Ergodic Properties of Markov Processes

Exercises for week 2

Exercise 1 Show that if $\mathscr{F}'$ is the trivial σ-algebra, *i.e.* $\mathscr{F}' = \{\phi, \Omega\}$, then $X' = \mathbf{E}(X \mid \mathscr{F}')$ is constant and equal to the expectation of X.

Exercise 2 Show that continuous functions are Borel-measurable. Give an example of a Borel-measurable function from **R** to **R** which is not continuous.

Exercise 3 Let $\Omega = [0,1]^2$, $\mathbf{P}(dx, dy) = (x + y)\,dx\,dy$, and let (X, Y) be a pair of random variables defined by $X(x, y) = x$ and $Y(x, y) = y$. Let $\mathscr{F}_Y$ be the σ-algebra generated by Y. Find an explicit expression for $\mathbf{E}(X \mid \mathscr{F}_Y)$ and give a function f such that $\mathbf{E}(X \mid \mathscr{F}_Y) = f \circ Y$.

Exercise 4 Show that $\mathscr{F}_1 \vee \mathscr{F}_2$ can equivalently be characterised by the expressions:

- $\mathscr{F}_1 \vee \mathscr{F}_2 = \sigma\{A \cup B \mid A \in \mathscr{F}_1 \text{ and } B \in \mathscr{F}_2\}$,
- $\mathscr{F}_1 \vee \mathscr{F}_2 = \sigma\{A \cap B \mid A \in \mathscr{F}_1 \text{ and } B \in \mathscr{F}_2\}$,

where $\sigma\mathscr{G}$ denotes the smallest σ-algebra containing $\mathscr{G}$.

Exercise 5 Let $\Omega = \{1, \ldots, 6\}^3$. We interpret elements of Ω as the possible outcomes of throwing a dice three times. Describe the σ-algebra $\mathscr{F}$ corresponding to knowing the value of the largest of the three throws.

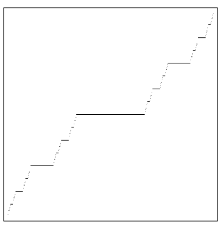
* **Exercise 6** Show the following elementary properties of conditional expectations:

- If $\mathscr{F}_1 \subset \mathscr{F}_2$, then $\mathbf{E}(\mathbf{E}(X \mid \mathscr{F}_2) \mid \mathscr{F}_1) = \mathbf{E}(\mathbf{E}(X \mid \mathscr{F}_1) \mid \mathscr{F}_2) = \mathbf{E}(X \mid \mathscr{F}_1)$.
- Find an example that shows that in general $\mathbf{E}(\mathbf{E}(X \mid \mathscr{F}_2) \mid \mathscr{F}_1) \neq \mathbf{E}(\mathbf{E}(X \mid \mathscr{F}_1) \mid \mathscr{F}_2)$.
- If Y is $\mathscr{F}_1$-measurable, then $\mathbf{E}(XY \mid \mathscr{F}_1) = Y\,\mathbf{E}(X \mid \mathscr{F}_1)$.
- If $\mathscr{F}_1 \subset \mathscr{F}_2$, and Y is $\mathscr{F}_2$-measurable then $\mathbf{E}(Y\mathbf{E}(X \mid \mathscr{F}_2) \mid \mathscr{F}_1) = \mathbf{E}(XY \mid \mathscr{F}_1)$.

Hint For the first part, use the fact that if $\mathscr{F}_1 \subset \mathscr{F}_2$, then any $\mathscr{F}_1$-measurable function is also $\mathscr{F}_2$-measurable.

** **Exercise 7** You have probably seen Lebesgue measurable functions defined through the property that $f^{-1}(A)$ is Lebesgue measurable for every open set A. Show that every Borel measurable function is also Lebesgue measurable but that the converse is not true in the case of functions from **R** to **R**.

Show that if $f : \mathcal{X} \to \mathcal{Y}$ and $g : \mathcal{Y} \to \mathcal{Z}$ are Borel measurable functions, then $g \circ f$ is also Borel measurable. This property is *not* true for Lebesgue measurable functions. Try to find a *continuous* function $f : \mathbf{R} \to \mathbf{R}$ and a Lebesgue measurable function g (you can take an indicator function for g) such that $g \circ f$ is not Lebesgue measurable.



Hint: Remember that every measurable set A of positive Lebesgue measure contains a subset $A' \subset A$ which is *not* Lebesgue measurable. (Take this statement for granted if you haven't seen it before.) Another useful ingredient for the construction of f is the Cantor function D (also called Devil's staircase), depicted here. Use the fact that if C is the Cantor set, then $D(C)$ is a set of Lebesgue measure 1.