Moduli space of symplectic connections of Ricci type on T^{2n}; a formal approach

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February 2002

Abstract

We consider analytic curves ∇^t of symplectic connections of Ricci type on the torus T^{2n} with ∇^0 the standard connection. We show, by a recursion argument, that if ∇^t is a formal curve of such connections then there exists a formal curve of symplectomorphisms ψ_t such that $\psi_t \cdot \nabla^t$ is a formal curve of flat T^{2n}-invariant symplectic connections and so ∇^t is flat for all t. Applying this result to the Taylor series of the analytic curve, it means that analytic curves of symplectic connections of Ricci type starting at ∇^0 are also flat.

The group G of symplectomorphisms of the torus (T^{2n}, ω) acts on the space $\mathscr{E}$ of symplectic connections which are of Ricci type. As a preliminary to studying the moduli space $\mathscr{E}/G$ we study the moduli of formal curves of connections under the action of formal curves of symplectomorphisms.

Research of the first three authors supported by an ARC of the communauté française de Belgique.
Mathematics Subject Classification (1991): 53C05, 58C35, 53C57.
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1 Introduction

On any symplectic manifold (M,ω) the space $\mathscr{S}$ of symplectic connections is an infinite dimensional affine space whose corresponding vector space is the space of completely symmetric 3-tensors on M. To encode some geometry into a symplectic connection it thus seems reasonable to introduce a selection rule for symplectic connections. A variational principle associated to a Lagrangian density, which is an invariant quadratic polynomial in the curvature, has been considered in [1]; the symplectic connections satisfying the Euler–Lagrange equations are said to be *preferred*. The symplectomorphism group G of (M,ω) acts naturally on $\mathscr{S}$ and stabilises the subspace $\mathscr{P}$ of preferred symplectic connections. The first question we wanted to address is to give a description of the moduli space $\mathscr{P}/G$ of preferred connections modulo the action of symplectomorphisms. Such a description was given in [1] when (M,ω) is a closed surface; but, up to now, very little has been done in the higher dimensional situation.

We have observed that a linear condition on the curvature (the vanishing of one of its irreducible components – the non-Ricci component, W) implies the Euler–Lagrange equations. Furthermore, this condition seems to imply that many of the properties of the surface situation extend to the higher-dimensional case. We have called symplectic connections satisfying this curvature condition *connections of Ricci type* (all symplectic connections in dimension 2 are of Ricci type). This condition is preserved by symplectomorphisms and so we modify our initial question to the following one: give a description of the space $\mathscr{E}$ of Ricci type connections and its moduli space $\mathscr{E}/G$.

This paper is devoted to this modified question in the case where M is a torus T^{2n} and ω a T^{2n}-invariant symplectic structure. Although we do not answer this question, we are able, in a formal setting made precise below, to show that the moduli space is infinite dimensional and to give a partial description of it.

If ∇^t is a formal curve of symplectic connections, we shall denote by W^t the W part of the curvature of ∇^t. We prove

Theorem *Let ∇^t be a formal curve of symplectic connections on (T^{2n},ω) such that ∇^0 is the standard flat connection on T^{2n}, and such that $W^t = 0$. Then the formal curvature R^t of ∇^t vanishes and there exists a formal curve of symplectomorphisms ψ_t such that $\widetilde{\nabla}^t := \psi_t.\nabla^t$ is a formal curve of flat T^{2n}-invariant symplectic connections.*

This implies

Theorem *Let ∇^t be an analytic curve of analytic symplectic connections on (T^{2n},ω) such that ∇^0 is the standard flat connection on T^{2n}, and such that $W^t = 0$. Then the curvature R^t of ∇^t vanishes.*

For the moduli space in the formal setting, we show:

Proposition *For two curves $\widetilde{\nabla^t}$ and $\widetilde{\nabla'^t}$ of invariant flat connections of Ricci-type on $(\mathbb{R}^{2n},\Omega)$ with $\widetilde{\nabla^0} = \widetilde{\nabla'^0}$ the trivial connection, there always exists a formal curve of symplectomorphisms $\widetilde{\psi}_t$ so that $\widetilde{\psi}_t \cdot \widetilde{\nabla^t} = \widetilde{\nabla'^t}$.*

Theorem *The moduli space of formal curves of Ricci-type symplectic connections starting with the standard flat connection on (T^{2n},ω) under the action of formal curves of symplectomorphisms is described by the space of formal curves of linear maps $A^t\colon\mathbb{R}^{2n}\to\mathfrak{sp}(2n,\mathbb{R})[\![t]\!]$ satisfying $A^t(X)A^t(Y)=0$ and $A^t(X)Y=A^t(Y)X$, modulo the action of $Sp(2n,\mathbb{Z})$.*

The plan of the paper is as follows. In §2 we recall some general properties of symplectic connections having Ricci-type curvature. In §3 we introduce the notion of formal curves of connections and we show that the properties of §2 are still true for a formal curve of symplectic connections with Ricci-type curvature. In §4, we analyse the $W^t=0$ condition at order 1 and order 2 for $\nabla^t=\nabla^0+\sum_{k=1}^{\infty}t^kA^{(k)}$ a formal curve of Ricci-type symplectic connections on T^{2n} with ∇^0 the standard flat connection; in particular, we show that there exists a function $U^{(1)}$ and a completely symmetric, T^{2n}-invariant 3-tensor $Q^{(1)}$ on T^{2n} such that $\underline{A}^{(1)}=(\nabla^0)^3U^{(1)}+Q^{(1)}$ and we show that $\nabla'^t=\nabla^0+t\overline{Q}^{(1)}$ (with $\omega(\overline{Q}^{(1)}(X)Y,Z)=Q^{(1)}(X,Y,Z)$) defines a curve of invariant flat symplectic connections on (T^{2n},ω). This remark can be formulated in a slightly different way: given $\nabla^t=\nabla^0+A^{(t)}$ a smooth curve of Ricci-type symplectic connections then, up to a symplectomorphism, the tangent vector to this family of connections lies in the finite dimensional space of flat T^{2n}-invariant symplectic connections. §5 is devoted to a proof of a recurrence lemma which implies the first theorem. In §6 we study the question of when two formal curves of flat invariant connections on T^{2n} are equivalent by a formal curve of symplectomorphisms.

Thanks: We would like to thank Boguslaw Hajduk and Aleksy Tralle who pointed out a mistake in an earlier version of this paper.

2 Ricci Type Curvature

A symplectic connection ∇ on a symplectic manifold (M,ω) is a linear connection having no torsion and for which ω is parallel ($\nabla\omega=0$). The curvature endomorphism R of ∇ is

$$R(X,Y)Z=\left(\nabla_X\nabla_Y-\nabla_Y\nabla_X-\nabla_{[X,Y]}\right)Z$$

for vector fields X,Y,Z on M. The symplectic curvature tensor

$$R(X,Y;Z,T)=\omega(R(X,Y)Z,T)$$

is antisymmetric in its first two arguments, symmetric in its last two and satisfies the first Bianchi identity

$$\underset{X,Y,Z}{\circlearrowleft\!\!\!\!\!\!\bigoplus}\ R(X,Y;Z,T)=0$$

where $\circlearrowleft\!\!\!\!\!\!\bigoplus$ denotes the sum over the cyclic permutations of the listed set of elements. The second Bianchi identity takes the form

$$\underset{X,Y,Z}{\circlearrowleft\!\!\!\!\!\!\bigoplus}\ (\nabla_XR)(Y,Z)=0.$$

The Ricci tensor r is the symmetric 2-tensor

$$r(X,Y) = \mathrm{Trace}[Z \mapsto R(X,Z)Y].$$

If $\dim M = 2n \geq 4$, the curvature R of such a connection has 2 irreducible components under the action of the symplectic group $Sp(2n,\mathbb{R})$. We denote them by E and W:

$$R = E + W.$$

The E component encodes the information contained in the Ricci tensor of ∇ and is called the Ricci part of the curvature tensor. It is given by

$$\begin{aligned} E(X,Y;Z,T) &= \frac{-1}{2(n+1)}\Big[2\omega(X,Y)r(Z,T) + \omega(X,Z)r(Y,T) + \omega(X,T)r(Y,Z) \\ &\qquad - \omega(Y,Z)r(X,T) - \omega(Y,T)r(X,Z)\Big]. \end{aligned}$$

The curvature is said to be *of Ricci type* if the W component vanishes, i.e. when $R = E$.

