49911,\ 93000,\ 165951,\ 289644,\ 492207,\ 821848,\ 1347750,$ $2177286, \ldots)$

$B_\infty(q^5) = (q^{-2}; 1,\ 1,\ 2,\ 3,\ 6,\ 5,\ 9,\ 13,\ 19,\ 27,\ 33,\ 44,\ 60,\ 75,\ 102,\ 123,\ 159,$ $201,\ 255,\ 318,\ 393,\ 484,\ 605,\ 734,\ 913,\ 1091,\ 1334,\ 1615,\ 1950, \ldots)$

and the resulting modular equations are

$p = 7:\ \ A^2 - B^8 + 14AB^3 + 43B^6 - 70AB - 475B^4 + 1325B^2 = 0,$

$p = 11:\ \ B^2 - A^5 - 11A^2B - 11A^4 - 110AB - 30A^3 - 275B - 125A - 629 = 0,$

$p = 13;\ \ A^4 - B^{14} + 26A^3B^3 + 221A^2B^6 + 624AB^9 + 274B^{12} - 78A^3B - 1066A^2B^4 - 4264AB^7 - 21267B^{10} - 1859A^2B^2 - 6760AB^5 + 516752B^8 + 11200A^2 + 62400AB^3 - 5189595B^6 - 26000AB + 24476050B^4 - 54513625B^2 + 46962500.$

The modular equations from here on, although able to be calculated, become somewhat unwieldy.

7. Schläfli-type Modular Equations for Signature Seven

For $N = 7$, the Hauptmodul is u^4, $S_N = 4$ and $P = 28$, which is even. The group $\Gamma^0(7)$ is generated by S^7, TST, $S^{-5}TS^{-3}$, $S^{-3}TS^{-5}$ represented by

$$\begin{pmatrix} 1 & 7 \\ 0 & 1 \end{pmatrix},\quad \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix},\quad \begin{pmatrix} -5 & 14 \\ 1 & -3 \end{pmatrix},\quad \begin{pmatrix} -3 & 14 \\ 1 & -5 \end{pmatrix},$$

all with odd lower right entries, as required.

The signs induced by the Fricke involution are $\left(\frac{7}{p}\right)$. These depend on p modulo 28, whilst p' depends on p modulo 4. Thus we break into cases modulo 28.

I. For $\mathbf{p \equiv 1, 9, 25}$ (mod 28) Here $k = 2$ and $l = 1$. The Fricke involution induces $+1$. We choose

$$A_c = (uv_c)^2 + 7^2/(uv_c)^2$$
$$B_c = (v_c/u) + (u/v_c).$$

II. For $\mathbf{p \equiv 3, 19, 27}$ (mod 28)

Here $k = 1$ and $l = 2$. The Fricke involution induces $+1$. Thus we choose

$$A_c = (uv_c) + 7/(uv_c)$$
$$B_c = (v_c/u)^2 + (u/v_c)^2.$$

III. For $\mathbf{p \equiv 5, 13, 17}$ (mod 28)

Here $k = 2$ and $l = 1$. The Fricke involution induces -1. We choose

$$A_c = (uv_c)^2 + 7^2/(uv_c)^2$$
$$B_c = (v_c/u) - (u/v_c).$$

The q-series of these are related in the correct way, with A_∞ being taken to A_0 and B_∞ to B_0 (since (v_∞/u) goes to $-(u/v_0)$ under the transformation $\tau \to \tau/7$, etc.).

IV. For $\mathbf{p \equiv 11, 15, 23}$ (mod 28)

Here $k = 1$ and $l = 2$. The Fricke involution induces -1. Again uv_∞ becomes $-v_0u$ sending A_∞ to $-A_0$ whilst B_∞ is sent to B_0. It turns out that this problem can be fixed by using the same method as for signature three. That is, we simply square P_c. We choose

$$A_c = (uv_c)^2 + 5^2/(uv_c)^2$$
$$B_c = (v_c/u)^2 + (u/v_c)^2.$$

The relevant q-series for small primes p are now given.

For $p = 3$:

$A_\infty(q^7) = (q^{-1}; 1, -1, \ldots)$

$B_\infty(q^7) = (q^{-1}; 1, 2, \ldots)$

For $p = 5$:

$A_\infty(q^7) = (q^{-3}; 1, -2, -1, 2, \ldots)$

$B_\infty(q^7) = (q^{-1}; 1, 1, 1, 4, \ldots)$

For $p = 11$:

$A_\infty(q^7) = (q^{-6}; 1, -2, -1, 2, 1, 2, -2, 2, -6, -4, 5, 0, 57, 98, 24, 472, 969, 1780, 3186, 5312, 8869, 14216, 22550, 34962, 53782, 80842, 120785, 177576, 259172, 373692, 535135, \ldots)$

$B_\infty(q^7) = (q^{-5}; 1, 2, 5, 10, 20, 36, 65, 108, 181, 290, 462, 708, 1088, 1632, 2425, 3546, 5143, 7370, 10484, 14758, 20662, 28646, 39492, 54024, 73561, 99428, 133823, 179054, 238515, 316062, 417166, \ldots)$

