Distinguishing smooth functions by a finite number of point values, and a version of the Takens embedding theorem

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Abstract

We prove a general result showing that a finite-dimensional collection of smooth functions whose differences cannot vanish to infinite order can be distinguished by their values at a finite collection of points; this theorem is then applied to the global attractors of various dissipative parabolic partial differential equations. In particular for the one-dimensional complex Ginzburg-Landau equation and for the Kuramoto-Sivashinsky equation we show that a finite number of measurements at a very small number of points (two and four respectively) serve to distinguish between different elements of the attractor: this gives an infinite-dimensional version of the Takens time-delay embedding theorem.

Key words: Global attractor, Determining nodes, Navier-Stokes equations, Gevrey regularity

1 Introduction

In recent years a large number of papers have been devoted to proving the existence of finite-dimensional attracting sets for various dissipative partial differential equations. Although the two-dimensional Navier-Stokes equations of fluid dynamics are perhaps the outstanding example (Ladyzhenskaya, 1975;

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Foias & Temam, 1979) there are numerous other well-known equations that possess such attractors (see e.g. Babin & Vishik, 1992; Hale, 1988; Temam, 1988/1996).

However, it is not immediately clear what the existence of such a set within some abstract phase space has to say about the original physical situations that these models are supposed to represent. Indeed, without some way to deduce “physical consequences” from their existence, such attractors could remain merely mathematical curiosities.

One use that has been found for attractors is as an indirect way of comparing the long-term dynamics of different systems: Stuart & Humphries (1996) compare the dynamics of equations and their numerical approximations; Mielke (1997) relates the behaviour of equations on ever larger subsets of $\mathbb{R}^n$ to their behaviour of the whole space; Caraballo et al. (1998) consider deterministic systems perturbed by a small noise.

More generally, the attractor provides a distinguished set of solutions which may have more desirable mathematical properties than the set of all possible solutions. For instance, solutions on the attractor can be defined both forwards and backwards in time, even though in general the solution of parabolic equations can only be expected to exist for $t \geq 0$. Also, the effect of parabolic smoothing often means that these solutions will be extremely smooth, enjoying bounds on their derivatives which are uniform over all the elements of the attractor: in this case all elements of the attractor are solutions in a classical sense.

In two previous papers (Friz & Robinson, 2001; Friz et al., 2001) we used the existence of a finite-dimensional global attractor consisting entirely of analytic functions to show that a finite number of point observations can be used to distinguish between different elements of the attractor. This was a conjecture due to Foias & Temam, 1984, and previously shown only for systems with inertial manifolds (Foias & Titi, 1991). In the context of fluid dynamics, where the Navier-Stokes equations are the mathematical model, this result uses the abstract dynamical systems framework in order to deduce consequences valid in the original physical domain: a fluid flow can be “fully resolved” using only a finite number of point observations of velocity (see Robinson (2001b) for a more detailed discussion).

In this paper we only require that the difference of any two solutions on the attractor has ‘finite order of vanishing’ (see Section 2.2 for details), replacing the assumption of analyticity. Remarkably, while obtaining greater generality in our results, the proofs are significantly simpler. We are also able to combine our main theorem with results due to Kukavica (1992) and Foias & Kukavica

(1995) in order to prove a result reminiscent of the Takens time-delay embedding theorem (Takens, 1981) for the complex Ginzburg-Landau equation and the Kuramoto-Sivashinsky equation.

2 Statement of the main theorem

The two main assumptions in our theorem are that the attractor has finite fractal dimension in $L^2(\Omega, \mathbb{R}^d)$, and consists of functions whose differences cannot vanish to infinite order. The first two parts of this section introduce these two ideas, while the third states the theorem.

2.1 The fractal dimension

The fractal dimension of a set X , measured in a Banach space $\mathcal{B}$, is defined as follows. Let $N_{\mathcal{B}}(X, \epsilon)$ be the minimum number of balls of radius ϵ (in the norm of $\mathcal{B}$) necessary to cover the set X . Then the fractal dimension $d_f(X; \mathcal{B})$ is given by

$$d_f(X; \mathcal{B}) = \limsup_{\epsilon \rightarrow 0} \frac{\log N_{\mathcal{B}}(X, \epsilon)}{-\log \epsilon}.$$

If $N_{\mathcal{B}}(X, \epsilon) \sim \epsilon^{-d}$ then this expression simply captures the exponent d : this is why it gives a sensible generalization of our intuitive idea of dimension. [In finite-dimensional spaces this definition is equivalent to the “box-counting dimension” (cf. Falconer, 1990, chapter 3). However, in infinite-dimensional spaces the unit cube contains elements with arbitrary large length, and a covering by balls is more sensible.]

We note here, for later use, two simple properties of the fractal dimension. First, it is clear from the definition that if $\mathcal{B}_1$ and $\mathcal{B}_2$ are two Banach spaces and $X \subset \mathcal{B}_2 \subset \mathcal{B}_1$ then

$$\|u\|_{\mathcal{B}_1} \leq k\|u\|_{\mathcal{B}_2} \quad \text{for all } u \in \mathcal{B}_2 \quad \Rightarrow \quad d_f(X; \mathcal{B}_1) \leq d_f(X; \mathcal{B}_2). \quad (1)$$

Also, if $\mathcal{B}_1$ and $\mathcal{B}_2$ are two arbitrary Banach spaces and $f: \mathcal{B}_2 \rightarrow \mathcal{B}_1$ is θ -Hölder³ then

$$d_f(f(X); \mathcal{B}_1) \leq \frac{d_f(X; \mathcal{B}_2)}{\theta}. \quad (2)$$

³ Throughout this paper we use this expression to mean that $\|f(u) - f(v)\|_{\mathcal{B}_1} \leq C\|u - v\|_{\mathcal{B}_2}^\theta$ for some constant C . In particular we do not require our “ θ -Hölder functions” to be bounded.

(For a proof and more on the fractal dimension see Eden et al. (1994, Chapter 10) or Robinson (2001a, Chapter 13)).

2.2 Order of vanishing of functions

Let Ω be an open connected set in $\mathbb{R}^m$. If $u \in C^\infty(\Omega, \mathbb{R}^d)$ then the order of vanishing of u at $x \in \Omega$ is the smallest integer k such that $\partial^\alpha u(x) \neq 0$ for some multi-index α with $|\alpha| = k$. We say that u has finite order of vanishing in Ω if the order of vanishing of u is finite at every $x \in \Omega$.

Note that while this definition does not require that the order of vanishing of u be uniformly bounded in Ω , nevertheless the order of vanishing of u is uniformly bounded on any compact subset K of Ω . Arguing by contradiction, suppose not; then there is a sequence $x_j \in K$ with the order of vanishing of u at x_j at least j . Since K is compact, x_j has a subsequence that converges to some $x^* \in K$; it follows that u vanishes to infinite order at x^* , a contradiction.

2.3 The main theorem

Although we will apply our main result to the attractors of various dissipative partial differential equations, we point out here that the theorem in fact treats only a collection of functions with particular properties.

Essentially, our theorem says that different functions lying on an attractor with finite fractal dimension d can be distinguished by comparing a finite number of their point values, with the number of observations required, k , comparable to the attractor dimension ($k \geq 16d + 1$): if u and v are elements of the attractor and

$$u(x_j) = v(x_j) \quad j = 1, \dots, k$$

then we must have $u(x) = v(x)$ throughout Ω . [For a similar result in the context of purely analytic systems see Sontag (2002).]

Theorem 1 *Let $\mathcal{A}$ be a compact subset of $L^2(\Omega, \mathbb{R}^d)$ with finite dimension $d_f(\mathcal{A})$ that, for each $r \in \mathbb{N}$ and for every compact subset K of Ω , is a bounded subset of $C^r(K, \mathbb{R}^d)$. Assume also that $u - v$ has finite order of vanishing for all $u, v \in \mathcal{A}$ with $u \neq v$. Then for $k \geq 16d_f(\mathcal{A}) + 1$ almost every set $\mathbf{x} = (x_1, \dots, x_k)$ of k points in Ω makes the map $E_{\mathbf{x}}$, defined by*

$$E_{\mathbf{x}}[u] = (u(x_1), \dots, u(x_k))$$

one-to-one between X and its image.