Lemma 2.1 *Let (M,ω) be a symplectic manifold of dimension $2n \geq 4$. If the curvature of a symplectic connection ∇ on M is of Ricci type then there is a 1-form u such that*

$$(\nabla_X r)(Y,Z) = \frac{1}{2n+1}\left(\omega(X,Y)u(Z) + \omega(X,Z)u(Y)\right).$$

Conversely, if there is such a 1-form u, the "Weyl" part of the curvature, $W = R - E$ satisfies

$$\underset{X,Y,Z}{\circlearrowleft\!\!\!\!\!\!\bigoplus} (\nabla_X W)(Y,Z;T,U) = 0.$$

Proof The property follows from the second Bianchi's identity, see [2]. ∎

Corollary 2.2 *A symplectic manifold with a symplectic connection whose curvature is of Ricci type is locally symmetric if and only if the 1-form u, defined in the lemma, vanishes.*

Denote by ρ the linear endomorphism such that

$$r(X,Y) = \omega(X,\rho Y).$$

The symmetry of r is equivalent to saying that ρ is in the Lie algebra of the symplectic group $Sp(TM,\omega)$. For an integer $p > 1$, define

$$\overset{(p)}{r}(X,Y) = \omega(X,\rho^p Y).$$

It is symmetric when p is odd and antisymmetric when p is even.

Lemma 2.3 *Let (M,ω) be a symplectic manifold with a symplectic connection ∇ with Ricci-type curvature. Then, the following identities hold:*

(i) There is a function b such that

$$\nabla u = -\frac{1+2n}{2(1+n)} \overset{(2)}{r} + b\omega.$$

(ii) The differential of the function b is given by

$$db = \frac{1}{1+n} i(\overline{u}) r$$

where $\overline{u}$ is the vector field such that $i(\overline{u})\omega = u$, so that

(iii)

$$b + \frac{2n+1}{4(1+n)} \operatorname{Trace} \rho^2$$

is a constant when M is connected.

Proof These identities follow from Lemma 2.1, see [2]. ■

Let the torus T^{2n} be endowed with a T^{2n}-invariant symplectic structure ω. Let ∇ be a symplectic connection on (T^{2n}, ω) which is of Ricci-type. The group G of symplectomorphisms of (T^{2n}, ω) acts on the set $\mathscr{E}$ of symplectic connections with $W = 0$. We are interested in the set of orbits of G in $\mathscr{E}$, i.e. in $\mathscr{E}/G$.

We now consider the symplectic vector space $(\mathbb{R}^{2n}, \Omega)$ and view Ω as a translation invariant symplectic structure. A symplectic connection on $\mathbb{R}^{2n}$ will be determined by its values on translation invariant vector fields. If, in addition, the connection ∇ is translation invariant then $B(X)Y := \nabla_X Y$ (for invariant vector fields X, Y) defines a linear map $B\colon \mathbb{R}^{2n} \to \mathfrak{sp}(2n, \mathbb{R})$ which completely determines ∇. The only condition on B is that $\Omega(B(X)Y, Z)$ is completely symmetric.

Proposition 2.4 *Let ∇ be a translation invariant symplectic connection on $(\mathbb{R}^{2n}, \Omega)$ and let $B(X)Y = \nabla_X Y$ as above. If ∇ is of Ricci type and $2n \geq 4$, then ∇ is flat and $B(X)B(Y) = 0$.*

Proof Since B is constant, the curvature endomorphism is given by

$$R(X, Y) = [B(X), B(Y)]$$

and so the Ricci tensor is given by

$$r(X, Y) = \operatorname{Trace}(B(X)B(Y)).$$

It is easy to see that symplectic curvature tensors $R(X, Y; Z, T)$ are, in fact, determined by the terms of the form $R(X, Y; X, Y)$ so that the equation $W = 0$ is equivalent to $R(X, Y; X, Y) = -\frac{2}{n+1}\Omega(X, Y) r(X, Y)$, and in the present case this has the form

$$(n+1)\Omega(B(X)X, B(Y)Y) = -2\Omega(X, Y) r(X, Y).$$

Polarising the equation in X we have

$$\begin{aligned}(n+1)\Omega(T,B(X)B(Y)Y) &= \Omega(X,Y)r(T,Y)+\Omega(T,Y)r(X,Y)\\ &= \Omega(X,Y)\Omega(T,\rho Y)+\Omega(T,Y)\Omega(X,\rho Y),\end{aligned}$$

so that $W=0$ is equivalent to

$$(n+1)B(X)B(Y)Y=\Omega(X,Y)\rho Y+\Omega(X,\rho Y)Y.$$

Polarising this in Y we have

$$2(n+1)B(X)B(Y)Z=\Omega(X,Y)\rho Z+\Omega(X,\rho Y)Z+\Omega(X,Z)\rho Y+\Omega(X,\rho Z)Y \quad (*).$$

Now choose dual bases X^i, X_i for $\mathbb{R}^{2n}$ with $\Omega(X^i,X_j)=\delta^i_j$ then an easy calculation shows

$$\rho=\sum_i B(X^i)B(X_i).$$

If we multiply $(*)$ by $B(X^i)$, set $X=X_i$ and sum we get

$$(n+1)\rho B(Y)Z=-B(Y)\rho Z-B(Z)\rho Y.$$

Alternatively we may substitute $B(X_i)Z$ for Z in (*), set $Y=X^i$ and sum to give

$$(n+1)B(X)\rho Z=-\rho B(Z)X+B(Z)\rho X.$$

Adding the two equations after setting $X=Y$ we see that

$$\rho B(X)=-B(X)\rho$$

and hence that

$$(n-1)\rho B(X)=0.$$

Thus if $2n\geq 4$

$$\rho B(X)=B(X)\rho=0 \quad \Rightarrow \rho^2=0.$$

Substituting ρZ for Z in (*) we have

$$0=r(X,Y)\rho Z+r(X,Z)\rho Y$$

and setting $Z=Y$, applying $\Omega(X,\,.\,)$ we get finally

$$0=r(X,Y)^2.$$

Thus the Ricci tensor vanishes, and hence ∇ is flat.

Putting $\rho=0$ in (*) yields $B(X)B(Y)=0$. ∎

3 Formal curves

Definition 3.1 *A* **formal curve of symplectic connections** *on a symplectic manifold (M,ω) is a formal power series*

$$\nabla^t = \nabla + \sum_{k=1}^{\infty} t^k A^{(k)}$$

where ∇ is a symplectic connection on M, and the $A^{(k)}$ are $(2,1)$ tensors such that

$$\underline{A}^{(k)}(X,Y,Z) := \omega(A^{(k)}(X)Y,Z) \tag{3.1}$$

is totally symmetric.

Definition 3.2 *A* **formal curve of symplectomorphisms** *is a homomorphism of Poisson algebras*

$$\psi_t\colon C^\infty(M) \longrightarrow C^\infty(M)[\![t]\!], \qquad \psi_t = \psi^{(0)} + \sum_{k=1}^{\infty} t^k \psi^{(k)}$$

such that $\psi^{(0)}\colon C^\infty(M) \longrightarrow C^\infty(M)$ is an isomorphism.

The leading term $\psi^{(0)}$ of a formal curve of symplectomorphisms is given by composition with a symplectomorphism $\psi^{(0)}(f) = f \circ \sigma = \sigma^*(f)$ so that we may take such a term out as a common factor and write $\psi_t = \sigma^* \circ \phi_t$ and $\phi_t = \mathrm{id} + \sum_{k\geq 1} t^k \phi^{(k)}$.