For $p = 13$:

$A_\infty(q^7) = (q^{-7}; 1, -2, -1, 2, 1, 2, -2, 2, -6, -4, 5, 2, 4, -4, 61, 84, 234, 500, 981, 1776, 3173, 5320, \ldots)$

$B_\infty(q^7) = (q^{-3}; 1, 1, 2, 3, 5, 7, 10, 15, 22, 28, 39, 50, 70, 87, 119, 152, 196, 247, 317, 394, 499, 625, \ldots)$

The associated modular equations are

$p = 3 : A = B - 3,$

$p = 5 : A = B^3 - 5B^2 + 3B - 5,$

$p = 11 : A^5 - B^6 + 22A^4B + 187A^3B^2 + 726A^2B^3 + 1155AB^4 + 396B^5 + 22A^4 + 308A^3B + 825A^2B^2 - 5808AB^3 - 23688B^4 + 250A^3 + 990A^2B - 12936AB^2 - 94908B^3 + 6776A^2 + 60764AB + 221531B^2 + 45665A + 135520B + 122850 = 0,$

$p = 13 : A^3 - B^7 + 13A^2B^2 + 52AB^4 + 39B^6 - 39AB^3 - 345B^5 + 13A^2 + 117AB^2 - 65B^4 + 195AB + 1299B^3 - 121A - 1105B^2 + 2255B - 1573 = 0.$

8. Schläfli-type Modular Equations for Signature Thirteen

For $N = 13$, the Hauptmodul is u^2, $S_N = 2$ and $P = 26$, which is even. The group $\Gamma^0(13)$ is generated by S^{13}, TST, $S^{-11}TS^7$, S^3TS^9, S^5TS^{-5}, S^7TS^{-11}. Again it is easy to check that these are represented by matrices with odd lower right entries. The signs induced by the Fricke involution are $\left(\frac{13}{p}\right) = \left(\frac{p}{13}\right)$. These depend on p modulo 13, whilst p' depends on p modulo 2. Thus we break into cases modulo 26.

I. For $\mathbf{p \equiv 1, 3, 9, 17, 23, 25 \pmod{26}}$ Here $k = 1$ and $l = 1$. The Fricke involution induces +1. Thus we choose

$$A_c = (uv_c) + 13/(uv_c)$$
$$B_c = (v_c/u) + (u/v_c).$$

II. For $\mathbf{p \equiv 5, 7, 11, 15, 19, 21 \pmod{26}}$ Here $k = 1$ and $l = 1$. The Fricke involution induces -1. We easily see that A_∞ is taken to $-A_0$ whilst B_∞ is taken to

B_0 by $\tau \to \tau/13$. We again have the problem that we have previously mentioned. In this case we again use the method which worked for signature five, i.e. we change the signs from the expected -1 to +1 in the definition of A_c and B_c. That is, we choose

$$A_c = (uv_c) + 13/(uv_c)$$
$$B_c = (v_c/u) + (u/v_c).$$

This again allows everything to permute as required, except that we again need to choose our polynomials $F(x, y)$ so that each of its terms has total order of the same parity.

The relevant q-series are

For $p = 3$:

$$A_\infty(q^{13}) = (q^{-2}; 1, -1, -1, \ldots)$$
$$B_\infty(q^{13}) = (q^{-1}; 1, 1, 3, \ldots)$$

For $p = 5$:

$$A_\infty(q^{13}) = (q^{-3}; 1, -1, -1, 0, 0, 0, 14, 15, 26, 39, 63, 105, 155, \ldots)$$
$$B_\infty(q^{13}) = (q^{-2}; 1, 1, 2, 3, 6, 5, 9, 13, 19, 27, 33, 44, 60, \ldots)$$

For $p = 7$:

$A_\infty(q^{13}) = (q^{-4}; 1, -1, -1, 0, 0, 1, 0, 0, 14, 14, 26, 39, 63, 92, 140, 207, 300, 416, 586, 793, 1092, 1443, 1966, 2574, 3406, \ldots)$

$B_\infty(q^{13}) = (q^{-3}; 1, 1, 2, 3, 5, 7, 12, 13, 20, 28, 39, 52, 70, 91, 117, 150, 196, 247, 317, 396, 503, 617, 774, 955, 1186, \ldots)$

The associated modular equations can now be determined.

$p = 3: \quad A = B^2 - 3B - 5,$

$p = 5: \quad A^4 - B^6 + 10A^3B + 45A^2B^2 + 100AB^3 + 106B^4 - 32A^2 - 160AB - 329B^2 + 260 = 0,$

$p = 7: \quad A^6 - B^8 + 14A^5B + 91A^4B^2 + 336A^3B^3 + 735A^2B^4 + 882AB^5 + 463B^6 - 50A^4 - 448A^3B - 1736A^2B^2 - 3108AB^3 - 2211B^4 + 625A^2 + 2450AB + 2725B^2 = 0.$

Again we must stop due to the amount of space required to record modular equations of higher degree.