Furthermore the point values of u at $(x_1, \dots, x_k)$ parametrize $\mathcal{A}$: the map $E_{\mathbf{x}}^{-1} : E_{\mathbf{x}}[\mathcal{A}] \rightarrow \mathcal{A}$ is continuous from $\mathbb{R}^{kd}$ into $L^2(\Omega, \mathbb{R}^d)$ and into $C^r(K, \mathbb{R}^d)$ for every $K \subset\subset \Omega$ and $r \in \mathbb{N}$.

“Almost every” is with respect to Lebesgue measure on Ω^k . Note that $\mathcal{A}$ consists of functions in $C^\infty(\Omega, \mathbb{R}^d)$; we make here the trivial observation that the condition on the order of vanishing is satisfied if all functions in $\mathcal{A}$ are real analytic: in particular this makes the results of our previous papers, Friz & Robinson (2001) and Friz et al. (2001), corollaries of what is presented here.

2.4 Antecedents

Our main theorem combines two classical approaches to studying the finite-dimensional nature of the asymptotic dynamics of dissipative equations.

One approach has been via general results that guarantee the existence of abstract parametrizations of finite-dimensional sets using a finite number of coordinates, or equivalent of embeddings of finite-dimensional sets into some $\mathbb{R}^k$: this approach goes back to Mañé (1981), whose result was subsequently improved by Foias & Olson (1996), and is currently at its most powerful in the form due to Hunt & Kaloshin (1999). A particular version of Hunt & Kaloshin’s result, which is a central element in the proof of our theorem, is given as Theorem 4 in this paper.

The other approach works not with the set of functions that form the attractor, but with the solutions themselves. Foias & Prodi (1967) introduced the notion of ‘determining modes’ for two solutions of the 2d Navier-Stokes equations: if $P_N u$ denotes the orthogonal projection of u onto the space spanned by the first N (generalized) Fourier modes, they showed that if N is sufficiently large then

$$|P_N u(t) - P_N v(t)| \rightarrow 0 \quad \text{as } t \rightarrow \infty \quad \Rightarrow \quad |u(t) - v(t)| \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

In a similar vein, Foias & Temam (1984) introduced the notion of ‘determining nodes’. A collection of points $\{x_1, \dots, x_k\}$ in the domain Ω is ‘determining’ if

$$\max_{j=1, \dots, k} |u(x_j, t) - v(x_j, t)| \rightarrow 0 \quad \text{as } t \rightarrow \infty$$

implies that

$$\sup_{x \in \Omega} |u(x, t) - v(x, t)| \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

Foias & Temam showed that there exists a δ such that if for every $x \in \Omega$

$$|x - x_j| < \delta \quad \text{for some } j \in \{1, \dots, k\}$$

then the collection of nodes is determining. In the same paper they made the conjecture which inspired our main result. This conjecture has previously been proved for systems that possess an inertial manifold by Foias & Titi (1991).

We note here that the theory of determining modes, nodes, and also ‘volume elements’ has been developed further in a series of papers by Jones & Titi (1992a & b, 1993), and set within a general framework by Cockburn et al. (1997).

3 Preparatory results

We now prove various results that will be needed for the proof of the main theorem. Essentially we show that there are “nice” parametrizations of various subsets of $(\mathcal{A} - \mathcal{A}) \setminus \{0\}$, and that we have good control of the zero sets of the functions in these subsets.

3.1 The thickness exponent

In order to obtain our parametrization we will make use of a result due to Hunt & Kaloshin (1999; Theorem 4 below) which relies on the notion of the “thickness exponent” of a set. If X is a subset of a Banach space $\mathcal{B}$, then the thickness exponent of X in $\mathcal{B}$, the quantity $\tau(X; \mathcal{B})$, is a measure of how well X can be approximated by linear subspaces of $\mathcal{B}$. Hunt & Kaloshin define the thickness exponent as follows: given any $\epsilon > 0$, denote by $d_{\mathcal{B}}(X, \epsilon)$ the dimension of the smallest linear subspace V of $\mathcal{B}$ such that all points in X lie within an ϵ neighbourhood of V with the convention that $d_{\mathcal{B}}(X, \epsilon) = \infty$ if such a V does not exist; then

$$\tau(X; \mathcal{B}) = \limsup_{\epsilon \rightarrow 0} \frac{\log d_{\mathcal{B}}(X, \epsilon)}{-\log \epsilon}. \quad (3)$$

We note here that if $X \subset \mathcal{B}_2 \subset \mathcal{B}_1$ for two Banach spaces $\mathcal{B}_1$ and $\mathcal{B}_2$ then

$$\|u\|_{\mathcal{B}_1} \leq k\|u\|_{\mathcal{B}_2} \quad \Rightarrow \quad \tau(X; \mathcal{B}_1) \leq \tau(X; \mathcal{B}_2), \quad (4)$$

cf. (1).

We will use the following simple result that provides an alternative and more practicable definition of the thickness exponent; we give the proof here since it seems to be of independent interest: essentially we show that if $\varepsilon_{\mathcal{B}}(X, n) \sim n^{-1/\tau}$ then τ is the thickness exponent of X .

Lemma 2 *Let X be a subset of a Banach space $\mathcal{B}$, and denote by $\varepsilon_{\mathcal{B}}(X, n)$ the minimum distance between X and any n -dimensional linear subspace of $\mathcal{B}$. Then*

$$\tau(X; \mathcal{B}) = \limsup_{n \rightarrow \infty} \frac{\log n}{-\log \varepsilon_{\mathcal{B}}(X, n)}. \quad (5)$$

PROOF. Denote by τ the thickness exponent calculated using (3) and by $\tilde{\tau}$ the right hand side of (5). Let $T > \tau$. Then there exists $\epsilon_0 \in (0, 1)$ such that

$$\frac{\log d_{\mathcal{B}}(X, \epsilon)}{-\log \epsilon} < T, \quad \epsilon \in (0, \epsilon_0]$$

which is same as

$$d_{\mathcal{B}}(X, \epsilon) < \frac{1}{\epsilon^T}, \quad \epsilon \in (0, \epsilon_0]. \quad (6)$$

Let $n_0 \in \mathbb{N}$ be such that $n_0 \geq 1/\epsilon_0^T$, and let $n \geq n_0$ be an arbitrary integer. Then we may choose $\epsilon \in (0, \epsilon_0]$ such that

$$\frac{1}{\epsilon^T} - 1 < n \leq \frac{1}{\epsilon^T}.$$

By (6), there is a subspace V_n of $\mathcal{B}$ with $\dim V_n = n$ such that X is included in the ϵ neighbourhood of V_n . This implies that

$$\varepsilon_{\mathcal{B}}(X, n) \leq \frac{1}{\epsilon} \leq (n+1)^{1/T}.$$

Therefore

$$\frac{\log n}{-\log \varepsilon_{\mathcal{B}}(X, n)} \leq T \frac{\log n}{\log(n+1)}$$

and since this holds for all integers $n \geq n_0$, we get $\tilde{\tau} < T$ which shows that $\tilde{\tau} \leq \tau$.

Now, taking any $T \in (0, \tau)$, there is a sequence $\epsilon_j \in (0, 1)$ converging to 0 such that

$$d_{\mathcal{B}}(X, \epsilon_j) \geq \frac{1}{\epsilon_j^T}.$$

Hence, $\varepsilon_{\mathcal{B}}(X, \lfloor 1/\epsilon_j^T \rfloor) \geq \epsilon_j$ where $\lfloor x \rfloor$ denotes the integer part of x . In particular, setting $n_j = \lfloor 1/\epsilon_j^T \rfloor$, we get

$$\frac{\log n_j}{-\log \varepsilon_{\mathcal{B}}(X, n_j)} \geq \frac{\log \lfloor 1/\epsilon_j^T \rfloor}{\log(1/\epsilon_j)} \geq \frac{\log(1/\epsilon_j^T - 1)}{\log(1/\epsilon_j)}$$

which shows

$$\tilde{\tau} \geq T$$

and since $T < \tau$ was arbitrary, we conclude $\tilde{\tau} \geq \tau$. $\square$

In general the thickness exponent of X measured in L^2 can be related to the smoothness of functions in X (Friz & Robinson, 1999); similar results are also true in other spaces, and lemma 3 below contains one result of this type (cf. Friz & Robinson, 2001; Friz et al. 2001).