If $\phi_t = \mathrm{id} + \sum_{k\geq 1} t^k \phi^{(k)}$ is a formal curve of symplectomorphisms beginning with the identity then the first order term $X^{(1)} = \phi^{(1)}$ is a symplectic vector field. Moreover, for any symplectic vector field, $\exp tX = \mathrm{id} + \sum_{k\geq 1} t^k/k! X^k$ is a formal curve of symplectomorphisms. A straightforward recursion argument then shows that any formal curve of symplectomorphisms beginning with the identity can be written in the form $\phi_t = \exp X_t$ where $X_t = \sum_{k\geq 1} t^k X^{(k)}$ is a formal curve of vector fields.

Definition 3.3 *A* **formal 1-parameter group of symplectomorphisms** *is a formal curve of symplectomorphisms ψ_t such that $\psi_{at} \circ \psi_{bt} = \psi_{(a+b)t}$ for all $a, b \in \mathbb{R}$.*

In order for this definition to make sense we first have to extend ψ_t by linearity over $\mathbb{R}[\![t]\!]$ to a morphism of $\mathbb{R}[\![t]\!]$-algebras. The definition then implies that $\psi^{(0)}$ is the identity and that $\psi^{(1)}(f) = X(f)$ for some symplectic vector field which we call **the infinitesimal generator** of ψ_t. It is easy to see that every formal 1-parameter group of symplectomorphisms has the form $\psi_t = \exp tX$. Moreover, a recursion shows that, if X_t is a formal curve of symplectic vector fields, we can find a second sequence of symplectic vector fields $Y^{(k)}$ such that

$$\exp X_t = \exp tY^{(1)} \circ \exp t^2 Y^{(2)} \circ \cdots \circ \exp t^k Y^{(k)} \circ \cdots$$

and so any formal curve of symplectomorphisms ψ_t can be factorised in two ways

$$\psi_t = \sigma^* \circ \exp X_t = \sigma^* \circ \phi_t^{(1)} \circ \phi_{t^2}^{(2)} \circ \cdots \circ \phi_{t^k}^{(k)} \circ \cdots$$

where the the $\phi_t^{(k)}$ are formal 1-parameter groups of symplectomorphisms.

Remark that a formal curve of symplectomorphisms ψ_t acts on a formal curve of vector fields X_t viewed as a $\mathbb{R}[\![t]\!]$-linear derivation of $C^\infty(M)[\![t]\!]$ by

$$(\psi_t \cdot X_t)f = \psi_t(X_t({\psi_t}^{-1}f)),$$

and acts on a formal curve of symplectic connections ∇^t by

$$(\psi_t \cdot \nabla^t)_X Y = \psi_t \cdot \left(\nabla^t_{{\psi_t}^{-1}\cdot X}{\psi_t}^{-1} \cdot Y\right). \tag{3.2}$$

Let ∇^t be a formal curve of symplectic connections on a symplectic manifold (M,ω) of dimension $2n$,

$$\nabla^t = \nabla + \sum_{k=1}^{\infty} t^k A^{(k)}.$$

We denote as in (3.1) by $\underline{A}^{(k)}$ the corresponding symmetric 3-tensors. The formal curvature endomorphism R^t of ∇^t is $R^t(X,Y) = \nabla^t_X \circ \nabla^t_Y - \nabla^t_Y \circ \nabla^t_X - \nabla^t_{[X,Y]}$ so that

$$R^t = R^\nabla + \sum_{k=1}^{\infty} t^k R^{(k)}$$

with

$$R^{(k)}(X,Y) = (\nabla_X A^{(k)})(Y) - (\nabla_Y A^{(k)})(X) + \sum_{\substack{p+q=k \\ p,q\geq 1}} [A^{(p)}(X), A^{(q)}(Y)]. \tag{3.3}$$

The symplectic curvature tensor $R^t(X,Y;Z,T) = \omega(R^t(X,Y)Z,T)$ is antisymmetric in its first two arguments, symmetric in its last two, satisfies the first Bianchi identity $\underset{X,Y,Z}{\circlearrowleft\!\!\!\!\!\bigoplus} R^t(X,Y;Z,T) = 0$ and the second Bianchi identity $\underset{X,Y,Z}{\circlearrowleft\!\!\!\!\!\bigoplus} (\nabla^t_X R^t)(Y,Z) = 0$.

The formal Ricci tensor is $r^t(X,Y) = \mathrm{Trace}[Z \mapsto R^t(X,Z)Y]$, so that

$$r^t = r^\nabla + \sum_{k=1}^{\infty} t^k r^{(k)}$$

where the $r^{(k)}$ are the symmetric tensors

$$r^{(k)}(X,Y) = \mathrm{Trace}[Z \mapsto (\nabla_Z A^{(k)})(X)Y] + \sum_{\substack{p+q=k \\ p,q\geq 1}} \mathrm{Trace}\, A^{(p)}(X)A^{(q)}(Y). \tag{3.4}$$

The Ricci part E^t of the formal curvature tensor is given by

$$\begin{aligned} E^t(X,Y;Z,T) \;=\; & \frac{-1}{2(n+1)}\Big[2\omega(X,Y)r^t(Z,T) + \omega(X,Z)r^t(Y,T) + \omega(X,T)r^t(Y,Z) \\ & - \omega(Y,Z)r^t(X,T) - \omega(Y,T)r^t(X,Z)\Big]. \end{aligned} \tag{3.5}$$

The formal curvature is said to be *of Ricci type* when $R^t = E^t$.

Lemma 3.4 *Let (M,ω) be a symplectic manifold of dimension $2n \geq 4$. If the formal curvature of a formal curve of symplectic connections ∇^t on M is of Ricci type then there exists a formal curve of 1-forms*

$$u^t = \sum_{k=0}^{\infty} t^k u^{(k)}$$

such that

$$(\nabla^t_X r^t)(Y,Z) = \frac{1}{2n+1}(\omega(X,Y)u^t(Z) + \omega(X,Z)u^t(Y)) \tag{3.6}$$

and there exists a formal curve of functions

$$b^t = \sum_{k=0}^{\infty} t^k b^{(k)}$$

such that

$$\nabla^t u^t = -\frac{1+2n}{2(1+n)} \overset{(2)}{r^t} + b^t\omega. \tag{3.7}$$

with $\omega(X,(\rho^t)Y) = r^t(X,Y) =$ and $\overset{(2)}{r^t}(X,Y) = \omega(X,(\rho^t)^2Y)$. Also

$$db^t = \frac{1}{1+n} i(\overline{u}^t) r^t. \tag{3.8}$$

Lemma 3.5 *Let ∇^t be a formal curve of translation invariant symplectic connections on $(\mathbb{R}^{2n},\Omega)$ and let $B^t(X)Y := \nabla^t_X Y$ (for invariant vector fields X,Y). If ∇^t is of Ricci type and $2n \geq 4$, then ∇^t is flat and $B^t(X)B^t(Y) = 0$.*

Proof We can copy in the formal series setting the proof of Lemma 2.4. Write $B^t = \sum_{k=0}^{\infty} t^k B^{(k)}$ where the $B^{(k)}$ are constant maps from $\mathbb{R}^{2n}$ to $sp(\mathbb{R}^{2n},\Omega)$. The formal curvature endomorphism is given by

$$R^t(X,Y) = [B^t(X), B^t(Y)] \quad \text{i.e.} \quad R^{(k)}(X,Y) = \sum_{\substack{p+q=k\\ p,q\geq 0}} [B^p(X), B^q(Y)]$$

and the formal Ricci tensor by

$$r^t(X,Y) = \mathrm{Trace}(B^t(X)B^t(Y)) \quad \text{i.e.} \quad r^{(k)}(X,Y) = \sum_{\substack{p+q=k\\ p,q\geq 0}} \mathrm{Trace}\, B^p(X)B^q(Y).$$

The equation $W^t = 0$ is again equivalent to $2(n+1)B^t(X)B^t(Y)Z = \Omega(X,Y)\rho^t Z + \Omega(X,\rho^t Y)Z + \Omega(X,Z)\rho^t Y + \Omega(X,\rho^t Z)Y$, i.e.