9. Schläfli-type Modular Equations of Degree Two

At this stage we note that there are not yet modular equations of degree two for any of the signatures and no modular equation of degree three for signature five. In this and the next section we deal with these cases separately, to remedy this situation. Here we begin with degree two.

For a fixed signature $N = 3, 5, 7$ or 13 we introduce the following functions

Definition 9.0.17. *Let*

$$u(\tau) = g_N(\tau) \qquad v_\infty(\tau) = g_N(2\tau)$$
$$v_0(\tau) = g_N(\tau/2) \qquad v_1(\tau) = g_N((\tau + N)/2).$$

We need to investigate the action of a set of generators for $\Gamma^0(N)$ on these functions. We note that any generator of the form $S^{\nu'}TS^\nu$ can be represented by $\begin{pmatrix} \nu' & \nu\nu' - 1 \\ 1 & \nu \end{pmatrix}$ and TST by $\begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}$.

We choose the following generators for the respective $\Gamma^0(N)$

For $N = 3$: S^3 and STS.

For $N = 5$: S^5, TST, $S^{-3}TS^3$ and S^3TS^7.

For $N = 7$: S^7, STS, S^9TS^{-3} and $S^{-3}TS^9$.

For $N = 13$: S^{13}, TST, $S^{-11}TS^7$, S^3TS^{-17}, S^5TS^{-5}, S^7TS^{15}, $S^{-5}TS^{-21}$ and S^9TS^3.

We will investigate the effect of these generators on composite functions of the form $(uv_c)^2$. Taking the square here has some advantages. One of these is that we do not need to calculate any Jacobi symbols which arise when we use (2). It is an especially useful simplification in the present case, since it will also turn out that we are only interested in functions permuted up to sign.

Firstly we will investigate the action of the generators on the function $(uv_\infty)^2$. For such a generator $A = \begin{pmatrix} \delta & \gamma \\ \beta & \alpha \end{pmatrix}$ we note that

$$2A\tau = \frac{\delta(2\tau) + (2\gamma)}{(\beta/2)(2\tau) + \alpha} = \frac{(2\delta)(\tau/2) + \gamma}{\beta(\tau/2) + (\alpha/2)} = \frac{(2\delta)\left(\frac{\tau+N}{2}\right) + (\gamma - N\delta)}{\beta\left(\frac{\tau+N}{2}\right) + \left(\frac{\alpha - N\beta}{2}\right)}.$$

There are two cases that will interest us:

In the first case, if β is even, we use the first expression for $2A\tau$ above. Note that the only time this case occurs is for the generator $A = S^N$, so that plugging in the values for $\alpha, \beta, \gamma, \delta$ we have

$$v_\infty\left(S^N\tau\right) = \nu(1, 0, 2N, 1)\, v_\infty(\tau).$$

Thus using our expression (2) for ν we have that

$$(uv_\infty)^2\left(S^N\tau\right) = \left(e^{-\frac{\pi i}{12}(N-1)(1+2)} uv_\infty(\tau)\right)^2 = \pm(uv_\infty)^2(\tau)$$

the sign depending on whether $N = 5, 13$ or $N = 3, 7$ respectively.

The second case of interest is when β is odd and α is odd, and $(\alpha - \beta N)/2$ is an odd integer (something we easily check for the other transformations listed above). In this case, we use the third expression for $aA\tau$ above.

Firstly we consider the transformation of v_∞. As usual, our expression (2), used in conjunction with the third expression for $2A\tau$ above, will yield a factor involving $\nu(\alpha', \beta', \gamma', \delta')$ where the $\alpha', \beta', \gamma', \delta'$ are the coefficients that one has in that expression. To save space, we concentrate on that part of the *index* of this factor, of the form

$$\mu'(\alpha', \beta', \gamma', \delta') = \alpha'(\gamma'/N + \beta') + ({\alpha'}^2 - 1)\beta'\delta'.$$

We also look at the corresponding expression $\mu(\alpha, \beta, \gamma, \delta)$ obtained from the factor $\nu(\alpha, \beta, \gamma, \delta)$ for the transformation of u.

In considering the transformation of $(uv_\infty)^2$ we are thus interested in the expression $K = 2(\mu + \mu')(\alpha, \beta, \gamma, \delta)$ where clearly we set $\alpha' = (\alpha - N\beta)/2$, $\beta' = \beta$, $\gamma' = \gamma - N\delta$, $\delta' = 2\delta$ as we have just explained. This expression is precisely

$$K = 2\alpha(\gamma/N + 1) + 2(\alpha^2 - 1)\delta + (\alpha - N)(\gamma/N - \delta + 1) + ((\alpha - N)^2 - 4)\delta$$

since β is always 1 in the transformations we are interested in.

Now since $(N - 1) \mid 24$ for $N = 3, 5, 7$ and 13, then $S_N = 24/(N - 1)$. Thus we are specifically interested in this expression K modulo $S_N = 12, 6, 4$ or 2 respectively. Thus we need to look at our expression K modulo 3 for $N = 3$ and 5, modulo 4 for $N = 3$ and 7 and modulo 2 for $N = 5$ and 13.