3.2 Properties of “ $\mathcal{A}_\phi$ ” (truncated versions of functions in $\mathcal{A}$)

In fact rather than dealing with $\mathcal{A}$ itself we will consider the collection of functions in $\mathcal{A}$ after a suitable truncation.

The domain Ω can be written in the form

$$\Omega = \bigcup_{n=1}^{\infty} K_n, \tag{7}$$

with each K_n a compact set whose interior $\dot{K}_n$ is a non-empty smooth connected set with smooth boundary. In what follows we will take K to be one of the sets K_j . Fix any smooth test function $\phi \in C^\infty(\mathbb{R}^m, [0, 1])$ such that

$$\phi \equiv 1 \quad \text{on } K$$

and

$$\text{supp } \phi \subset \subset \Omega$$

i.e., the support of ϕ is a compact subset of Ω . Denote by Ω_ϕ the interior of $\text{supp } \phi$, and set

$$\mathcal{A}_\phi = \{f\phi|_B : f \in \mathcal{A}\}$$

where B is a sufficiently large open ball such that $\Omega_\phi \subset\subset B$. Clearly, $\mathcal{A}_\phi$ consists of functions from B into $\mathbb{R}^d$.

We show that the fractal dimension of $\mathcal{A}_\phi$ in all the spaces $C^r(B, \mathbb{R}^d)$ is bounded by the fractal dimension of $\mathcal{A}$ in $L^2(\Omega, \mathbb{R}^d)$ and that its thickness exponent is zero in all these spaces.

Lemma 3 *Let $\mathcal{A}$ be a finite-dimensional compact subset of $L^2(\Omega, \mathbb{R}^d)$ that, for each $r \in \mathbb{N}$ and for every compact subset K of Ω , is a bounded subset of $C^r(K, \mathbb{R}^d)$. Then for any $r \in \mathbb{N}$, the fractal dimension of $\mathcal{A}_\phi$ in $C^r(B, \mathbb{R}^d)$ is less than or equal to that of $\mathcal{A}$ in $L^2(\Omega, \mathbb{R}^d)$, and the thickness exponent of $\mathcal{A}_\phi$ in $C^r(B, \mathbb{R}^d)$ is zero.*

PROOF.

Let A denote the d -component negative Laplacian on B with Dirichlet boundary conditions. It follows from standard Sobolev embedding results and the theory of elliptic regularity (see Evans (1998) for example) that

$$\|u\|_{C^r(B, \mathbb{R}^d)} \leq C \|u\|_{H^{r+(d/2)+1}(B, \mathbb{R}^d)} \leq C \|u\|_{D(A^{(r+(d/2)+1)/2})} \quad (8)$$

[with $C = C(m, d)$], so that using (1) and (4) it is sufficient to prove the lemma with $C^r(B, \mathbb{R}^d)$ replaced by $D(A^r)$. We note here that since functions in $\mathcal{A}$ enjoy uniform bounds on their derivatives (when restricted to compact subsets of Ω), so do functions in $\mathcal{A}_\phi$ and hence $\mathcal{A}_\phi$ is uniformly bounded in $D(A^r)$ for each $r \in \mathbb{N}$,

$$|A^r u| \leq R_r \quad \text{for all} \quad u \in \mathcal{A}_\phi. \quad (9)$$

We start with the fractal dimension. If $s > r$, then for any $u \in D(A^s)$ we have the interpolation inequality

$$|A^r u| \leq |u|^{1-(r/s)} |A^s u|^{r/s}.$$

It follows from (9) that the identity map from $\mathcal{A}_\phi$ onto itself is Hölder continuous as a map from $L^2(\Omega, \mathbb{R}^d)$ into $D(A^r)$ with Hölder exponent as close to 1 as required. That the fractal dimension of $\mathcal{A}_\phi$ in the space $D(A^r)$ is bounded by $d_f(\mathcal{A}; L^2)$ is a consequence of (2).

In order to show that the thickness exponent is zero, let P_n denote the projection onto the space spanned by the first n eigenfunctions of A ,

$$P_n u = \sum_{j=1}^n (u, w_j) w_j$$

and set $Q_n = I - P_n$. Recall that the n th eigenvalue of A satisfies $\lambda_n \sim n^{2/d}$ (see e.g. Davies, 1995). For any $k \in \mathbb{N}$ we have

$$\begin{aligned} |A^j Q_n u| &= |A^{-k} Q_n (A^{k+j} u)| \leq \|A^{-k} Q_n\|_{\text{op}} |A^{k+j} u| \leq \lambda_{n+1}^{-k} R_{k+j} \\ &\leq C(n+1)^{-2k/d} R_{k+j} \leq Cn^{-2k/d} R_{k+j}, \end{aligned}$$

where R_j is the bound from (9).

Therefore,

$$\text{dist}_{D(A^j)}(\mathcal{A}_\phi, P_n D(A^j)) \leq \frac{C R_{k+j}}{n^{2k/d}}$$

which shows that $\varepsilon_{D(A^j)}(\mathcal{A}_\phi, n) \leq Cn^{-2k/d}$, and hence using lemma 2 that

$$\tau(\mathcal{A}_\phi; D(A^j)) \leq \frac{d}{2k}.$$

Since this holds for any k , we get $\tau(\mathcal{A}_\phi; D(A^j)) = 0$, and the result follows. $\square$

We will use the fact that the thickness exponent of $\mathcal{A}_\phi$ is zero in order to apply the following theorem which is a particular case of a result due to Hunt & Kaloshin (1999). [Foias & Olson (1996) proved a similar result, but with no bound on the Hölder exponent of the parametrization.]

Theorem 4 *Let X be a bounded subset of a Banach space $\mathcal{B}$ with finite fractal dimension $d_f(X)$ and thickness exponent zero. Then, if N is an integer with $N > 2d_f(X)$, and*

$$0 < \theta < 1 - \frac{2d_f(X)}{N},$$

there exists a parametrization of X in terms of N coordinates which is θ -Hölder from $\mathbb{R}^N$ into $\mathcal{B}$.

We note that the parameters in fact range over a subset Π of $\mathbb{R}^N$ that is homeomorphic to X .

3.3 Zero sets of functions that vanish to finite order

The fact that functions in $(\mathcal{A}_\phi - \mathcal{A}_\phi) \setminus \{0\}$ vanish to finite order enables us to obtain good control on their zero sets (Theorem 6, below). In the course of the argument will need the following Hölder implicit function theorem.

Proposition 5 *Let $E \subset \mathbb{R}^D$, and let $u(x; \varepsilon)$ be a function from $\mathbb{R} \times E$ into $\mathbb{R}$ which is Hölder continuous in ε with exponent θ ,*

$$|u(x; \varepsilon_1) - u(x; \varepsilon_2)| \leq C|\varepsilon_1 - \varepsilon_2|^\theta,$$

and differentiable in x with u_x jointly continuous in (x, ε) at (x_0, ε_0) . Then if

$$u(x_0; \varepsilon_0) = 0 \quad \text{and} \quad u_x(x_0; \varepsilon_0) \neq 0$$

there exists a neighbourhood of (x_0, ε_0) in $\mathbb{R} \times E$, and a θ -Hölder function $x(\varepsilon)$ such that

$$u(x(\varepsilon), \varepsilon) = 0$$

and $(x(\varepsilon), \varepsilon)$ is the unique zero of $u(x, \varepsilon)$ in this neighbourhood for each ε sufficiently close to ε_0 .

PROOF. Essentially the proof follows Hale (1969, Chapter 0, Theorem 3.3). Without loss of generality we set $x_0 = 0$ and $\varepsilon_0 = 0$ and write

$$u(x; \varepsilon) = u_x(0; 0)x + F(x; \varepsilon),$$

where $F(x; \varepsilon) = u(x; \varepsilon) - u_x(0; 0)x$. It follows that a solution of $u(x; \varepsilon) = 0$ is precisely a fixed point of the map

$$T_\varepsilon[x] \equiv -\frac{F(x; \varepsilon)}{u_x(0; 0)}.$$

It is straightforward to show, using the continuity of u_x in x and ε , that for each fixed ε with $|\varepsilon|$ sufficiently small, T_ε is a contraction (with constant $\lambda < 1$ independent of ε) on a sufficiently small neighbourhood of the origin in $\mathbb{R}$: hence T_ε has a unique fixed point $x(\varepsilon)$ for all ε with $|\varepsilon|$ sufficiently small.