$$\sum_{\substack{p+q=k\\ p,q\geq 0}} 2(n+1)B^{(p)}(X)B^{(q)}(Y)Z = \Omega(X,Y)\rho^{(k)}Z + \Omega(X,\rho^{(k)}Y)Z + \Omega(X,Z)\rho^{(k)}Y + \Omega(X,\rho^{(k)}Z)Y. \tag{3.9}$$

Choosing dual bases X^i, X_i for $\mathbb{R}^{2n}$ with $\Omega(X^i,X_j) = \delta^i_j$ then $\rho^t = \sum_i B^t(X^i)B^t(X_i)$, i.e. $\rho^{(k)} = \sum_{p+q=k}\sum_i B^{(p)}(X^i)B^{(q)}(X_i)$. If we multiply (3.9) by $B^{(k')}(X^i)$, set $X = X_i$ and sum over i and over $k,k' \geq 0$ so that $k+k' = K$ we get

$$(n+1) \sum_{\substack{q'+q=K\\ q,q'\geq 0}} \rho^{(q')}B^{(q)}(Y)Z = \sum_{\substack{k'+k=K\\ k',k\geq 0}} \left(-B^{(k')}(Y)\rho^{(k)}Z - B^{(k')}(Z)\rho^{(k)}Y\right).$$

This can be written in terms of formal series

$$(n+1)\rho^t B^t(Y)Z = -B^t(Y)\rho^t Z - B^t(Z)\rho^t Y.$$

Alternatively we may substitute $B^{(s)}(X_i)Z$ for Z in (3.9), set $Y = X^i$ and sum to give

$$(n+1)B^t(X)\rho^t Z = -\rho^t B^t(Z)X + B^t(Z)\rho^t X.$$

Adding the two equations after setting $X = Y$ as before, we see that $\rho^t B^t(X) = -B^t(X)\rho^t$, so $(n-1)\rho^t B^t(X) = 0$ and, if $2n \geq 4$, $\rho^t B^t(X) = B^t(X)\rho^t = 0$ thus $(\rho^t)^2 = 0$. This in turn implies $r^t = 0$, hence $R^t = 0$ and ∇ is flat. Putting $\rho^t = 0$ in 3.9 yields $B^t(X)B^t(Y) = 0$. ■

4 Curves of Ricci Type Connections on the Torus

Consider the torus T^{2n} endowed with a T^{2n}-invariant symplectic structure ω. Let ∇^0 be the standard flat, T^{2n}-invariant symplectic connection on (T^{2n}, ω). Let

$$\nabla^t = \nabla^0 + \sum_{k=1}^{\infty} t^k A^{(k)}$$

be a formal curve of symplectic connections such that $W(t) = 0$. We denote as before (3.1) by $\underline{A}^{(k)}$ the corresponding symmetric 3-tensors ($\underline{A}^{(k)}(X,Y,Z) = \omega(A^{(k)}(X)Y,Z)$).

We consider, as given by Lemma 3.4, the corresponding formal curve of 1-forms $u^t = \sum_{k=0}^{\infty} t^k u^{(k)}$ and the formal curve of functions $b^t = \sum_{k=0}^{\infty} t^k b^{(k)}$; clearly $u^{(0)} = 0$ and $b^{(0)} = 0$ since $r^{\nabla^0} = 0$.

Lemma 4.1 *If $\nabla^t = \nabla^0 + \sum_{k=1}^{\infty} t^k A^{(k)}$ is a formal curve of symplectic connections such that $W(t) = 0$, then the formal curvature vanishes at order 1 in t (i.e. one has $b^{(1)} = 0$, $u^{(1)} = 0$, $r^{(1)} = 0$, $R^{(1)} = 0$). Furthermore, there exists a function $U^{(1)}$ and a completely symmetric, T^{2n}-invariant 3-tensor $Q^{(1)}$ on T^{2n} such that*

$$\underline{A}^{(1)} = (\nabla^0)^3 U^{(1)} + Q^{(1)}.$$

Proof Denote by x^a ($1 \leq a \leq 2n$) the standard angle variables on T^{2n} and by ∂_a the corresponding T^{2n}-invariant vector fields on T^{2n} (the standard flat connection is defined by $\nabla^0_{\partial_a}\partial_b = 0$).

At order 1, since $b^{(0)} = 0$, $u^{(0)} = 0$, $r^0 = 0$, we have:

(i) $db^{(1)} = 0$ by (3.8), so $b^{(1)}$ is a constant;

(ii) $du^{(1)} = b^{(1)}\omega$ by (3.7); but ω is not exact by compactness of T^{2n} so $b^{(1)} = 0$ and $\nabla^0 u^{(1)} = 0$ thus $u^{(1)}(X)$ is a constant for any T^{2n}-invariantvector field X on T^{2n};

(iii) the equation (3.6) at order 1 yields $(\nabla^0 r^1)$ as a combination of products of ω and u^1 so that $\partial_a(r^{(1)}(\partial_b, \partial_c))$ is a constant; the periodicity of the angles x^a implies then that $\partial_a(r^{(1)}(\partial_b, \partial_c)) = 0$ so $u^{(1)} = 0$ and $r^{(1)}(\partial_b, \partial_c) = a^{(1)}_{ab}$ is a constant.

The definition of the (formal) Ricci tensor (3.4) at order 1 yields $a^{(1)}_{ab} = -\partial_q A^{(1)q}{}_{ab}$; hence, for each value of the indices a, b, the $2n$-form $a^{(1)}_{ab}\omega^n$ is exact; this implies

$$a^{(1)}_{ab} = 0 \quad \text{so} \quad r^{(1)} = 0 \quad \text{and thus} \quad R^{(1)} = 0.$$

The definition of the (formal) curvature tensor (3.3) at order 1 gives $R^{(1)}_{abcd} = \partial_a \underline{A}^{(1)}_{bcd} - \partial_b \underline{A}^{(1)}_{acd}$. Hence, for each value of the indices c, d the 1-form $\underline{A}^{(1)}_{\cdot cd}$ is closed, so there exist functions k_{cd} on T^{2n} and constants $Q^{(1)}_{bcd}$ such that:

$$\underline{A}^{(1)}_{bcd} = \partial_b k^{(1)}_{cd} + Q^{(1)}_{bcd}.$$

Since ∇^t is symplectic, $\underline{A}^{(1)}_{bcd}$ is totally symmetric; the fact that $\underline{A}^{(1)}_{bcd} - \underline{A}^{(1)}_{cbd} = 0$ implies

$$\partial_b k^{(1)}_{cd} - \partial_c k^{(1)}_{bd} = -Q^{(1)}_{bcd} + Q^{(1)}_{cbd}.$$

When d is fixed, the left-hand side is an exact 2-form. The right-hand side is T^{2n}-invariant. Since there are no non-zero exact T^{2n}-invariant forms, this implies

$$Q^{(1)}_{bcd} = Q^{(1)}_{cbd}, \qquad \partial_b k^{(1)}_{cd} - \partial_c k^{(1)}_{bd} = 0.$$

Similarly $\underline{A}^{(1)}_{bcd} - \underline{A}^{(1)}_{bdc} = 0$ gives

$$\partial_b k^{(1)}_{cd} - \partial_b k^{(1)}_{dc} = -Q^{(1)}_{bcd} + -Q^{(1)}_{bdc}.$$

In this case, when c and d are fixed, the left-hand side is an exact 1-form, while the right-hand side is T^{2n}-invariant. For the same reason as above, we deduce that both members vanish:

$$Q^{(1)}_{bcd} = Q^{(1)}_{bdc} \qquad k^{(1)}_{cd} - k^{(1)}_{dc} = \text{constant}.$$