Since $\alpha\delta - \beta\gamma = 1$ and $\beta = 1$ we make frequent use of the identity $\gamma = \alpha\delta - 1$. We also note that $N|\gamma = \alpha\delta - 1$, α and δ are always odd, and thus γ is always even.

Working modulo 4 for $N = 3, 7 \equiv -1 \pmod 4$ we have

$$\begin{aligned} K &\equiv -2\alpha(\gamma+1) + 2(\alpha^2-1)\delta + (\alpha+1)(-\gamma-\delta+1) + ((\alpha+1)^2-4)\delta \\ &\equiv \alpha\gamma - \alpha - \alpha^2\delta - 2\delta + \alpha\delta - \gamma + 1 \equiv -2\alpha - 2\delta + 2 \equiv 2 \pmod 4. \end{aligned}$$

Working modulo 2 for $N = 5, 11$ we have

$$K \equiv (\alpha-1)(\gamma-\delta+1) + (\alpha-1)^2\delta \equiv 0 \pmod 2.$$

Working modulo 3 for $N = 3$ we have

$$K \equiv -\alpha\delta \pmod 3.$$

But $3 = N|\alpha\delta - 1$ so $M \equiv -1 \pmod 3$.

Modulo 3 for $N = 5$ we have

$$K \equiv -2\alpha\delta - \gamma - 2 \equiv -1 \pmod 3.$$

So in all cases $K \equiv 2 \pmod{S_N}$.

Next we look at the action of all the generators on $(uv_0)^2$. It is clear that S^N always takes this function to $\zeta_{S_N}^{-2}(uv_1)^2$.

Now we consider the action of the other transformations. We note

$$\frac{A\tau}{2} = \frac{\delta(\tau/2) + (\gamma/2)}{(2\beta)(\tau/2) + \alpha}.$$

In a manner similar to the argument above, we look at the following expression modulo S_N

$$M = 2\alpha(\gamma/N + 1) + 2(\alpha^2-1)\delta + \alpha(\gamma/N + 4) + 4(\alpha^2-1)\delta.$$

Modulo 4 this clearly becomes

$$M \equiv \gamma \equiv \alpha\delta - 1 \equiv 0 \pmod 4$$

by checking each of the individual transformations listed above for $N = 3$ and 7.

Modulo 2 the expression becomes for $N = 5$ and 13

$$M \equiv N\alpha\gamma \equiv 0 \pmod 2.$$

Finally we have for $N = 3$ and 5 that $M \equiv 0 \pmod 3$.

Thus in all cases $M \equiv 0 \pmod{S_N}$.

Finally we investigate the effect of our transformations on $(uv_1)^2$. It is clear that S^N takes this expression to $\zeta_{S_N}^{-4}(uv_0)^2$.

For the other cases we note

$$\frac{A\tau + N}{2} = \frac{\left(\frac{\delta+\beta N}{2}\right)(2\tau) + (\gamma + \alpha N)}{\beta(2\tau) + (2\alpha)}.$$

This represents a transformation of 2τ by an element of $\Gamma^0(N)$, but we note that 2α is not odd. Thus we decompose the transformation into two others in $\Gamma^0(N)$ of the required form

$$\begin{pmatrix} \frac{\delta+\beta N}{2} & \gamma + \alpha N \\ \beta & 2\alpha \end{pmatrix} = \begin{pmatrix} \frac{\delta+\beta N}{2} & \gamma + \alpha N - N\left(\frac{\delta+\beta N}{2}\right) \\ \beta & 2\alpha - N\beta \end{pmatrix} \begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix}.$$

Here the relevant expression will be

$$P = 2\alpha(\gamma/N - 1) + 2(\alpha^2 - 1)\delta \\ + (2\alpha - N)(2\gamma/N + 2\alpha - (\delta + N) - 2) + ((2\alpha - N)^2 - 1)(\delta + N) + 2.$$

Firstly modulo 4, for $N = 3, 7 \equiv -1 \pmod 4$

$$P \equiv -2\alpha - 2\alpha N + N(\delta + N) + 2N + 2 \equiv -\delta + 1 \equiv 0 \pmod 4$$

checking all cases for $N = 3, 7$.

Clearly $P \equiv 0 \pmod 2$ in all relevant cases.

Finally, modulo 3 for $N = 3$ we have

$$P \equiv \alpha^2 - 2\alpha\delta + 2 \equiv 1 \pmod 3,$$

and for $N = 5$

$$P \equiv -\alpha\delta - 2\gamma + 1 \equiv 0 \pmod 3.$$

Thus again, $P \equiv 0 \pmod{S_N}$ in all cases, except for $N = 3$ where we have $P \equiv 4 \pmod{S_N}$.