That $x(\varepsilon)$ is Hölder continuous in ε follows easily. Indeed, observe that

$$|T_{\varepsilon_1}[x] - T_{\varepsilon_2}[x]| \leq \frac{|u(x; \varepsilon_1) - u(x; \varepsilon_2)|}{|u_x(0; 0)|} \leq \frac{C}{|u_x(0; 0)|} |\varepsilon_1 - \varepsilon_2|^\theta,$$

using the Hölder continuity of $u(x; \varepsilon)$ in ε . Then we have

$$\begin{aligned} |x(\varepsilon_1) - x(\varepsilon_2)| &\leq |T_{\varepsilon_1}[x(\varepsilon_1)] - T_{\varepsilon_1}[x(\varepsilon_2)]| + |T_{\varepsilon_1}[x(\varepsilon_2)] - T_{\varepsilon_2}[x(\varepsilon_2)]| \\ &\leq \lambda |x(\varepsilon_1) - x(\varepsilon_2)| + \frac{C}{|u_x(0; 0)|} |\varepsilon_1 - \varepsilon_2|^\theta, \end{aligned}$$

from which

$$|x(\varepsilon_1) - x(\varepsilon_2)| \leq \frac{C}{(1 - \lambda)|u_x(0; 0)|} |\varepsilon_1 - \varepsilon_2|^\theta$$

follows. $\square$

We now apply this to generalize a lemma from a paper of Yamazato (1983) concerning the zero sets of real analytic functions (cf. Federer (1991) and Sontag (1996)); the proof below is significantly simpler than that in Friz & Robinson (2001).

Theorem 6 *Let K be a compact connected subset of $\mathbb{R}^m$. Suppose that for every $p \in \Pi \subset \mathbb{R}^N$ the function $w = w(x; p)$,*

$$w: K \times \Pi \rightarrow \mathbb{R}^d,$$

has order of vanishing at most $M < \infty$, and is such that $\partial^\alpha w(x; p)$ depends on p in a θ -Hölder way for all $|\alpha| \leq M$. Then the zero set of $w(x; p)$, i.e.,

$$\{(x, p) : w(x, p) = 0\},$$

viewed as a subset of $K \times \Pi \subset \mathbb{R}^m \times \mathbb{R}^N$, is contained in a countable union of manifolds of the form

$$(x_i(x', p), x'; p), \tag{10}$$

where $x' = (x_1, \dots, x_{i-1}, x_{i+1}, x_m)$ and x_i is a θ -Hölder function of its arguments.

(The same result is true if w has finite order of vanishing within an open set Ω , and all partial derivatives depend Hölder continuously on p .)

PROOF. We treat the case $d = 1$, since the zero set of a function into $\mathbb{R}^d$ is a subset of the set of zeros of any one component; we denote by Z the zero set of w , and by Z_j the collection of all points in Z with order of vanishing exactly j .

Now take a point $(y; p_0) \in Z_j$. Then $\partial^\alpha u(y; p_0) = 0$ for every α with $|\alpha| = j - 1$ (a finite collection), while for each such α we have $\partial_i \partial^\alpha u(y; p_0) \neq 0$ for some i . We can therefore apply proposition 1 to the function

$$u(x_i; \underbrace{(x'; p)}_\varepsilon) = w(x; p), \quad \text{where } x' = (x_1, \dots, x_{i-1}, x_{i+1}, x_m),$$

to deduce that all zeros of $\partial^\alpha w$ within a neighbourhood of $(y; p_0)$ are contained in a manifold of the form

$$(x_i(x'; p), x'; p),$$

with x_i a θ -Hölder function of its arguments.

It follows that every point in Z_j has an open neighbourhood U such that $Z_j \cap U$ is contained in a finite collection of θ -Hölder manifolds. Thus, using Lindelöf's Theorem (see Kuratowski (1968, Chapter II, Section 17)), a countable collection of these neighbourhoods still covers Z_j , and hence Z_j is contained in a countable union of θ -Hölder manifolds. Since $Z = \cup_{j=1}^M Z_j$ by assumption the same is true of Z itself. $\square$

3.4 The Hausdorff dimension

The structure of the zero set of functions in $\mathcal{A}_\phi - \mathcal{A}_\phi$ allows us to obtain good bounds on its Hausdorff dimension. We recall that the Hausdorff dimension of a subset X of $\mathbb{R}^n$ is defined as

$$d_H(X) = \inf\{d \geq 0 : \mathcal{H}^d(X) = 0\},$$

where $\mathcal{H}^d$ is the d -dimensional Hausdorff measure,

$$\mathcal{H}^d(X) = \lim_{\epsilon \rightarrow 0} \inf \left\{ \sum_i r_i^d : X \subseteq \cup_i B(x_i, r_i) \text{ with } r_i \leq \epsilon \right\}$$

(here $B(x, r)$ is a ball centred at x with radius r ; see Falconer (1985 or 1990) for further details). We will require the following four properties:

- (1) If $X \subset \mathbb{R}^n$ and $f: X \rightarrow \mathbb{R}^m$ is a θ -Hölder function then the Hausdorff dimension of the graph of f ,

$$G = \{(x, f(x)) : x \in X\} \subset \mathbb{R}^{n+m} = \mathbb{R}^n \times \mathbb{R}^m,$$

satisfies

$$d_H(G) \leq n + (1 - \theta)m. \tag{11}$$

- (2) Hausdorff dimension is stable under countable unions,

$$d_H\left(\bigcup_{j=1}^{\infty} X_j\right) = \sup_j d_H(X_j). \tag{12}$$

- (3) Hausdorff dimension does not increase under the application of bounded linear maps L ,

$$d_H(LX) \leq d_H(X). \tag{13}$$

- (4) A set in $\mathbb{R}^n$ with Hausdorff dimension strictly less than n has zero Lebesgue measure.

For (1) see Friz et al. (2001), for (2) and (3) see Falconer (1985 or 1990); (4) follows from the definition of Hausdorff dimension, since n -dimensional Hausdorff measure is proportional to n -dimensional Lebesgue measure (see Theorem 1.12 in Falconer, 1985).

4 Proof of the main theorem

We can now prove the theorem: recall that we have to show that almost every choice of k points $\mathbf{x} = (x_1, \dots, x_k)$ from Ω makes the map

$$E_{\mathbf{x}}[u] = (u(x_1), \dots, u(x_k))$$

one-to-one between $\mathcal{A}$ and its image.

PROOF. If $E_{\mathbf{x}}$ fails to be one-to-one for a set of $\mathbf{x}$ of positive measure in Ω^k , then it must fail to be one-to-one for a set of positive measure in K_i for some i . So it suffices to show that for each i almost every collection of k points in K_i makes $E_{\mathbf{x}}$ one-to-one between $\mathcal{A}$ and its image.

For this, it suffices to show that for almost every choice of k points in $\dot{K}_i$ the map $E_{\mathbf{x}}$ is one-to-one between $\mathcal{A}_{\phi}$ and its image (where ϕ is chosen equal to 1 on K_i); if ϕu and ϕv are equal on $\dot{K}_i$ then the difference of u and v must be identically zero on $\dot{K}_i$, and since the difference of functions in $\mathcal{A}$ have finite order of vanishing, $u - v$ can only be zero on $\dot{K}_i$ if $u = v$ throughout Ω .

So we take a fixed compact set $K = K_i$ and with ϕ chosen as in section 2 we consider $W = (\mathcal{A}_{\phi} - \mathcal{A}_{\phi}) \setminus \{0\}$. If $E_{\mathbf{x}}$ is to be one-to-one on $\mathcal{A}_{\phi}$ then it should be non-zero on W .

Now divide W into the countable union

$$W = \bigcup_{j=0}^{\infty} W_j$$

where W_j consists of functions whose order of vanishing is at most j . If for each j almost every collection of points $\mathbf{x}$ in $\dot{K}_i$ makes $E_{\mathbf{x}}$ non-zero on W_j , then clearly almost every collection of such points makes $E_{\mathbf{x}}$ non-zero on all of W .

