MOAC WORKSHEET FOURIER SERIES, FOURIER TRANSFORM, & SAMPLING

Working through the following exercises you will glean a quick overview/review of a few essential ideas that you will need in the MOAC *course. You should consult a textbook for a more in-depth, rigorous treatment. Harder exercises start with the words "Can you…".*

Exercise 1 Explain why the function

$$x(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos k\omega_0 t + b_k \sin k\omega_0 t \tag{1}$$

repeats itself every $2\pi/\omega_0$ time units, that is, $x(t + 2\pi/\omega_0) = x(t)$ for all values of t.

§ 1 Such functions are called T_0-*periodic* where $T_0 = 2\pi/\omega_0$.

Exercise 2 Show that a T_0-periodic function is also $2T_0$-periodic, $3T_0$-periodic, and so on.

§ 2 A periodic function thus has infinitely many periods, which are integral multiples of the *fundamental period* T_0.

Exercise 3 Can you explain why the fundamental period of a given periodic function $x(t)$ is defined to be the *smallest* positive value of T such that $x(t+T) = x(t)$?

Exercise 4 Let $\tilde{k}$ denote some selected value of k. Evaluate $\int_{-T_0/2}^{+T_0/2} x(t) \cos k\omega_0 t dt$ with $x(t)$ given by equation (1). You should obtain $a_{\tilde{k}} T_0/2$. (Hint: you can use integration by parts (twice!), or use *Euler's formulas* $\cos\theta = (e^{\theta} + e^{-\theta})/2$ and $\sin\theta = (e^{\theta} - e^{-\theta})/(2i)$ where $i^2 = -1$.)

§ 3 Your result

$$a_k = \frac{2}{T_0} \int_{-T_0/2}^{+T_0/2} x(t) \cos k\omega_0 t dt \tag{2}$$

indicates that the coefficients (as and bs) in equation (1) can be found by "integrating together" the function $x(t)$ with the cosines and sines of the corresponding frequencies (so a_k goes with a cosine of frequency $k\omega_0$, and so on).

Exercise 5 Also establish

$$a_k = \frac{2}{T_0} \int_{-T_0/2}^{+T_0/2} x(t) \cos k\omega_0 t dt \tag{3}$$

and

$$a_0 = \frac{2}{T_0} \int_{-T_0/2}^{+T_)/2} x(t) dt \ . \tag{4}$$

§ 4 The idea of a *Fourier series representation* is to represent *any* periodic function with fundamental period T_0 in the form of a weighed sum of cosines and sines, as in equation (1).

Exercise 6 Let $x(t)$ be a periodic function with fundamental period T_0, such that $x(t) = 0$ for $-T_0/2 \leq t \leq 0$ and $x(t) = 1$ for $0 \leq t \leq T_0/2$. Sketch a graph of $x(t)$ for $x = -2T_0$ to $x = +2T_0$.

Exercise 7 With $x(t)$ as in exercise 6, show that $a_0 = 1$. (Hint: equation (4).)

Exercise 8 With $x(t)$ as in exercise 6, show that $a_1 = 0$. (Hint: equation (3); remember that $\omega_0 = 2\pi/T_0$, and refer to sketches of $\cos\omega_0 t$ and $\sin\omega_0 t$ on the interval $[-T_0/2, +T_0/2]$.)

Exercise 9 With $x(t)$ as in exercise 6, show that $a_k = 0$ for $k = 1, 2, 3, \ldots$.

Exercise 10 With $x(t)$ as in exercise 6, show that $b_1 = \frac{2}{\pi}$. (Hint: equation (2); remember that $\cos\pi = -1$.)

Exercise 11 With $x(t)$ as in exercise 6, show that $b_2 = 0$. (Hint: equation (2); remember that $\cos 2\pi = +1$.)

Exercise 12 Can you show, with $x(t)$ as in exercise 6, that $b_k = \frac{2}{k\pi}$ when k is odd, and $b_k = 0$ when k is even?

§ 5 The coefficients (as and bs) which you can calculate for a given periodic function, using equations (2)—(4), contain *all* information about that function. For instance, the quantity $A_k = \sqrt{a_k^2 + b_k^2}$ tells you how much of frequency $k\omega_0$ "is present" in $x(t)$; A_k is the *amplitude* at frequency $k\omega_0$. The amplitudes together (for all values of k) make up the *amplitude spectrum* of $x(t)$.

Exercise 13 When $x(t) = \cos\omega_0 t$, show that $A_1 = 1$ and $A_k = 0$ for $k \neq 1$. (Hint: you already did most of the work in exercises 4 and 5.)

Exercise 14 Show that the *phase-shifted* function $x(t) = \cos(\omega_0 t - \varphi)$ has the same amplitude spectrum as $x(t) = \cos\omega_0 t$. (Hint: use the formulas $\cos(a - b) = \sin a \sin b + \cos a \cos b$ and $\sin^2\varphi + \cos^2\varphi = 1$.)

§ 6 Different signals may thus have the same amplitude spectrum. To determine a function fully, another number must be provided at each frequency. This is the *phase* $\varphi_k = -\tan^{-1}(b_k/a_k)$. The phases together (for all values of k) make up the *phase spectrum* of $x(t)$.