Thus we have enough information to determine what roots of unity appear when we apply the various transformations to our functions $(uv_c)^2$. However, some of these roots of unity are inconvenient to work with. It is possible to obtain far simpler ones by modifying the definition of the function v_1. This will only affect the roots of unity that appear in transformations that take $(uv_1)^2$ to $(uv_0)^2$ and vice versa, or $(uv_1)^2$ to $(uv_\infty)^2$ and vice versa.

We change our definition to the following

$$v_1(\tau) = \zeta_{S_N}^{-1} g_N((\tau + N)/2).$$

This has the effect of changing the root of unity that appears when $(uv_0)^2$ is transformed by S^N, to 1, and the root of unity appearing when $(uv_1)^2$ is transformed by S^N, to $\zeta_{S_N}^6$. This is -1 in the cases $N = 3, 7$ and 1 in the cases $N = 5, 13$. In a similar way, the roots of unity appearing in the transformations between $(uv_\infty)^2$ and $(uv_1)^2$ are now ± 1 in the cases $N = 3, 7$, cube roots of unity for $N = 5$ and always 1 for $N = 13$.

We now combine this information with the information we had already obtained for the other roots of unity (which have not changed as a result of the new definition). This allows us to obtain a set of functions P_c which are invariant *up to sign* under all relevant transformations. In addition, using the fact that u^{S_N} is always completely invariant under all relevant transformations, we can construct further functions Q_c which are also permuted in the same way and with the same signs as the P_c.

For $N = 3$ choose $P_c = (uv_c)^2$ and $Q_c = (u/v_c)^6$.

For $N = 5$ choose $P_c = (uv_c)^3$ and $Q_c = (u/v_c)^3$.

For $N = 7$ choose $P_c = (uv_c)^2$ and $Q_c = (u/v_c)^2$.

For $N = 13$ choose $P_c = (uv_c)$ and $Q_c = (u/v_c)$.

Our aim now, is to construct functions A_c and B_c permuted *up to sign* by $\Gamma^0(N)$ and by the Fricke involution. Thus we look at the action of the latter.

Clearly we have

$$v_\infty(-N/\tau) = g_N(-2N/\tau) = \sqrt{N}/g_N(\tau/2) = \sqrt{N}/v_0(\tau).$$

For the action on v_1 we solve

$$\begin{pmatrix} 1 & N \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & -N \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \tau \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & -N \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \delta & \gamma \\ \beta & \alpha \end{pmatrix} \begin{pmatrix} 1 & \pm N \\ 0 & 2 \end{pmatrix}.$$

We obtain

$$\alpha = (1 \pm N)/2, \quad \beta = -1, \quad \gamma = \mp N \text{ and } \delta = 2. \tag{30}$$

For $N = 3, 7$ we pick the negative sign in the third matrix on the right, otherwise we pick the positive sign, so that α is always odd. If we pick the negative sign, the third matrix on the right hand side represents $(\tau - N)/2$ which we express as $S^{-N}((\tau + N)/2)$. This extra transformation by S^{-N} introduces an additional root of unity, i.e. ζ_{S_N}.

Ignoring this additional complication for now, the roots of unity that we need to consider are

$$\left(\frac{N}{|(1 \pm N)/2|}\right) \nu((1 \pm N)/2, -1, \mp N, 2) = \zeta_{S_N}^{(\pm 1 - 2 + N \pm 3N + N^2)/2},$$

from applying (2) to (30). Evaluating these and factoring in the extra ζ_{S_N} (for $N = 3, 7$) that we ignored, we obtain ζ_{S_N} for $N = 3$, $-\zeta_{S_N}^{22} = -\zeta_{S_N}^{4}$ for $N = 5$, $\zeta_{S_N}^{17} = \zeta_{S_N}$ for $N = 7$ and $-\zeta_{S_N}^{110} = -1$ for $N = 13$.

Now bearing in mind the definition of v_1 and that after applying the Fricke involution the roots of unity above end up in the denominator, we see that applying the Fricke involution to v_1 yields the roots of unity $\zeta_{S_N}^{-3}$ for $N = 3$, -1 for $N = 5$, $\zeta_{S_N}^{-3}$ for $N = 7$ and -1 for $N = 13$. In all cases we see that the Fricke involution induces a -1 when transforming any of the P_c or Q_c defined above.

Next we note that in every case the transformation $\tau \to \tau/2$ sends the coefficients of the q-series of P_∞ to those of P_0 and those of Q_∞ to those of Q_0, whilst the transformation $\tau \to \tau + N$ sends P_0 to P_1 and Q_0 to Q_1.

Putting all this together

For $N = 3$ choose $A_c = P_c + 3^2/P_c$ and $B_c = Q_c + 1/Q_c$.

For $N = 5$ choose $A_c = P_c + 3^3/P_c$ and $B_c = Q_c + 1/Q_c$.

For $N = 7$ choose $A_c = P_c + 3^2/P_c$ and $B_c = Q_c + 1/Q_c$.

For $N = 13$ choose $A_c = P_c + 3/P_c$ and $B_c = Q_c + 1/Q_c$.