For a fixed j , the results of section 3 imply that $d_f(W_j; C^r(B, \mathbb{R}^d)) \leq 2d_f(\mathcal{A}; L^2)$ and $\tau(W_j; C^r(B, \mathbb{R}^d)) = 0$ for all j and r . Thus, using Theorem 4, for any

$$N > 4d_f(\mathcal{A}) \quad \text{and} \quad \theta < 1 - (4d_f(\mathcal{A})/N)$$

there exists a parametrization $w(x; p)$ of W_j in terms of N coordinates $p \in \Pi \subset \mathbb{R}^N$ which is θ -Hölder into $C^j(B, \mathbb{R}^d)$. The mapping

$$p \mapsto w(x; p)|_K$$

is therefore θ -Hölder into $C^j(K, \mathbb{R}^d)$. It follows that all the derivatives of u up to order j depend in a θ -Hölder way on the parameter p .

Now, suppose that $\mathbf{x} = (x_1, \dots, x_k)$ is a set of k points in $\dot{K}$ for which $E_{\mathbf{x}}$ is zero somewhere on W_j . Then there must exist a $p \in \Pi$ such that

$$w(x_i; p) = 0 \quad \text{for all} \quad i = 1, \dots, k.$$

Theorem 6 guarantees that the zeros of w , considered as a subset of $\dot{K} \times \Pi$, are contained in a countable collection of sets, each of which is the graph of a θ -Hölder function,

$$(x', x_j(x'; \varepsilon); \varepsilon),$$

where $x' = (x_1, x_{j-1}, x_{j+1}, x_m)$. Each of these manifolds has $(m-1) + N$ free parameters.

It follows that collections of k such zeros (considered as a subset of $\dot{K}^k \times \Pi$) are contained in the product of k such manifolds. Since the coordinate p is common to each of these, they are the graphs of θ -Hölder functions from a subset of $\mathbb{R}^{N+(m-1)k}$ into $\mathbb{R}^k$. Equation (11) shows that each of these sets has Hausdorff dimension at most

$$N + (m-1)k + k(1 - \theta),$$

and using (12) the same goes for the whole countable collection.

The projection of this collection onto $\dot{K}^k$ enjoys the same bound on its dimension (13), and so to make sure that these ‘bad choices’ do not cover $\dot{K}^k \subset \mathbb{R}^{dk}$ we need

$$N + (m-1)k + k(1 - \theta) < mk.$$

This is certainly true if

$$k > \frac{N}{\theta}$$

and since θ can be chosen arbitrarily close to $1 - (4d_f(\mathcal{A})/N)$, it follows that

$$k > \frac{N^2}{N - 4d_f(\mathcal{A})}$$

will suffice. Choosing the integer value of N with $8d_f(\mathcal{A}) - \frac{1}{2} \leq N < 8d_f(\mathcal{A}) + \frac{1}{2}$ shows that $k \geq 16d_f(\mathcal{A}) + 1$ suffices.

Since the collection of ‘bad choices’ is a subset of $\mathbb{R}^{km}$ with Hausdorff dimension less than km it follows (from fact (4) in section 3.4) that almost every choice (wrt Lebesgue measure) makes $E_{\mathbf{x}}$ non-zero on W_j , and the main part of the theorem holds.

The continuity of $E_{\mathbf{x}}^{-1}$ into $L^2(\Omega, \mathbb{R}^d)$ follows since $\mathcal{A}$ is compact and $E_{\mathbf{x}}$ is one-to-one between $\mathcal{A}$ and its image. Continuity into $C^r(K, \mathbb{R}^d)$ is then a consequence of the boundedness of

$$\mathcal{A}|_K = \{u|_K : u \in \mathcal{A}\}$$

in $C^j(K, \mathbb{R}^d)$ for any j , the Sobolev interpolation inequality

$$\|u\|_{H^s} \leq C \|u\|_{L^2}^{1-(s/k)} \|u\|_{H^k}^{s/k},$$

and the Sobolev embedding $H^s \subset C^r$ for $s > r + (d/2)$.

□

5 First applications: spatial nodes

We now apply Theorem 1 to the global attractors of various dissipative partial differential equations. These are compact, globally attracting, invariant sets for the semigroup generated by solutions of the equations (see, for example, Babin & Vishik (1992); Hale (1988); Ladyzhenskaya (1991); Robinson (2001); Temam (1988/1996)). As such they can be viewed as representing all possible “long-term configurations” of the system, together with their dynamics.

5.1 Analytic attractors

Our previous results (Friz & Robinson (2001), Friz et al. (2001)) treated attractors consisting entirely of analytic functions: since the difference of any two analytic functions is also analytic, the condition on the finite order of vanishing is automatically satisfied in this case. Furthermore the estimates yielding the analyticity give the required uniform estimates on the derivatives. For more details see Friz & Robinson (2001) for the case of periodic boundary conditions (2d Navier-Stokes equations with analytic forcing; the complex Ginzburg-Landau equation in 1d and 2d; the 1d Kuramoto-Sivashinsky

equation; reaction-diffusion equations with analytic nonlinearity) and Friz et al. (2001) for the 2d Navier-Stokes equations with Dirichlet boundary conditions and analytic forcing.

5.2 A smooth reaction-diffusion equation

In this section, we provide an example of a non-analytic equation where Theorem 1 leads to existence and density of instantaneous determining nodes. Consider the reaction diffusion equation (RDE)

$$u_t - \Delta u + f(u, x) = 0 \tag{14}$$

on a bounded domain $\Omega \subset \mathbb{R}^m$, where

$$f(u, x) = \sum_{j=0}^n a_j(x) u^j$$

with $n \geq 3$ odd and

$$a_n(x) \geq \delta > 0, \quad x \in \Omega$$

to ensure dissipativity. We also assume that $a_j \in C^\infty(\mathbb{R}^m)$ for all $j = 0, \dots, n$. We choose one of the following boundary conditions:

- (i) $\Omega = [0, 1]^m$ is a periodic domain; or
- (ii) Ω is a convex C^1 domain with Dirichlet boundary conditions ($u|_{\partial\Omega} = 0$); or
- (iii) Ω is a $C^{2,\alpha}$ domain for some $\alpha \in (0, 1]$, with Dirichlet boundary conditions $u|_{\partial\Omega} = 0$;

Given one of these conditions we have the following unique continuation property: let v be a smooth solution of the equation

$$\partial_t v - \Delta v + \sum_{j=1}^m W_j(x, t) \partial_j v + V(x, t) v = 0$$

in $\Omega \times (0, T_0)$ for some $T_0 > 0$ where V and W_j are bounded and measurable; then $v(\cdot, t)$ has finite order of vanishing in Ω for every $t \in (0, T_0)$. In the case of boundary condition (i) the proof is given in Kurata (1994) and also in Poon (1996). When we consider (ii) we apply Theorem 1.1' in Poon (1996). For case (iii) we use combination of two methods from Poon's papers. Namely, we follow the argument in his 1996 paper but with the standard heat kernel substituted

by the Neumann heat kernel as in the 2000 paper: in view of the Remark on p. 530 in Poon (1996) and results in Poon (2000), the proof extends to this situation as well.

The relevance of this statement to the RDE is given by the following simple fact: if u_1 and u_2 are two solutions of the RDE (not necessarily with same initial condition) then the difference satisfies

$$\partial_t v - \Delta v + V(x, t)v = 0$$

where

$$V(x, t) = \int_0^1 \frac{\partial f}{\partial u}(\tau u_1(x, t) + (1 - \tau)u_2(x, t)) d\tau.$$

We now recall some basic facts about existence and uniqueness of solutions and existence of the global attractor. For every initial datum $u_0 \in H = L^2(\Omega)$ (in the case of periodic boundary conditions, $H = L^2_{\text{per}}(\Omega)$), there exists a unique solution $u \in C([0, \infty), H) \cap L^2([0, \infty), H^1(\Omega))$ (with $H^1(\Omega)$ substituted by $H^1_{\text{loc}}(\Omega)$ in the case of periodic boundary conditions) of (14) with

$$u(x, 0) = u_0(x), \quad x \in \Omega.$$

The solution u satisfies $u \in C^\infty(\Omega \times (0, \infty))$ in the case of Dirichlet boundary conditions and $u \in C^\infty(\mathbb{R}^m \times (0, \infty))$ in the case of periodic boundary conditions.