Hence $Q^{(1)}_{bcd}$ is completely symmetric. Furthermore, for each fixed index d, the 1-form $k^{(1)}_{\cdot d}$ is closed. Hence there exist functions $S^{(1)}_d$ and constants T_{cd} such that

$$k^{(1)}_{cd} = \partial_c S^{(1)}_d + T^{(1)}_{cd}.$$

The fact that $k^{(1)}_{cd} - k^{(1)}_{dc}$ is a constant implies for the 1-form $S^{(1)}_{\cdot}$ that $dS^{(1)}$ is T^{2n}-invariant, thus $S^{(1)}$ is closed. Hence there exists a function $U^{(1)}$ and constants $V^{(1)}_d$ such that

$$S^{(1)}_d = \partial_d U^{(1)} + V^{(1)}_d.$$

Substituting, we have:

$$\underline{A}^{(1)}_{bcd} = \partial^3_{bcd} U^{(1)} + Q^{(1)}_{bcd}.$$

■

Lemma 4.2 *If $\nabla^t = \nabla^0 + \sum_{k=1}^{\infty} t^k A^{(k)}$ is a formal curve of symplectic connections such that $W(t) = 0$, then the curvature vanishes at order 2 in t, (i.e. $b^{(2)} = 0$, $u^{(2)} = 0$, $r^{(2)} = 0$, $R^{(2)} = 0$).*

Writing $\underline{A}^{(1)} = (\nabla^0)^3 U^{(1)} + Q^{(1)}$ as in Lemma 4.1, the formula $\nabla'^t = \nabla^0 + t\overline{Q}^{(1)}$, where $\omega(\overline{Q}^{(1)}(X)Y, Z) = Q^{(1)}(X, Y, Z)$, defines a curve of invariant flat symplectic connections on (T^{2n}, ω).

Furthermore, there exist a function $U^{(2)}$ and a T^{2n}-invariant, completely symmetric tensor $Q^{(2)}$ such that

$$\underline{A}^{(2)}_{bcd} = \underset{bcd}{\circlearrowright\!\!\!\!\!\bigoplus} U^{(1)p}{}_b(Q^{(1)}_{pcd} + \frac{1}{2}U^{(1)}_{pcd}) + \frac{1}{2}U^{(1)p}U^{(1)}_{pbcd} + \partial^3_{bcd}U^{(2)} + Q^{(2)}_{bcd}$$

where

$$U^{(1)}_{p_1\ldots p_k} = \partial^k_{p_1\ldots p_k}U^{(1)} \qquad U^{(1)p}{}_{q_1\ldots q_k} = \partial^{k+1}_{qq_1\ldots q_k}U^{(1)}\omega^{qp} \qquad \omega^{pq}\omega_{ql} = \delta^p_l.$$

PROOF At order 2, since $b^{(0)} = b^{(1)} = 0$, $u^{(0)} = u^{(1)} = 0$, $r^{(0)} = r^{(1)} = 0$

(i) $db^{(2)} = 0$ by (3.8), so $b^{(2)}$ is a constant;

(ii) $du^{(2)} = b^{(2)}\omega$ by (3.7); so $b^{(2)} = 0$ and $\nabla^0 u^{(2)} = 0$;

(iii) the equation (3.6) at order 2 yields that $\partial_a(r^{(2)}(\partial_b, \partial_c))$ is a constant; again this implies $u^{(2)} = 0$ and $r^{(2)}(\partial_b, \partial_c) = a^{(2)}_{ab}$ is a constant.

The definition of the (formal) Ricci tensor yields $a^{(2)}_{ab} = -\partial_q A^{(2)q}{}_{ab} + A^{(1)p}{}_{qb}A^{(1)q}{}_{ap}$; Using lemma 4.1 with $Q^{(1)p}{}_{qb} = Q^{(1)}{}_{qbk}\omega^{kp}$:

$$A^{(1)p}{}_{qb}A^{(1)q}{}_{ap} = Q^{(1)p}{}_{qb}Q^{(1)q}{}_{ap} + \partial_q(Q^{(1)q}{}_{ap}U^{(1)p}{}_b) + \partial_p(U^{(1)q}{}_aQ^{(1)p}{}_{qb}) + \partial_q(U^{(1)p}{}_bU^{(1)q}{}_{ap}).$$

Hence:

$$a^{(2)}_{ab} = Q^{(1)p}{}_{qb}Q^{(1)q}{}_{ap} - \partial_q(A^{(2)q}{}_{ab} - U^{(1)p}{}_bQ^{(1)q}{}_{ap} - U^{(1)p}{}_aQ^{(1)q}{}_{pb} - U^{(1)p}{}_bU^{(1)q}{}_{ap}).$$

Since there are no exact, non-zero, T^{2n}-invariant $2n$-form on T^{2n}, we have

$$a^{(2)}_{ab} = Q^{(1)p}{}_{qb}Q^{(1)q}{}_{ap}, \qquad \partial_q(A^{(2)q}{}_{ab} - U^{(1)p}{}_bQ^{(1)q}{}_{ap} - U^{(1)p}{}_aQ^{(1)q}{}_{pb} - U^{(1)p}{}_bU^{(1)q}{}_{ap}) = 0.$$

The definition of the (formal) curvature tensor at order 2 gives $R^{(2)}_{abcd} = \partial_a\underline{A}^{(2)}_{bcd} - \partial_b\underline{A}^{(2)}_{acd} + A^{(1)p}_{bc}\underline{A}^{(1)}_{apd} - A^{(1)p}_{ac}\underline{A}^{(1)}_{bpd}$. Using lemma 4 we get

$$\begin{aligned} R^{(2)}_{abcd} \;=\; & \partial_a(\underline{A}^{(2)}_{bcd} + U^{(1)}{}_{pd}Q^{(1)p}{}_{bc} - U^{(1)p}{}_cQ^{(1)}{}_{bpd} - U^{(1)p}{}_cU^{(1)}{}_{bpd}) \\ & - \partial_b(\underline{A}^{(2)}_{acd} + U^{(1)}{}_{pd}Q^{(1)p}{}_{ac} - U^{(1)p}{}_cQ^{(1)}{}_{apd} - U^{(1)p}{}_cU^{(1)}{}_{apd}) \\ & + Q^{(1)p}{}_{bc}Q^{(1)}{}_{apd} - Q^{(1)p}{}_{ac}Q^{(1)}{}_{bpd}. \end{aligned}$$

The $W^{(2)} = 0$ condition says that:

$$R^{(2)}_{abcd} = -\frac{1}{2(n+1)}\left[2\omega_{ab}a^{(2)}_{cd} + \omega_{ac}a^{(2)}_{bd} + \omega_{ad}a^{(2)}_{bc} - \omega_{bc}a^{(2)}_{ad} - \omega_{bd}a^{(2)}_{ac}\right].$$

The fact that there does not exist a non-zero T^{2n}-invariant exact 2-form implies on one hand:

$$\begin{aligned} & \partial_a(\underline{A}^{(2)}_{bcd} + U^{(1)}{}_{pd}Q^{(1)p}{}_{bc} - U^{(1)p}{}_cQ^{(1)}{}_{bpd} - U^{(1)p}{}_cU^{(1)}{}_{bpd}) \\ - \; & \partial_b(\underline{A}^{(2)}_{acd} + U^{(1)}{}_{pd}Q^{(1)p}{}_{ac} - U^{(1)p}{}_cQ^{(1)}{}_{apd} - U^{(1)p}{}_cU^{(1)}{}_{apd}) = 0, \end{aligned}$$

and on the other hand:

$$\begin{aligned} {Q^{(1)}}^p{}_{bc}{Q^{(1)}}_{apd} - {Q^{(1)}}^p{}_{ac}{Q^{(1)}}_{bpd} &= -\frac{1}{2(n+1)}\Big[2\omega_{ab}a^{(2)}_{cd} + \omega_{ac}a^{(2)}_{bd} + \omega_{ad}a^{(2)}_{bc} \\ &\qquad - \omega_{bc}a^{(2)}_{ad} - \omega_{bd}a^{(2)}_{ac}\Big], \end{aligned}$$

where $a^{(2)}_{ab} = {Q^{(1)}}^p{}_{qb}{Q^{(1)}}^q{}_{ap}$.