Now the A_c and B_c are permuted up to sign by $\Gamma^0(N)$ and by the Fricke involution. Because of the possibility of a sign change, we must construct the polynomial $F(x, y)$ only with terms whose total orders have the same parity. This process also preserves the coefficients of the q-series of the F_c under the transformations listed above as required.

We now compute the relevant q-series.

For $N = 3$

$$A_0(q^6) = (q^{-1}; 1, 0, \ldots)$$
$$B_0(q^6) = (q^{-1}; 1, 0, \ldots)$$

For $N = 5$

$$A_\infty(q^{10}) = (q^{-3}, 1, 0, -3, 0, \ldots)$$
$$B_\infty(q^{10}) = (q^{-1}; 1, 0, 4, 0, \ldots)$$

For $N = 7$

$$A_\infty(q^{14}) = (q^{-3}, 1, 0, -2, 0, \ldots)$$

$$B_\infty(q^{14}) = (q^{-1}; 1,\ 0,\ 3,\ 0, \ldots)$$

For $N = 13$

$$A_\infty(q^{26}) = (q^{-3}, 1,\ 0, -1,\ 0, \ldots)$$
$$B_\infty(q^{26}) = (q^{-1}; 1,\ 0,\ 2,\ 0, \ldots)$$

These yield the modular equations

$N = 3,\ p = 2:\ A = B$

$N = 5,\ p = 2:\ A = B^3 - 15B$

$N = 7,\ p = 2:\ A = B^3 - 11B$

$N = 13,\ p = 2:\ A = B^3 - 7B.$

upon comparing q-series.

10. A Modular Equation of Signature Five, Degree Three

In order to obtain the last outstanding modular equation we define the functions

Definition 10.0.18.

$$u(\tau) = \mathfrak{h}_1(\tau), \quad v_\infty(\tau) = \mathfrak{h}_1(3\tau), \quad v_0(\tau) = \mathfrak{h}_1(\tau/3),$$
$$v_1(\tau) = \mathfrak{h}_1((\tau+5)/3), \quad v_2(\tau) = \mathfrak{h}_1((\tau+10)/3),$$

where $\mathfrak{h}_1(\tau)$ is the function of signature 5 defined in section 2.

We choose the following generators for $\Gamma^0(5)$

$$S^5,\ TST,\ S^2TS^8,\ S^8TS^2$$

where again we think of $S^{v'}TS^v$ as being represented by $\begin{pmatrix} \nu' & \nu\nu'-1 \\ 1 & \nu \end{pmatrix}$, etc.

Firstly we determine the action of S^5 on our functions. This can be done easily by referring to the expressions for the modular transformations of the functions of section 2. We construct the following table

	u	v_∞	v_0	v_1	v_2
$\tau \to \tau + 5$	$\zeta_6^{-1}\, u$	$-v_\infty$	v_1	v_2	$\zeta_6^{-1}\, v_0$

In particular, if we apply the transformation S^5 to $(uv_c)^2$ for the various subscripts c the root of unity that appears is ζ_3^{-1} for $(uv_\infty)^2$, $(uv_0)^2$ and $(uv_1)^2$, and ζ_3 for $(uv_2)^2$.

Now we deal with the three transformations that remain. Firstly we apply them to v_∞. We note that if the transformation being applied is given by the matrix

$$A = \begin{pmatrix} \delta & \gamma \\ \beta & \alpha \end{pmatrix},$$

then the transformation of the argument of $v_\infty = \mathfrak{h}_1(3\tau)$, by A, can be expressed

$$3A\tau = \frac{(3\delta)\left(\frac{\tau+5}{3}\right) + (\gamma - 5\delta)}{\beta\left(\frac{\tau+5}{3}\right) + \left(\frac{\alpha-5\beta}{3}\right)}.$$

Note that for each of the transformations we are considering, $\alpha - 5\beta$ is divisible by 3. It is also easy to calculate that the action of TST, S^2TS^8 and S^8TS^2 on u gives $\zeta_6\, u$, $\zeta_3^{-1}\, u$ and $\zeta_3^{-1}\, u$ respectively. We will denote the roots of unity that appear here by ζ_u in each case.

We are now in a position to calculate

$$(uv_\infty)(s^{v'}TS^v)^2 = \zeta_u^2\,\nu_5((\alpha-5\beta)/3,\beta,\gamma-5\delta,3\delta)^2\,(uv_1)(\tau)^2. \tag{31}$$

Examining the expression for ν_5 as given by theorem (4.0.11) we see that the first part of ν_5^2 has the form $\left(e^{-\frac{\pi i}{12}(N-1)}\right)^2 = \zeta_3^{-1}$. Thus we are only interested in the rest of the exponent of $\nu_5((\alpha-5\beta)/3,\beta,\gamma-5\delta,3\delta)$ modulo 3. But this is given by

$$\frac{\alpha-5\beta}{3}\left(\frac{\gamma-5\delta}{5}+\beta\right)+\left(\left(\frac{\alpha-5\beta}{3}\right)^2-1\right)3\beta\delta \equiv \frac{\alpha-5}{3}(1-\delta) \pmod 3,$$

since $\beta = 1$, $1/5 \equiv 2 \pmod 3$, $3|\alpha-5$ and $3|\gamma$ for all of the transformations we are interested in.