The equation has a global attractor $\mathcal{A}$, which can be characterized as the largest bounded invariant set. Moreover, due to the strong dissipative nature of the nonlinearity, it is the largest invariant subset of H and attracts all solutions starting in H at a uniform rate. The attractor also has finite fractal dimension in $L^2(\Omega, \mathbb{R}^d)$, $d_f(\mathcal{A})$, which can be bounded above by a method due to Constantin & Foias (1985).

A direct application of Theorem 1 gives the following statement.

Theorem 7 *Let $k \geq 16d_f(\mathcal{A}) + 1$. Then for almost every set of k points $\mathbf{x} = (x_1, \dots, x_k)$ in Ω , the mapping*

$$E_{\mathbf{x}}: \mathcal{A} \rightarrow \mathbb{R}^k$$

defined by

$$E_{\mathbf{x}}[u] = (u(x_1), \dots, u(x_k))$$

is one-to-one between $\mathcal{A}$ and its image.

We had hoped to apply Theorem 1 to the 2d Navier-Stokes equations with a smooth forcing term. However, we currently do not know whether the difference of two solutions of these equations cannot vanish to infinite order.

6 Space-time nodes and Takens-type theorems

First we prove a generalized version of Theorem 1 which will allow us to translate our previous result (valid for almost every collection of k spatial points) into results valid for almost every choice of k points in space-time, and even for almost every choice of k times at a small number of spatial points: these statements are made precise below.

Let $\Omega_1 \subset \mathbb{R}^{m_1}$ and $\Omega_2 \subset \mathbb{R}^{m_2}$ be two open connected sets.

Theorem 8 *Let X be a compact subset of $L^2(\Omega_1, \mathbb{R}^{d_1})$ with finite fractal dimension, $d_f(X)$. Let Y be a subset of $L^2(\Omega_2, \mathbb{R}^{d_2})$ that is bounded in $C^r(K, \mathbb{R}^{d_2})$ for each $r \in \mathbb{N}$ and each compact subset K of Ω_2 . Assume that $u - v$ has finite order of vanishing for every $u, v \in Y$ such that $u \not\equiv v$. Also, assume that there exists a one-to-one map $\Sigma: X \rightarrow Y$ that is Lipschitz from $L^2(\Omega_1, \mathbb{R}^{d_1})$ into $L^2(\Omega_2, \mathbb{R}^{d_2})$. Then for every $k \geq 16d_f(X) + 1$ almost every set $\mathbf{y} = (y_1, \dots, y_k)$ of k points in Ω_2 makes the map*

$$u \mapsto ((\Sigma u)(y_1), \dots, (\Sigma u)(y_k))$$

one-to-one between X and its image.

The proof of this theorem is essentially a repetition of the proof of Theorem 1, but working throughout with Y rather than with X . Clearly if the map $E: Y \rightarrow \mathbb{R}^{kd_2}$ defined by $v \mapsto (v(y_1), \dots, v(y_k))$ is one-to-one between Y and its image, then $E \circ \Sigma$ is one-to-one between X and its image.

6.1 Almost every collection in space-time

Suppose that the set X consists of solutions of a partial differential equation that we write in an (extremely) abstract form as

$$\frac{du}{dt} = F(u); \quad (15)$$

we assume that F is a local C^∞ function of u and its derivatives, i.e. that for each $k \in \mathbb{N}$ and every compact set $K \subset \Omega$ there exists a $k' \in \mathbb{N}$ and a compact set $K' \subset \Omega$ such that

$$\max_{|\alpha| \leq k} \sup_{x \in K} |D^\alpha F(u)| \leq \max_{|\beta| \leq k'} \sup_{x \in K'} |D^\beta u|. \quad (16)$$

Denote by $S(t)u_0$ the solution at time t of (15) with initial condition u_0 . The next result appears to have many conditions, but they are readily satisfied by many well-known examples of partial differential equations.

Corollary 9 *Suppose that $\mathcal{A}$ is an invariant set under the dynamics of a PDE (15) satisfying (16). Assume also that for each $t > 0$ the solution operator $S(t)$ restricted to $\mathcal{A}$ is (i) Lipschitz from $L^2(\Omega; \mathbb{R}^d)$ into itself,*

$$|S(t)u_0 - S(t)v_0| \leq L(t)|u_0 - v_0| \quad \text{with} \quad L(t) \in L^2(0, T), \quad (17)$$

and (ii) injective, i.e. if $S(t)u_0 = S(t)v_0$ then $u_0 = v_0$. Then provided that the assumptions of Theorem 1 hold and that $k > 16d_f(\mathcal{A}) + 1$, for any $T > 0$ almost every collection of k points $\{(x_j, t_j)\}_{j=1}^k$ from $\Omega \times [0, T]$ make the map

$$u \mapsto (u(x_1, t_1), \dots, u(x_k, t_k))$$

one-to-one between $\mathcal{A}$ and its image.

(The notation $u(x, t)$ above is shorthand for $[S(t)u](x)$.)

PROOF. Let $X = \mathcal{A}$, and define $\Sigma : L^2(\Omega, \mathbb{R}^d) \rightarrow L^2(\Omega \times [0, T], \mathbb{R}^d)$ by

$$[\Sigma u](t) = S(t)u \quad \text{for every } t \in [0, T];$$

set $Y = \Sigma(X)$. If $\Sigma u = \Sigma v$ then $S(t)u = S(t)v$ for almost every $t \in [0, T]$; in particular $S(\tau)u = S(\tau)v$ for some $\tau > 0$ and the injectivity property of $S(t)$ implies that $u = v$.

Clearly Σ is a Lipschitz continuous map from $L^2(\Omega; \mathbb{R}^d)$ into $L^2(\Omega \times [0, T]; \mathbb{R}^d)$, since

$$\begin{aligned} \|[\Sigma u] - [\Sigma v]\|_{L^2(\Omega \times [0, T])}^2 &\leq \int_0^T \|S(t)u - S(t)v\|_{L^2(\Omega)}^2 dt \\ &\leq \left(\int_0^T L(t)^2 dt \right) \|u_0 - v_0\|_{L^2(\Omega)}^2, \end{aligned}$$

while condition (16) ensures that Y is a bounded subset of $C^r(K \times [0, T]; \mathbb{R}^d)$ for every $r \in \mathbb{N}$ and every compact subset K of Ω . Finally, suppose that the difference of two functions in Y vanishes to infinite order at a space-time point (ξ, τ) . Then in particular the function of space $[\Sigma u](\tau) - [\Sigma v](\tau)$ vanishes to infinite order at ξ . Since $\mathcal{A}$ is invariant under $S(t)$, $[\Sigma u](\tau)$ and $[\Sigma v](\tau)$ are both elements of $\mathcal{A}$, and hence cannot vanish to infinite order at ξ unless they are equal. It now follows, using the injectivity property for $0 \leq t < \tau$, and the uniqueness of solutions (a consequence of (17)) for $\tau < t \leq T$, that $[\Sigma u](t) = [\Sigma v](t)$ for all $0 \leq t \leq T$, and hence that $\Sigma u = \Sigma v$. Theorem 8 now yields the result as stated. $\square$

The theorem in this form applies to all the examples of section 5. A related result, using a somewhat different argument, can be found in Robinson (2001b).

6.2 Two Takens time delay embedding theorems

Takens' celebrated time delay embedding theorem (Takens, 1980, see also Sauer et al., 1993) guarantees, under various genericity assumptions, that a finite number of repeated observations at equally spaced time intervals⁴ are sufficient to distinguish between different elements of the attractor of a finite-dimensional dynamical system: if the attractor has dimension d then for a prevalent set of Lipschitz functions $h : \mathbb{R}^n \rightarrow \mathbb{R}$ and all T sufficiently small, the map

$$u \mapsto H[u] = (h[u(0)], h[u(T)], h[u(2T)], \dots, h[u(2dT)])$$

is 1 – 1 on the attractor (this formulation is from Hunt & Kaloshin, 1999).

⁴ In fact these time intervals need not be equally spaced, see remark 2.9 in Sauer et al. (1993), but it is much easier to reconstruct the underlying dynamics if they are by using a simple shift on the time series.

However, this time-delay embedding theorem has only been proved for finite-dimensional systems. Here we apply Theorem 8 to deduce Takens-type theorems for two particular infinite-dimensional examples. [For a related, but weaker, result for the 2d Navier-Stokes equations see Robinson (2001b).]