This last relation tells us that the T^{2n}-invariant connection defined by $\nabla^0 + tQ^{(1)}$ (which is symplectic because of the complete symmetry) has a W tensor which is zero. Lifting everything to $\mathbb{R}^{2n}$ and applying lemma 3 we get that the corresponding curvature vanishes identically. Hence:

$$a^{(2)}_{ab} = 0, \qquad {Q^{(1)}}^p{}_{bc}{Q^{(1)}}_{apd} - {Q^{(1)}}^p{}_{ac}{Q^{(1)}}_{bpd} = 0.$$

This in turn implies

$$r^{(2)} = 0, \qquad R^{(2)} = 0.$$

The first relation tells us that there exist functions $k'^{(2)}_{cd}$ and constants ${Q^{(2)}}_{bcd}$ such that

$$\underline{A}^{(2)}_{bcd} - {U^{(1)}}^p{}_c{Q^{(1)}}_{bpd} - {U^{(1)}}^p{}_d{Q^{(1)}}_{bpc} - {U^{(1)}}^p{}_c{U^{(1)}}_{bpd} = \partial_b k'^{(2)}_{cd} + {Q^{(2)}}_{bcd}.$$

This can be rewritten as

$$\underline{A}^{(2)}_{bcd} - \underset{bcd}{\circlearrowleft\!\!\!\!\bigoplus} {U^{(1)}}^p{}_b({Q^{(1)}}_{pcd} + \frac{1}{2}{U^{(1)}}_{pcd}) - \frac{1}{2}{U^{(1)}}^p{U^{(1)}}_{pbcd} = \partial_b k^{(2)}_{cd} + {Q^{(2)}}_{bcd} \tag{4.10}$$

with

$$k^{(2)}_{cd} = k'^{(2)}_{cd} - {U^{(1)}}^p{Q^{(1)}}_{pcd} + \frac{1}{2}{U^{(1)}}^p{}_c{U^{(1)}}_{pd} - \frac{1}{2}{U^{(1)}}^p{U^{(1)}}_{pcd}.$$

Indeed we have ${U^{(1)}}^p{}_c{U^{(1)}}_{bpd} = \frac{1}{2}{U^{(1)}}^p{}_c{U^{(1)}}_{bpd} + \frac{1}{2}\partial_b({U^{(1)}}^p{}_c{U^{(1)}}_{pd}) + \frac{1}{2}{U^{(1)}}^p{}_d{U^{(1)}}_{bpc}$ and also $\frac{1}{2}{U^{(1)}}^p{}_b{U^{(1)}}_{cpd} = \frac{1}{2}\partial_b({U^{(1)}}^p{U^{(1)}}_{cpd}) - \frac{1}{2}{U^{(1)}}^p\partial_b{U^{(1)}}_{cpd}$.

Now the left hand side of the equation 4.10 is totally symmetric in its indices (bcd) so the same reasoning as in Lemma 4.1 shows that $Q^{(2)}$ is totally symmetric and there exists a function $U^{(2)}$ so that $\partial_b k^{(2)}_{cd} = \partial^3_{bcd}U^{(2)}$. Substituting, we find:

$$\underline{A}^{(2)}_{bcd} = \underset{bcd}{\circlearrowleft\!\!\!\!\bigoplus} {U^{(1)}}^p{}_b({Q^{(1)}}_{pcd} + \frac{1}{2}{U^{(1)}}_{pcd}) + \frac{1}{2}{U^{(1)}}^p{U^{(1)}}_{pbcd} + \partial^3_{bcd}U^{(2)} + {Q^{(2)}}_{bcd}$$

which ends the proof of the lemma. ■

5 A Recurrence Lemma

Lemma 5.1 *Let ∇^t be a formal curve of symplectic connections on (T^{2n}, ω) such that $\nabla^{(0)} = \nabla^0$, and $W^t = 0$. Assume that, for all orders $l < k$, $\underline{A}^{(l)}$, and thus $r^{(l)}$, $u^{(l)}$, $b^{(l)}$ are T^{2n}-invariant. Then, at order k, $r^{(k)}$, $u^{(k)}$, $b^{(k)}$ are T^{2n}-invariant, and there exist a function $U^{(k)}$ on T^{2n} and a T^{2n}-invariant completely symmetric 3 tensor $Q^{(k)}$ such that*

$$\underline{A}^{(k)} = \partial^3 U^{(k)} + Q^{(k)}.$$

PROOF Assume that, up to order $k-1$ (included), $\underline{A}^{(l)}_{abc}$, $r^{(l)}_{ab}$, $u^{(l)}_a$, $b^{(l)}$ are T^{2n}-invariant. Then, at order k, we have

$$(i) \quad R^{(k)}_{abcd} = \partial_a \underline{A}^{(k)}_{bcd} - \partial_b \underline{A}^{(k)}_{acd} + \sum_{\substack{s+s'=k\\ s,s'>0}} A^{(s)p}{}_{bc}\underline{A}^{s'}_{apd} - A^{(s)p}{}_{ac}\underline{A}^{(s')}{}_{bpd};$$

$$(ii) \quad r^{(k)}_{ac} = -\partial_q A^{(k)q}{}_{ac} + \sum_{\substack{s+s'=k\\ s,s'>0}} A^{(s)p}{}_{qc}A^{(s')q}{}_{ap};$$

$$(iii) \quad \partial_c r^{(k)}_{ab} - \sum_{\substack{s+s'=k\\ s,s'>0}} A^{(s)p}{}_{ca}r^{(s')}_{pb} + \Gamma^{(s)p}{}_{cb}r^{(s')}_{ap} = \frac{1}{2n+1}(\omega_{cb}u^{(k)}_a + \omega_{ca}u^{(k)}_b);$$

$$(iv) \quad \partial_b u^{(k)}_a - \sum_{\substack{s+s'=k\\ s,s'>0}} A^{(s)p}{}_{ba}u^{(s')}_p = -\frac{1+2n}{2(1+n)} \sum_{\substack{s+s'=k\\ s,s'>0}} r^{(s)}_{bc}r^{(s')c}{}_a + b^{(k)}\omega_{ba};$$

$$(v) \quad \partial_a b^{(k)} = \frac{1}{1+n} \sum_{\substack{s+s'=k\\ s,s'>0}} \overline{u}^{(s)c}r^{(s')}_{ca}.$$

Relation (v) implies that $db^{(k)}$ is T^{2n}-invariant. Hence $db^{(k)} = 0$ and $b^{(k)}$ is a constant. Antisymmetrising (iv) we get that $du^{(k)} - b^{(k)}\omega$ is a T^{2n}-invariant 2-form, hence $du^{(k)} = 0$ and

$$b^{(k)}\omega_{ba} - \frac{1+2n}{2(1+n)} \sum_{\substack{s+s'=k\\ s,s'>0}} r^{(s)}_{bc}r^{(s')c}{}_a = 0.$$

Also

$$\partial_b u^{(k)}_a = \sum_{\substack{s+s'=k\\ s,s'>0}} A^{(s)p}{}_{ba}u^{(s')}_p.$$

Using periodicity again and the fact that the right hand side is a constant, we see that the $u^{(k)}_a$ are constants. Relation (iii) tells us, for the same reason, that the $r^{(k)}_{ab}$ are constants. Finally from (i) and the $W^t = 0$ condition, we get that $\partial_a \underline{A}^{(k)}_{bcd} - \partial_b \underline{A}^{(k)}_{acd}$ is a constant hence

$$\partial_a \underline{A}^{(k)}_{bcd} - \partial_b \underline{A}^{(k)}_{acd} = 0. \tag{5.11}$$

The reasoning of Lemma 4.1 applies to equation (5.11) so there exist a function $U^{(k)}$ on T^{2n} and a T^{2n}-invariant completely symmetric 3 tensor $Q^{(k)}$ such that

$$\underline{A}^{(k)} = \partial^3 U^{(k)} + Q^{(k)}.$$

■

We can now proceed to the proof of the main theorem.