Exercise 15 Can you tell what the phase spectrum of a function looks like if its Fourier series representation contains only sine terms (as with for instance the function of exercise 6)?

§ 7 The idea of 'picking out the frequencies' of a function by integrating that function together with sines and cosines, as in equations (2)—(4), is so nice that we would like to be able to apply this idea even when $x(t)$ is not periodic (or when perhaps it is, but we do not know the fundamental period). Since the function is aperiodic, we resolve to integrate from $-\infty$ to $+\infty$, and to do this for all possible frequencies (that is, for all real values of ω). This gives a function of ω:

$$X_{\cos}(\omega) = \int_{-\infty}^{+\infty} x(t)\cos\omega t dt \tag{5}$$

and, similarly,

$$X_{\sin}(\omega) = \int_{-\infty}^{+\infty} x(t)\sin\omega t dt \ . \tag{6}$$

Exercise 16 Assume that the aperiodic function $x(t)$ is non-zero only in a finite neighbourhood of $t = 0$:

$$x(t) = 0 \quad \text{for } |t| > fracT_1 2 \ . \tag{7}$$

Show that

$$X_{\cos}(\omega) = \int_{-T_1/2}^{+T_1/2} x(t)\cos\omega t dt$$

on this assumption, and write down a similar expression for $X_{\sin}(\omega)$.

Exercise 17 Suppose that you try to assign, somewhat arbitrarily, a fundamental period T_0 to a function satisfying condition (7). Let $\omega_0 = 2\pi/T_0$ and explain why $X_{\cos}(k\omega_0) = a_k T_0/2$ is sure to be correct only if you choose the fundamental period such that $T_0 \geq T_1$. (Hint: combine equation (2) with the previous exercise.)

§ 8 The foregoing exercises suggest that an aperiodic function $x(t)$ satisfying condition (7) can be represented by a Fourier series, as follows:

$$x_{T_0}(t) = \frac{2}{T_0}\sum_{k=1}^{\infty}\left(X_{\cos}(k\omega_0)\cos k\omega_0 t + X_{\sin}(k\omega_0)\sin k\omega_0 t\right) \tag{8}$$

where $a_0 = (2/T_0)\int_{-T_1/2}^{+T_1/2} x(t)dt$ and care has been taken to choose $T_0 \geq T_1$.

Exercise 18 Explain why $x_{T_0}(t)$, defined by equation (8), is not quite the same as $x(t)$. (Hint: first explain why $x_{T_0}(t)$ is T_0-periodic, and why $x(t)$ is not.)

Exercise 19 Can you see why we have $\lim_{T_0\to\infty} x_{T_0}(t) = x(t)$?

Exercise 20 Show that

$$\lim_{T_0\to\infty} x_{T_0}(t) = \lim_{T_0\to\infty}\frac{1}{\pi}\sum_{k=1}^{\infty}\left(X_{\cos}(k\omega_0)\cos k\omega_0 t + X_{\sin}(k\omega_0)\sin k\omega_0 t\right)\omega_0 \ .$$

(Hint: show that the term with a_0 vanishes and work the ω_0 into the integrand.)

Exercise 21 Can you establish the following result?

$$x(t) = \frac{1}{\pi}\int_0^{\infty}\left(X_{\cos}(\omega)\cos\omega t + X_{\sin}(\omega)\sin\omega t\right)d\omega \ .$$

(Hint: combine the results of the previous two exercises. Let $\omega_0 = \Delta\omega$ and $\omega = k\Delta\omega$ and consider the sum as a Riemann sum.)

Exercise 22 Can you now establish the following key result?

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\omega)e^{i\omega t}d\omega \quad \text{where} \quad X(\omega) = \int_{-\infty}^{+\infty} x(t)e^{-i\omega t}dt \ . \tag{9}$$

(Hint: combine the results of the previous exercise with Euler's formulas $\cos\theta = (e^{\theta} + e^{-\theta})/2$ and $\sin\theta = (e^{\theta} - e^{-\theta})/(2i)$ where $i^2 = -1$.)

Exercise 23 In equation (9), do you have to assume that $x(t)$ satisfies the condition (7)? (Hint: you took the limit $T_0 \to \infty$.)

§ 9 The function $X(\omega)$ is the *Fourier transform* of $x(t)$; they are related to each other by (9). The function $|X(\omega)|$ is the *magnitude spectrum* of $x(t)$; the function $\arg X(\omega)$ the *phase spectrum* (for a complex number $z = Ae^{i\vartheta}$, $|z| = A$ and $\arg z = \vartheta$).

§ 10 In instrumentation, t is time and $x(t)$ is interpreted as the *signal* on the *transmission line* between the probe (sensor) and the signal processing equipment. Thus the signal $x(t)$ often represents the voltage on a wire coming from the probe (or the light intensity in a fiber, or whatever physical encoding of the signal is most convenient).

In many instances, the phenomena underlying the signal $x(t)$ are much more apparent, and hence easier to tell apart, from the signals Fourier transform $X(\omega)$ then from the signal itself. This accounts for the widespread use of Fourier transforms in instrumentation and signal processing.