6.2.1 The complex Ginzburg-Landau equation

First we consider the complex Ginzburg-Landau equation (CGLE)

$$u_t - (1 + i\nu)u_{xx} + (1 + i\mu)|u|^2u - au = 0 \quad (18)$$

with initial condition

$$u(x, 0) = u_0(x) \quad (19)$$

subject to periodic boundary conditions with respect to $\Omega = [0, 1]$.

It is known that for every initial datum $u_0 \in L^2_{\text{per}}(\Omega)$, there exists a unique solution $u(x, t) = (S(t)u_0)(x)$ of (18)–(19) defined for all $(x, t) \in \mathbb{R} \times [0, \infty)$ (see Temam, 1988). The solution is a real-analytic function of $(x, t) \in \mathbb{R} \times (0, \infty)$ (Jolly et al., 1990). Also, for every $a, b \in (0, \infty)$ such that $a < b$ the mapping

$$L^2_{\text{per}}(\Omega) \rightarrow C^r_{\text{per}}(\Omega \times [a, b])$$

defined by

$$u_0 \mapsto (S(t)u_0)(x)$$

is continuous.

The complex Ginzburg-Landau equation possesses a global attractor $\mathcal{A}$ which can be characterized as the largest bounded invariant subset of $L^2_{\text{per}}(\Omega)$. Due to the strong dissipation inherent in the nonlinearity, it is the maximal invariant subset of $L^2_{\text{per}}(\Omega)$ and attracts every set in $L^2_{\text{per}}(\Omega)$ at a uniform rate. It consists of real-analytic functions and has finite fractal dimension $d_f(\mathcal{A})$ in $L^2_{\text{per}}(\Omega)$ (Ghidaglia & Hron, 1987).

It was shown in Kukavica (1992) that there exists a number $\delta_0 > 0$ such that if x_1, x_2 is an arbitrary pair of different points with $|x_1 - x_2| \leq \delta_0$, then for any two solutions u_1 and u_2 belonging to the global attractor $\mathcal{A}$,

$$u_1(x_j, t) = u_2(x_j, t), \quad j = 1, 2, \quad \text{for every } t \geq 0$$

implies that

$$u_1(x, t) = u_2(x, t), \quad x \in \Omega, \quad \text{for every } t \geq 0.$$

We say that x_1 and x_2 are a set of “determining nodes”. (The constant δ_0 can be explicitly computed in terms of μ , ν , and a .) By combining this with Theorem 8 we obtain the following result.

Theorem 10 *Let x_1 and x_2 be two points with $|x_1 - x_2| \leq \delta_0$ (δ_0 as above), choose $T_0 > 0$, and let $k \geq 16d_f(\mathcal{A}) + 1$. Then for almost every set of k times*

$$\mathbf{t} = (t_1, t_2, \dots, t_k)$$

where $t_1, \dots, t_k \in [0, T_0]$ the mapping

$$E_{\mathbf{t}}: \mathcal{A} \rightarrow \mathbb{R}^{2k}$$

defined by

$$E_{\mathbf{t}}(u) = ([S(t_1)u](x_1), \dots, [S(t_k)u](x_1), [S(t_1)u](x_2), \dots, [S(t_k)u](x_2))$$

is one-to-one between $\mathcal{A}$ and its image.

This means, in particular, that there exist $0 \leq t_1 < t_2 < \dots < t_k$ such that if $u_1(x, t)$ and $u_2(x, t)$ are two solutions belonging to the global attractor $\mathcal{A}$ with

$$u_1(x_1, t_j) = u_2(x_1, t_j), \quad j = 1, \dots, k$$

and

$$u_1(x_2, t_j) = u_2(x_2, t_j), \quad j = 1, \dots, k$$

then $u_1 \equiv u_2$.

Note that by the invariance of the global attractor $\mathcal{A}$, we may replace the interval $[0, T_0]$ with any $[a, b]$ where $-\infty < a < b < \infty$. Since $\mathbb{R} = \cup_{n \in \mathbb{N}} [-n, n]$, Theorem 10 holds also if we replace $[0, T_0]$ by $\mathbb{R}$.

PROOF. [Proof of Theorem 10] We will apply Theorem 8 with $X = \mathcal{A}$ and with Y and Σ chosen as follows. Let $\Omega_1 = (-1, 1)$ (any other open interval

containing $[-1/2, 1/2]$ would do) and $\Omega_2 = (1, T_0 + 1)$. We choose $d_1 = 1$ and $d_2 = 2$. We define

$$\Sigma: X \rightarrow C(\Omega_2, \mathbb{R}^2)$$

by

$$(\Sigma(u_0))(t) = ([S(t)u_0](x_1), [S(t)u_0](x_2)) \quad (20)$$

for $t \in \Omega_2 = (1, T_0 + 1)$ (note that $S(t)u_0$ is a (joint) analytic function of $x \in \mathbb{R}$ and $t > 0$ and can thus be evaluated at $x = x_1$ and $x = x_2$). Due to the space-time analyticity of $(S(t)u_0)(x)$, it is clear that the mapping Σ has the required continuity property. The space Y is chosen to be the image of X under the mapping (20). It only remains to check that Σ is one-to-one and that Y satisfies the unique continuation property required in Theorem 8. Let $u_0, v_0 \in \mathcal{A}$ be two initial data, and let $u(\cdot, t) = S(t)u_0$ and $v(\cdot, t) = S(t)v_0$ be the corresponding solutions. If $\Sigma(u_0) = \Sigma(v_0)$, then

$$u(x_j, t) = v(x_j, t), \quad j = 1, 2, \quad t \in (1, T_0 + 1).$$

By analyticity,

$$u(x_j, t) = v(x_j, t), \quad j = 1, 2, \quad t \geq 0 \quad (21)$$

Then the determining nodes result recalled above implies that

$$u(x, t) = v(x, t), \quad x \in \mathbb{R}, \quad t \geq 0 \quad (22)$$

which shows that Σ is one-to-one.

The proof of the unique continuation property is almost identical. Suppose that $\Sigma(u) - \Sigma(v)$ vanishes to infinite order at some $t \in \Omega_2 = (1, T_0 + 1)$. Then by space-time analyticity we have (21) which, using the determining nodes result once more, implies (22). This argument shows that the finite order of vanishing assumption holds.

Theorem 8 then implies that E_t is one-to-one for almost every choice of $t_1, \dots, t_k \in (1, T_0 + 1)$. By the invariance of the global attractor $\mathcal{A}$, we conclude that E_t is one-to-one for almost every choice of $t_1, \dots, t_k \in (0, T_0)$ and hence for almost every choice of $t_1, \dots, t_k \in [0, T_0]$, as asserted. $\square$

If $a > 4\pi^2$, then the use of two nodes is necessary. Given any $x_0 \in \mathbb{R}$, consider the two explicit solutions

$$u_j(x, t) = \sqrt{a - 4\pi^2} \exp\left(2\pi i(-1)^j(x - x_0) - 4\pi^2 i \nu t - a \mu i t + 4\pi^2 \mu i t\right)$$

for $j = 1, 2$. Then

$$u_1(x_0, t) = u_2(x_0, t) = \sqrt{a - 4\pi^2} \exp\left(-4\pi^2 i \nu t - a \mu i t + 4\pi^2 \mu i t\right)$$

for all $t \in \mathbb{R}$ while clearly $u_1(\cdot, t)$ and $u_2(\cdot, t)$ are not identical.

However, only one spatial node is required in the case of Dirichlet boundary conditions

$$u(0, t) = u(1, t) = 0, \quad t \geq 0.$$

In this case, we can choose any $x_1 \in (0, \delta_0]$ and $x_2 = 0$. Since all solutions automatically agree at $x_2 = 0$, we get the following statement.

Theorem 11 *Take an arbitrary point x_1 with $0 < x_1 \leq \delta_0$, choose $T_0 > 0$, and let $k \geq 16d_f(\mathcal{A}) + 1$. Then for almost every set of points*

$$\mathbf{t} = (t_1, t_2, \dots, t_k), \quad t_j \in [0, T_0]$$

the mapping

$$E_{\mathbf{t}}: \mathcal{A} \rightarrow \mathbb{R}^{2k}$$

defined by

$$E_{\mathbf{t}}[u] = \left((S(t_1)u)(x_1), \dots, (S(t_k)u)(x_1), \right)$$

is one-to-one between $\mathcal{A}$ and its image.