Theorem 5.2 *Let ∇^t be a formal curve of symplectic connections on (T^{2n}, ω) with ∇^0 the standard connection, and $W^t = 0$. Then there exists a formal curve of symplectomorphisms ψ_t such that $\widetilde{\nabla}^t := \psi_t.\nabla^t$ is a formal curve of symplectic connections which is T^{2n}-invariant and has $\widetilde{W}^t = 0$, hence is flat. In particular, ∇^t is flat.*

PROOF If $\nabla^t = \nabla^0 + \sum_{k=0}^{\infty} t^p A^{(p)}$ is any formal curve of symplectic connections, one defines as in 3.2 the action of a formal curve ψ_t of symplectomorphisms on ∇^t:

$$(\psi_t \cdot \nabla^t)_X Y = \psi_t \cdot \left(\nabla^t_{\psi_t^{-1}\cdot X}\psi_t^{-1} \cdot Y\right).$$

Consider a formal one-parameter group $\psi_f(t)$ of symplectomorphisms generated by a hamiltonian vector field X_f $(i(X_f)\omega = df)$ and consider the formal curve of symplectomorphisms defined by $\psi^k_f(t) = \psi_f(t^k)$. Write

$$\psi^k_f(t) \cdot \nabla^t = \nabla^0 + \sum_{p=0}^{\infty} t^p \widetilde{A}^{(p)}$$

then $\widetilde{A}^{(p)} = A^{(p)}, \forall p < k$ and

$$\widetilde{A}^{(k)}_X Y = A^{(k)}_X Y + [X_f, \nabla^0_Y Z] - \nabla^0_{[X_f,Y]} Z - \nabla^0_Y [X_f, Z].$$

Observe that $[X_f, \nabla^0_Y Z] - \nabla^0_{[X_f,Y]} Z - \nabla^0_Y [X_f, Z] = R^0(X_f, Y)Z + ((\nabla^0)^2 X_f)(Y, Z)$ and $\omega(((\nabla^0)^2 X_f)(Y, Z), T) = ((\nabla^0)^3 f)(Y, Z, T)$.

Assume now that the curve $\nabla_t = \nabla^0 + \sum_{k=0}^{\infty} t^p A^{(p)}$ is a curve of symplectic connections on the torus (T^{2n}, ω) and that ∇^0 is the standard flat connection.

At order 1, we have seen in Lemma 4.1 that $\underline{A}^{(1)} = (\nabla^0)^3 U^{(1)} + Q^{(1)}$ so choosing $f_1 = -U^{(1)}$ and $\psi^{(1)}(t) = \psi_{f_1}(t)$ as defined above we see that

$$\psi^{(1)}(t) \cdot \nabla^t = \nabla^0 + t\overline{Q}^{(1)} + \sum_{p=2}^{\infty} t^p \widetilde{A}^{(p)}$$

with $\omega(\overline{Q}^{(1)}(X)Y, Z) = Q^{(1)}(X, Y, Z)$.

Assume now that one has found a formal curve of symplectomorphisms $\psi^{(k-1)}(t)$ so that

$$\psi^{(k-1)}(t) \cdot \nabla^t = \nabla^0 + \sum_{p=1}^{k-1} t^p \overline{Q}^{(p)} + \sum_{p=k}^{\infty} t^p \widetilde{A}^{(p)}$$

where the $\overline{Q}^{(p)}$ are T^{2n}-invariant.

At order k, we have seen in Lemma 5.1 that $\underline{A}^{(k)} = (\nabla^0)^3 U^{(k)} + Q^{(k)}$ where $Q^{(k)}$ is T^{2n}-invariant, so choosing $f_k = -U^{(k)}$, $\psi^k_{f_k}(t)$ as defined above and $\psi^{(k)}(t) = \psi_{f_k}(t^k) \circ \psi^{(k-1)}(t)$ we see that

$$\psi^{(k)}(t) \cdot \nabla^t = \psi_{f_k}(t^k) \cdot \psi^{(k-1)}(t) \cdot \nabla^t = \nabla^0 + \sum_{p=1}^{k} t^p \overline{Q}^{(p)} + \sum_{p=k+1}^{\infty} t^p \widetilde{A}^{(p)}$$

with $\omega(\overline{Q}^{(k)}(X)Y, Z) = Q^{(k)}(X, Y, Z)$. By induction this proves that one can build a formal curve of symplectomorphisms

$$\psi(t) = \ldots \circ \psi_{(f_k)}(t^k) \circ \ldots \circ \psi_{f_2}(t^2) \circ \psi_{f_1}(t)$$

so that $\widetilde{\nabla}(t) := \psi(t).\nabla(t)$ is a formal curve of symplectic connections which is T^{2n}-invariant and has $\widetilde{W}(t) = 0$. Lifting the connection to $\mathbb{R}^{2n}$ and using Lemma 3.5 shows that $\widetilde{\nabla}(t)$ has vanishing curvature. Since $\nabla(t) = (\psi(t))^{-1} \cdot \widetilde{\nabla}(t)$, its curvature is 0 so $\nabla(t)$ is flat. ■

The above theorem implies:

Theorem 5.3 *Let ∇^t be an analytic curve of analytic symplectic connections on (T^{2n}, ω) such that ∇^0 is the standard flat connection on T^{2n}, and such that $W^t = 0$. Then the curvature R^t of ∇^t vanishes.*

6 Equivalence of formal curves of connections

In this section we study the question of when two formal curves of flat invariant connections on T^{2n} are equivalent by a formal curve of symplectomorphisms. First we consider the question on $(\mathbb{R}^{2n}, \Omega)$. Here it is easy to answer.

The first case to consider is the case of a single flat invariant connection $\nabla^A = \nabla^0 + A$ on $(\mathbb{R}^{2n}, \Omega)$. We have seen that such a connection is given by a linear map $A\colon \mathbb{R}^{2n} \to \mathfrak{sp}(2n, \mathbb{R})$ satisfying $A(X)A(Y) = 0$ and $\Omega(A(X)Y, Z)$ completely symmetric. Define $\psi^A\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ by

$$\psi^A(x) = x - \frac{1}{2}A(x)x.$$

Proposition 6.1 *ψ^A is a symplectomorphism of $(\mathbb{R}^{2n}, \Omega)$ satisfying $\psi^A \cdot \nabla^0 = \nabla^A$.*

Proof It is enough to check that ψ^A is a symplectomorphism on constant vector fields. We make extensive use of the fact that $A(X)A(Y) = 0$. If X is a constant vector field then

$$\psi^A_* X_x = \frac{d}{dt}\psi^A(x + tX)\Big|_{t=0} = (X - A(x)X)_{\psi^A(x)},$$

thus $\psi^A \cdot X = X - A(\cdot)X$. Hence

$$\Omega(\psi^A \cdot X, \psi^A \cdot Y)(x) = \Omega(X - A(x)X, Y - A(x)Y) = \Omega(X, Y).$$

It is easy to see that ψ^{-A} is an inverse for ψ^A so that ψ^A is a symplectomorphism. Indeed, $t \mapsto \psi^{tA}$ is a 1-parameter group of symplectomorphisms with generator the symplectic vector field $(X_A)x = -\frac{1}{2}A(x)x_x$.

Finally, for constant vector fields X, Y

$$(\psi^A \cdot \nabla^0)_X Y = \psi^A \cdot (\nabla^0_{\psi^{-A}\cdot X}\psi^{-A} \cdot Y) = \psi^A \cdot ((X + A(\cdot)X)(A(\cdot)Y)).$$

But

$$(X + A(\cdot)X)(A(\cdot)Y)_x = \frac{d}{dt}A(x + t(X + A(x)X))Y\Big|_{t=0} = A(X)Y$$

so

$$(\psi^A \cdot \nabla^0)_X Y = \psi^A \cdot (A(X)Y) = A(X)Y = \nabla^A_X Y.$$

■

If $\nabla^t = \nabla^0 + A^t$ is a formal curve of invariant flat connections on $(\mathbb{R}^{2n}, \Omega)$ given by a curve of linear maps $A^t\colon \mathbb{R}^{2n} \to \mathfrak{sp}(2n, \mathbb{R})[\![t]\!]$ satisfying $A^t(X)A^t(Y) = 0$ and $\Omega(A^t(X)Y, Z)$ completely symmetric, we define a formal curve of vector fields X_{A^t} by

$$X_{A^t}(f)(x) = -\frac{1}{2}(A_t(x)x)_x f$$

and set

$$\psi_{A^t} = \exp X_{A^t}.$$

Proposition 6.2 *ψ_{A^t} is a formal curve of symplectomorphisms of $(\mathbb{R}^{2n}, \Omega)$ and $\psi_{A^t} \cdot \nabla^0 = \nabla^{A^t}$.*