In all cases we see that $1-\delta \equiv -1 \pmod 3$ and so in summary we see that the root of unity appearing on the right side of (31) is ζ_3^{-1} for TST, S^2TS^8 and 1 for S^8TS^2.

We perform a similar computation for the transformation of $(uv_0)^2$ by each of our transformations. We write

$$\frac{A\tau}{3} = \frac{\delta(\tau/3)+(\gamma/3)}{(3\beta)(\tau/3)+\alpha}.$$

For all our transformations, $3|\gamma$, and so we have

$$(uv_0)(S^{v'}TS^v)^2 = \zeta_u^2\,\nu_5(\alpha,3\beta,\gamma/3\delta)^2\,(uv_0)^2. \tag{32}$$

We examine the appropriate part of the exponent of ν_5^2 modulo 3.

$$\alpha(\gamma/15+3\beta)+(\alpha^2-1)3\beta\delta \equiv \alpha\gamma/15 \pmod 3.$$

Thus the root of unity appearing on the right side of (32) is ζ_3 for TST, and ζ_3^{-1} for S^2TS^8 and S^8TS^2.

Now for the transformation of $(uv_1)^2$ we write

$$\frac{A\tau+5}{3} = \frac{\delta\left(\frac{\tau+10}{3}\right)+\frac{(\gamma-10\delta+5)}{3}}{(3\beta)\left(\frac{\tau+10}{3}\right)+(\alpha-10\beta)}.$$

For all three transformations we have $3|(\gamma-10\delta+5)$ and so

$$(uv_1)(s^{v'}TS^v)^2 = \zeta_u^2\,\nu_5(\alpha-10\beta,3\beta,(\gamma-10\delta+5)/3,\delta)^2\,(uv_2)^2. \tag{33}$$

Again examining the appropriate part of the exponent of ν_5^2 modulo 3 we obtain

$$\begin{aligned}(\alpha-10\beta)\left(\frac{\gamma-10\delta+5}{15}+3\beta\right)&+((\alpha-10\beta)^2-1)3\beta\delta\\ &\equiv (\alpha-1)\left(\frac{\gamma-10\delta+5}{15}\right) \pmod 3.\end{aligned}$$

However in all cases $\alpha \equiv 2 \pmod 3$ and so this becomes $(\gamma-10\delta+5)/15 \pmod 3$.

Evaluating, we find that the root of unity on the right side of (33) is 1 for TST, ζ_3 for S^2TS^8 and ζ_3^{-1} for S^8TS^2.

Finally we transform $(uv_2)^2$. We write

$$\frac{A\tau+10}{3} = \frac{\left(\frac{\delta+10\beta}{3}\right)(3\tau)+(\gamma+10\alpha)}{\beta(3\tau)+(3\alpha)}.$$

In each case $3|(\delta+10\beta)$ and so

(34) $$(uv_2)(S^{v'}TS^v)^2 = \zeta_u^2\,\nu_5(3\alpha,\beta,\gamma+10\delta,(\delta+10\beta)/3)^2\,(uv_\infty)^2.$$

The appropriate part of the exponent is

$$3\alpha\left(\frac{\gamma+10\alpha}{5}+\beta\right)+\frac{(9\alpha^2-1)\beta(\delta+10\beta)}{3}\equiv-\frac{\delta+\beta}{3}\pmod 3.$$

Thus the root of unity on the right of (34) is ζ_3 for TST, ζ_3^{-1} for S^2TS^8 and ζ_3 for S^8TS^2.

It is clear from these computations that the values $P_c=(uv_c)^3$ are all permuted up to sign by all the transformations we have considered. Since u^6 is also invariant under all these transformations then the values $Q_c=(v_c/u)^3$ are all permuted up to sign by these transformations in exactly the same way and with the same signs as the P_c.

We now examine the action of the Fricke involution, $\tau\to-5/\tau$, on our functions. We know from theorem (4.0.16) that it sends the function u to $\sqrt{5}/u$. We use this to determine the action on the other functions.

Clearly

$$v_\infty(-5/\tau)=u(-15/\tau)=\sqrt{5}/u(\tau/5)=\sqrt{5}/v_0(\tau).$$

For the action on v_1 we solve

$$\begin{pmatrix}1&5\\0&3\end{pmatrix}\begin{pmatrix}0&-5\\1&0\end{pmatrix}\begin{pmatrix}\tau\\1\end{pmatrix}=\begin{pmatrix}0&-5\\1&0\end{pmatrix}\begin{pmatrix}\delta&\gamma\\\beta&\alpha\end{pmatrix}\begin{pmatrix}1&-10\\0&3\end{pmatrix}\begin{pmatrix}\tau\\1\end{pmatrix}.$$

Solving, we obtain

$$\alpha=-3,\quad\beta=-1,\quad\gamma=10,\quad\delta=3.$$

The second matrix on the right then represents a linear transformation and contributes the following root of unity

$$\left(\frac{5}{3}\right)e^{-\frac{\pi i}{3}[-3(10/5-1)+(3^2-1)(-3)]}=1.$$

Thus we can see that v_1 is sent to $\sqrt{5}/(\zeta_6\,v_1)$, the ζ_6 appearing since we need to change the argument $(\tau-10)/3$ to that of v_1, i.e. $(\tau+5)/3$.