PROOF. The result is simply a corollary of Theorem 10 since solutions of the CGLE with Dirichlet boundary conditions can be extended to odd periodic solutions of the CGLE (cf. Kukavica, 1992). $\square$

6.2.2 The Kuramoto-Sivashinsky equation

A similar result, but requiring more spatial points, can be proved for higher order dissipative partial differential equations. Here, using a result due to Foias and Kukavica (1995), we prove a Takens type theorem for the Kuramoto-Sivashinsky equation (KSE)

$$u_t + u_{xxxx} + u_{xx} + uu_x = 0 \quad (23)$$

with initial condition

$$u(x, 0) = u_0(x) \quad (24)$$

subject to periodic boundary conditions with respect to $\Omega = [0, L]$. As usual, we also require

$$\int_{\Omega} u_0(x) dx = 0; \quad (25)$$

we denote by $\dot{L}_{\text{per}}^2(\Omega)$ the subspace of $L_{\text{per}}^2(\Omega)$ consisting of functions with zero average as in (25). The equation enjoys the same properties as the CGLE mentioned above (existence and uniqueness; space-time analyticity; continuity of the solution map $S(t)$; existence of a finite-dimensional global attractor). Note that it is also possible to show the attractor for the KSE is the largest invariant subset of $\dot{L}_{\text{per}}^2(\Omega)$; however, in this case this is not immediate, see Kukavica & Malcok.

Foias & Kukavica (1995) showed that there exists a set $\{x_1, x_2, x_3, x_4\} \subset \mathbb{R}$ of four points that is determining, i.e. if u_1 and u_2 are two solutions belonging to the global attractor $\mathcal{A}$ such that

$$u_1(x_j, t) = u_2(x_j, t), \quad j = 1, 2, 3, 4, \quad \text{for every } t \geq 0$$

then

$$u_1(x, t) = u_2(x, t), \quad x \in \Omega, \quad \text{for every } t \geq 0.$$

In order to be determining, the points $x_1 < x_2 < x_3 < x_4$ have to be chosen in the following way: first, x_1 and x_4 need to be chosen so that $|x_1 - x_4| \leq \delta_0$ where $\delta_0 = \delta_0(L) > 0$; then we pick any x_2 and x_3 so that $|x_1 - x_2| + |x_3 - x_4| \leq \delta_1$ where $\delta_1 = \delta_1(\delta_0, L) > 0$.

Theorem 12 *Take a set of points $\{x_1, x_2, x_3, x_4\}$ that is determining for the KSE, choose any $T_0 > 0$, and let $k \geq 16d_f(\mathcal{A}) + 1$. Then for almost every set of k times*

$$\mathbf{t} = (t_1, t_2, \dots, t_k), \quad t_j \in [0, T_0], \quad j = 1, \dots, k$$

the mapping

$$E_{\mathbf{t}}: \mathcal{A} \rightarrow \mathbb{R}^{4k}$$

defined by

$$E_{\mathbf{t}}[u_0] = \left(\{[S(t_1)u_0](x_j)\}_{j=1}^4, \dots, \{[S(t_k)u_0](x_j)\}_{j=1}^4 \right)$$

is one-to-one between $\mathcal{A}$ and its image.

Instead of using values of $S(t)u_0$ at four points x_1, x_2, x_3 , and x_4 , we can instead use only one point x_1 and the mapping

$$E_t[u_0] = \left(\{\partial_x^k S(t)u_0(x_1)\}_{k=0}^3 \right).$$

We would like to thank P. Constantin to pointing this to us. This fact relies on the following: if

$$\partial_x^k u_1(x_1, t) = \partial_x^k u_2(x_1, t), \quad k = 0, 1, 2, 3, \quad t \geq 0$$

then $u_1(x, t) = u_2(x, t)$ for every $x \in \mathbb{R}$ and $t \geq 0$. This result is actually easier to prove than that in Foias & Kukavica (1995). If $v = u_1 - u_2$, then

$$\partial_x^k v(x_1, t) = 0, \quad t \geq 0 \tag{26}$$

for $k = 0, 1, 2, 3$, and so

$$\partial_t \partial_x^k v(x_1, t) = 0, \quad t \geq 0 \tag{27}$$

for $k = 0, 1, 2, 3$. But the equation for v then implies (26) for $k = 4$ which then gives (27) for $k = 4$. Continuing by induction, we obtain (26) for all $k \in \mathbb{N} \cup \{0\}$ which then by analyticity leads to $v \equiv 0$. A similar property also holds for the complex Ginzburg-Landau equation.

7 Determining modes

Finally, we mention an application of our result which is related to (but does not resolve) a conjecture of Foias & Temam concerning the finite-dimensional Galerkin projections of the attractor for the 2d Navier-Stokes equations. We would like to thanks E. Olson for suggesting this.

In 1967 Foias & Prodi proved that a finite number of modes are determining for the Navier-Stokes equations, in the sense that if N is large enough then

$$|P_N u(t) - P_N v(t)| \rightarrow 0 \quad \Rightarrow \quad |u(t) - v(t)| \rightarrow 0. \quad (28)$$

Foias & Temam then conjecture that in fact, for some N sufficiently large, solutions on the attractor are completely determined by their first N Fourier modes, i.e. if

$$P_N u = P_N v \quad \text{with} \quad u, v \in \mathcal{A}$$

then in fact $u = v$.

Here we prove a result on the attractor that is reminiscent of this conjecture, but requires a finite number of measurements of the first N modes.

Theorem 13 *Suppose that the first N modes are determining in the sense of (28), that the attractor fulfils the conditions of Theorem 1, and that solutions are analytic functions of time and Lipschitz continuous with respect to their initial condition,*

$$|S(t)u_0 - S(t)v_0| \leq K(T)|u_0 - v_0| \quad \text{for all} \quad t \in [0, T]. \quad (29)$$

Then for any $T > 0$, for almost every set of k times in $[0, T]$ (with $k > 16d_f(\mathcal{A})$) if $u, v \in \mathcal{A}$ and

$$P_N u(t_j) = P_N v(t_j) \quad \text{for all} \quad j = 1, \dots, k$$

then $u = v$.

PROOF. Define $\Sigma : L^2(\Omega_1, \mathbb{R}^{d_1}) \rightarrow L^2([0, T], \mathbb{R}^{N_{d_1}})$ by $\Sigma[u_0](t) = P_N u(t)$, $0 \leq t \leq T$ (we denote by $u(t)$ the solution with initial condition u_0 , $u(t) = S(t)u_0$). Take $X = \mathcal{A}$ and $Y = \Sigma[X]$. From (29) the map Σ is clearly Lipschitz continuous. That Σ is one-to-one follows since $\Sigma[u_0] = \Sigma[v_0]$ implies that

$$P_N u(t) = P_N v(t) \quad \text{for all} \quad t \in [0, T]. \quad (30)$$

Since $u(t)$ and $v(t)$ are analytic functions of time, so are $P_N u(t)$ and $P_N v(t)$, and so the equality in (30) implies equality for all $t \geq 0$. Since the first N modes are determining,

$$P_N u(t) = P_N v(t) \quad \text{for all } t \geq 0 \quad \Rightarrow \quad u(t) = v(t) \quad \text{for all } t \geq 0,$$

and in particular that $u_0 = v_0$. A similar argument shows that differences of elements of Y have finite order of vanishing. $\square$

8 Conclusion

We have shown that rather than the property of analyticity being fundamental to our “instantaneously determining nodes” results (cf. remarks in the introduction of Friz & Robinson (2001)), it is in fact the finite order of vanishing of differences of solutions which is central.

This has allowed us to extend our previous work to cover attractors of non-analytic reaction-diffusion equations, and to move from purely spatial nodes to distributed observations in space-time in a fairly natural way (cf. the arguments in Robinson (2001b), which while obtaining observations at different time points are much less elegant). We have also been able to prove a Takens type result for one-dimensional dissipative PDEs of various orders.

It would be interesting to prove that our ‘instantaneous determining nodes’ result holds for the 2D Navier-Stokes equation even when force is not analytic. The present paper reduces this result to showing that differences of solutions of the Navier-Stokes equations cannot vanish to infinite order. We believe that this open question is interesting in its own right.

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