Proof As the exponential of a derivation, ψ_{A^t} is invertible with inverse $\exp -X_{A^t} = \psi_{-A^t}$. Moreover $\psi_{A^t} \cdot X = \exp \operatorname{ad} X_{A^t} X$ and it is easy to verify that $\operatorname{ad} X_{A^t} X = A^t(\cdot)X$, $(\operatorname{ad} X_{A^t})^2 X = 0$ so that $\psi_{A^t} \cdot X = X - A^t(\cdot)X$ as before. Likewise $\psi_{-A^t} \cdot X = X + A^t(\cdot)X$ so that

$$(\psi_{A^t} \cdot \nabla^0)_X Y = \psi_{A^t} \cdot (\nabla^0_{\psi_{-A^t} \cdot X}(Y + A^t(\cdot)Y)) = A^t(X)Y.$$

■

In particular the above proves

Theorem 6.3 *For two curves $\widetilde{\nabla^t}$ and $\widetilde{\nabla'^t}$ of invariant flat connections of Ricci-type on $(\mathbb{R}^{2n}, \Omega)$ with $\widetilde{\nabla^0} = \widetilde{\nabla'^0}$ the trivial connection, there always exists a formal curve of symplectomorphisms $\widetilde{\psi_t}$ so that $\widetilde{\psi_t} \cdot \widetilde{\nabla^t} = \widetilde{\nabla'^t}$.*

Finally, we need to know what is the general form of a formal curve of symplectomorphisms of $(\mathbb{R}^{2n}, \Omega)$ which fixes the trivial connection ∇^0.

Proposition 6.4 *Let $\psi_t = \sigma^* \circ \exp X_t$ be a formal curve of symplectomorphisms with $\psi_t \cdot \nabla^0 = \nabla^0$ then $\sigma(x) = Cx + d$ and $(X_t)_x = (C_t(x) + d_t)_x$ where $C \in Sp(2n, \mathbb{R})$, $d \in \mathbb{R}^{2n}$, $C_t \in t\mathfrak{sp}(2n, \mathbb{R})[\![t]\!]$ and $d_t \in t\mathbb{R}^{2n}[\![t]\!]$.*

Proof Evaluation at $t = 0$ shows that $\sigma \cdot \nabla^0 = \nabla^0$ so that $\sigma(x) = Cx + d$ where $C \in Sp(2n, \mathbb{R})$ and $d \in \mathbb{R}^{2n}$. Hence $\exp X_t \cdot \nabla^0 = \nabla^0$. ∇^0 is the connection for which constant vector fields are parallel, so $(\exp X_t \cdot \nabla^0)_X Y = 0$ for constant vector fields X, Y. Hence $\nabla^0_{\exp -X_t \cdot X} \exp -X_t \cdot Y = 0$ and so $\nabla^0_X \exp -X_t \cdot Y = 0$. But the only parallel vector fields for ∇^0 are the constant fields, so $\exp -X_t \cdot Y$ is constant. The leading term is $-t[X^{(1)}, Y]$ and hence $[X^{(1)}, Y]$ is constant. Since $X^{(1)}$ is symplectic, this means $X^{(1)}_x = (C_1 x + d_1)_x$ where $C_1 \in \mathfrak{sp}(2n, \mathbb{R})$. Further $\exp tX^{(1)}$ preserves ∇^0 and $\exp -tX(1) \circ \exp X_t = \exp X'_t$ with $X'_t = O(t^2)$ so we can recurse to conclude that $(X_t)_x = (C_t(x) + d_t)_x$ for formal curves $C_t \in t\mathfrak{sp}(2n, \mathbb{R})[\![t]\!]$ and $d_t \in t\mathbb{R}^{2n}[\![t]\!]$. ■

Theorem 6.5 *Let ∇^t and ∇'^t be two curves of invariant flat connections on T^{2n} with $\nabla^0 = \nabla'^0$ the trivial connection and suppose that there is a formal curve of symplectomorphisms ψ_t with $\psi_t \cdot \nabla^t = \nabla'^t$ then there is an element $C \in Sp(2n, \mathbb{Z})$ such that as a symplectomorphism of T^{2n} we have $\nabla'^t = C \cdot \nabla^t$*

Proof We lift the connections and ψ_t to $\mathbb{R}^{2n}$ and denote the lifts by a tilde. $\widetilde{\psi}_t \cdot \widetilde{\nabla}^t = \widetilde{\nabla'}^t$. Then $\widetilde{\nabla}^t = \nabla^0 + A^t$, $\widetilde{\nabla'}^t = \nabla^0 + B^t$ where $A^t, B^t \colon \mathbb{R}^{2n} \to \mathfrak{sp}(2n, \mathbb{R})[\![t]\!]$ are linear with the usual properties. Thus

$$(\widetilde{\psi}_t \circ \psi_{A^t}) \cdot \nabla^0 = \psi_{B^t} \cdot \nabla^0$$

and hence

$$\widetilde{\psi}_t \circ \psi_{A^t} = \psi_{B^t} \circ \sigma^* \circ \exp X_t$$

where $\sigma(x) = Cx + d$ and $(X_t)_x = (C_t x + d_t)_x$.

Now $\psi_{B^t} \circ \sigma^* = \sigma^* \circ \sigma^{-1*} \circ \exp X_{B^t} \circ \sigma^* = \sigma^* \circ \exp \sigma \cdot X_{B^t}$ and

$$(\sigma \cdot X_{B^t})_x = (X_{C \cdot B^t})_x + ((C \cdot B^t)(x)d)_x - \frac{1}{2}((C \cdot B^t)(d)d)_x$$

and the last two terms are in the pronilpotent semidirect product $t\mathfrak{sp}(2n,\mathbb{R})[\![t]\!]+t\mathbb{R}^{2n}[\![t]\!]$. We can exponentiate this equation in the form

$$\exp\sigma\cdot X_{B^t}=\exp X_{C\cdot B^t}\exp Z_t$$

with $Z_t\in t\mathfrak{sp}(2n,\mathbb{R})[\![t]\!]+t\mathbb{R}^{2n}[\![t]\!]$. At order zero we see that σ must be the lift of ψ^0 and so must preserve the lattice: $C\in Sp(2n,\mathbb{Z})$. Then $\sigma^{-1}\circ\widetilde{\psi}_t$ descends to the torus and leads off with the identity, so is of the form $\exp L_t$ where L_t is a formal series of periodic vector fields on $\mathbb{R}^{2n}$. Thus we have, combining the terms in $\exp t\mathfrak{sp}(2n,\mathbb{R})[\![t]\!]+t\mathbb{R}^{2n}[\![t]\!]$ and renaming as Z_t,

$$\exp L_t=\exp X_{C\cdot B^t}\exp Z_t\exp -X_{A^t}.$$

Equating the coefficient of t on both sides we see that

$$L^{(1)}=X_{C\cdot B^{(1)}}+Z^{(1)}-X_{A^{(1)}}$$

and since linear and quadratic functions are never periodic we see that $C\cdot B^{(1)}=A^{(1)}$, and $L^{(1)}=Z^{(1)}$ is constant. A simple recursion (moving constant terms past $\exp X_{C\cdot B^t}$) suffices to see that $A^t=C\cdot B^t$. ∎

So we have:

Theorem 6.6 *The moduli space of curves of Ricci-type symplectic connections starting with the standard flat connection on (T^{2n},ω) under the action of formal curves of symplectomorphisms is described by the space of formal curves $A^t\colon\mathbb{R}^{2n}\to\mathfrak{sp}(2n,\mathbb{R})[\![t]\!]$ satisfying $A^t(X)A^t(Y)=0$ and $A^t(X)Y=A^t(Y)X$, modulo the action of $Sp(2n,\mathbb{Z})$.*

It is worth noting that a curve of Ricci type connections on the torus is equivalent to the constant curve at the trivial connection when lifted to $\mathbb{R}^{2n}$.

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