In particular, we see that the Fricke involution takes v_1^3 to $-(\sqrt{5}/v_1)^3$.

For the action on v_2 we solve

$$\begin{pmatrix}1&10\\0&3\end{pmatrix}\begin{pmatrix}0&-5\\1&0\end{pmatrix}\begin{pmatrix}\tau\\1\end{pmatrix}=\begin{pmatrix}0&-5\\1&0\end{pmatrix}\begin{pmatrix}\delta&\gamma\\\beta&\alpha\end{pmatrix}\begin{pmatrix}1&10\\0&3\end{pmatrix}\begin{pmatrix}\tau\\1\end{pmatrix}.$$

Solving, we obtain

$$\alpha=7,\quad\beta=-2,\quad\gamma=-10,\quad\delta=3.$$

The linear transformation contributes the root of unity

$$\left(\frac{5}{7}\right)e^{-\frac{\pi i}{3}[7(-10/5-2)+(7^2-1)(-6)]}=\zeta_6$$

Thus again we see that v_2^3 is taken to $-(\sqrt{5}/v_2)^3$.

We see that the action of the Fricke involution is not consistent across all our functions. The best that we can do without affecting our analysis of the actions of the modular transformations on the functions is to adjust the definitions of the functions v_1 and v_2 by some 6-th root of unity. However this does not help us in removing the inconsistency that we have here.

The simplest solution to this problem appears to be to define

$$A_c = (uv_c)^3 + 5^3/(uv_c)^3$$
$$B_c = (v_c/u)^3 + (u/v_c)^3$$

and to construct a polynomial $F(x, y)$ in $x = A_\infty$ and $y = B_\infty$ whose q-series vanishes up to and including the constant term but where all monomials have *even* total degree. Now when factors of -1 are induced by the Fricke involution they occur in pairs and cancel out.

The relevant q-series are

$$A_\infty(q^5) = (q^{-2}; 1, -3,\ 0,\ 2,\ 134, \ldots)$$
$$B_\infty(q^5) = (q^{-1}; 1,\ 3,\ 10,\ 16,\ 42, \ldots)$$

from which we obtain the modular equation

$N = 5,\ p = 3:\ A^2 - B^4 + 18AB + 85B^2 = 0.$

11. Evaluation of an Eta Quotient Using Modular Equations

In this final section we give a very simple example of application of the modular equations we have developed. We explicitly evaluate a specific quotient of the Dedekind eta function. Evaluation of such quantities is important in obtaining explicit generators of various number fields in explicit class field theory (see [3] for further details).

Our example will come from the functions of signature three. In particular we will make use of the modular equation of degree five for this signature.

We make the specific assignment $\tau = 1 - 1/\sqrt{-5}$. We note that $5\tau = 5 - 5/\sqrt{-5} = 5 + \sqrt{-5}$.

Plugging this value of τ into the modular equation of degree five and signature three we will end up with a polynomial relation between

$$\mathfrak{g}_1(\tau) = \mathfrak{g}_1(1 - 1/\sqrt{-5}) = \zeta_{12}^{-1}\,\mathfrak{g}_2(-1/\sqrt{-5}) = \zeta_{12}^{-1}\,\mathfrak{g}_3(\sqrt{-5})$$

and

$$\mathfrak{g}_1(5\tau) = \mathfrak{g}_1(5 + \sqrt{-5}) = \zeta_{12}^{-2}\,\mathfrak{g}_3(\sqrt{-5}).$$

Letting u be the first of these value and v the second, the appropriate modular equation yields

$$(uv)^2 + 9/(uv)^2 - (v/u)^3 + (u/v)^3 + 5 = 0.$$

In other words

$$-\mathfrak{g}_3(\sqrt{-5})^4 - 9/\mathfrak{g}_3(\sqrt{-5})^4 + 2i + 5 = 0,$$

Finally if we let $x = \mathfrak{g}_3(\sqrt{-5})^4$ then rearranging and squaring the previous equation yields the following irreducible polynomial expression

$$x^4 - 10x^3 + 47x^2 - 90x + 81 = 0.$$

Noting that

$$x = \mathfrak{g}_3(\sqrt{-5})^4 = \frac{\eta((\sqrt{-5}+2)/3)^4}{\eta(\sqrt{-5})^4},$$

we see that we have completed an evaluation of a non-trivial eta quotient.

12. Acknowledgements

Many thanks to Peter Stevenhagen and Chris Cummins for pointing out a number of important things to me, and to Robin Chapman for providing multitudinous corrections to various drafts of this manuscript.

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