The signal cannot be recorded in its entirety (this is true even of analog recording methods); the transmission line typically feeds into an accumulator which collects isolated values of the signal at regular intervals. This *sampling* process results in a sequence of *measurements* $x(kT_s)$ where $k \in \mathbb{Z}$ and T_s is the *measurement interval*, with associated *measurement frequency* $\omega_s = 2\pi/T_s$.

The basic sampling problem is to sample the signal sufficiently frequently to be able to reconstruct $X(\omega)$ and $x(t)$ from the sequence of measurements.

§ 11 The act of taking a measurement at time t_s is described by the function $\delta(t - t_s)$, which has the property that

$$\int_{-\infty}^{t} x(u)\delta(u - t_s)du = \begin{cases} 0 & \text{when } t < t_s \\ x(t_s) & \text{when } t \geq t_s \end{cases} \tag{10}$$

where this integral represents the state of the accumulator.

Exercise 24 Complete the following equation: $\int_{-\infty}^{t} \delta(u)du \quad$. (Hint: comparison with equation (10) shows that $t_s = 0$, $x(t) = 1$.)

Exercise 25 Can you derive the Fourier series representation of the sampling process with sampling frequency ω_s represented by $\sum_{k=-\infty}^{+\infty} \delta(u - kT_s)$? (Hint: calculate the as and bs, equations (2)—(4).)

Exercise 26 Can you rewrite your result of the previous exercise as follows?

$$\sum_{k=-\infty}^{+\infty} \delta(u - kT_s) = \frac{\omega_s}{2\pi} \sum_{k=-\infty}^{+\infty} e^{ik\omega_s t} \tag{11}$$

(Hint: remember, $\cos\vartheta = (e^{\vartheta} + e^{-\vartheta})/2$.)

§ 12 The Fourier transform of the sampled process is defined to be

$$X_s(\omega) = \sum_{k=-\infty}^{+\infty} x(kT_s) e^{-i\omega k T_s} \ . \tag{12}$$

Exercise 27 Verify that $X_s(\omega)$ can be calculated from the sequence of measurements.

Exercise 28 Can you derive the following result? X_s is the Fourier transform of

$$x_s(t) = \sum_{k=-\infty}^{+\infty} x(t)\delta(u - kT_s)$$

.

Exercise 29 Can you derive the following key relationship between X_s and X?

$$X_s(\tilde{\omega}) = \frac{1}{T_s} \int_{-\infty}^{+\infty} X(\omega)\delta_{\omega_s}(\tilde{\omega} - \omega) d\omega \ . \tag{13}$$

(Hint: combine equation (12) with the expression for $x(t)$ in (9); refer to exercise 26 to derive the formula $\delta_{\omega_s}(\omega) = \omega_s^{-1} \sum_{k=-\infty}^{+\infty} e^{ik2\pi\omega/\omega_s}$.)

Exercise 30 Can you explain, referring to equation (13), why $X_s(\omega)$ is not the same function as $X(\omega)$? (Hint: X_s is ω_s-periodic. Note that δ_{ω_s} 'picks up' values not just at $\tilde{\omega}$, but also at $\tilde{\omega} + k\omega_s$, where $k \in \mathbb{Z}$.)

§ 13 The Fourier transform of a *band-limited* signal satisfies the following condition:

$$X(\omega) = 0 \quad \text{for } |\omega| > \omega_M \tag{14}$$

that is, ω_M is the highest frequency occurring in the signal $x(t)$.

Exercise 31 Can you show that for a band-limited signal, you have $X_s(\omega) = X(\omega)$ for all values of ω, *provided that* $\omega_s \geq 2\omega_M$? (Hint: verify that the integration limits $-\infty$ to $+\infty$ can be replaced by $-\omega_s/2$ to $+\omega_s/2$ in equation (13), and show that the problem you identified in exercise 30 is now avoided.)

Exercise 32 Conclude that for a band-limited signal with maximum frequency ω_M, the Fourier transform can be calculated from the measurements

$$X(\omega) = \sum_{k=-\infty}^{+\infty} x(kT_s) e^{-i\omega k T_s} \tag{15}$$

provided $\omega_s \geq 2\omega_M$. (Hint: combine equations (12) and the previous exercise.)

§ 14 The sampling rate $2\omega_M$, where ω_M is the highest frequency in a band-limited signal, is known as the *Nyquist frequency*.

Exercise 33 Explain why the Nyquist frequency is the ideal sampling frequency. (Hint: consider the drawback of sampling at (much) lower and higher rates.)

Exercise 34 Can you derive the following formula? It expresses the original signal $x(t)$ in terms of only the measured values sampled at $t = kT_s$ ($k \in \mathbb{Z}$):

$$x(t) = \sum_{k=-\infty}^{+\infty} x(kT_s) \frac{\sin((t - kT_s)\omega_s/2)}{(t - kT_s)\omega_s/2} \tag{16}$$

where $\omega_s \geq 2\omega_M$. (Hint: combine equations (9) and (15), and use $\sin\vartheta = (e^{\vartheta} - e^{-\vartheta})/(2i)$.)