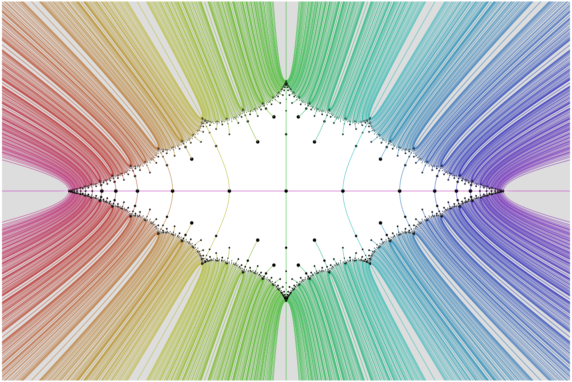
All about the Riley Slice
LMS Presidential Lecture 2019

Caroline Series

THE UNIVERSITY OF WARWICK



Picture courtesy Yasushi Yamashita (Nara), September 2019

The Riley Slice

The Riley slice is the name given to the space of all (non-elementary) subgroups of $SL(2,\mathbb{C})$ with two parabolic generators.

$SL(2,\mathbb{C})$ is the set of matrices $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a,b,c,d \in \mathbb{C}; ad - bc = 1$.

T acts on the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \infty$ by linear fractional transformations $T : z \mapsto \frac{az+b}{cz+d}$. Typically T has two fixed points: it is called parabolic if it has only one.

If S and T are parabolics with distinct fixed points, by normalising we can assume these are ∞ and 0 respectively. Further normalising we can assume $S : z \mapsto z+1, T = T_c : z \mapsto \dfrac{1}{cz+1}$, for some $c \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

Thus we identify the Riley slice with $\mathbb{C}^*$. Our object is to study the family of groups

$$G_c =< S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, T_c = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} > .$$

This family has been studied from various viewpoints and illustrates many of the powerful concepts introduced by Thurston.

Questions about G_c

As we have just seen, up to conjugation in $SL(2,\mathbb{C})$, any (non-trivial) group with two parabolic generators can be written

$$G_c = \langle S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, T_c = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \rangle.$$

$T : z \mapsto \frac{az+b}{cz+d}$ extends to an isometry of hyperbolic 3-space $\mathbb{H}^3$, which we think of as sitting above $\mathbb{C}$. The boundary $\partial\mathbb{H}^3$ of $\mathbb{H}^3$ is identified with $\hat{\mathbb{C}}$. The distance from any point in $\mathbb{H}^3$ to $\hat{\mathbb{C}}$ is infinite.

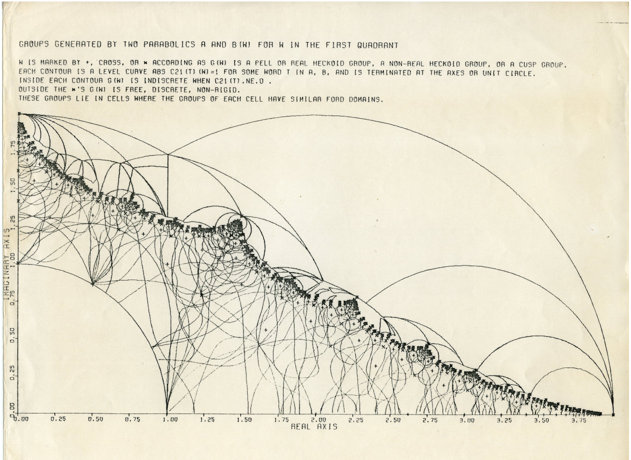
If $G \subset SL(2,\mathbb{C})$ is discrete as a subgroup of $SL(2,\mathbb{C})$ then $\mathbb{H}^3/G$ is a hyperbolic manifold or orbifold.

We are interested in questions such as:

- For which c is G_c discrete? For which c is G_c discrete and free?
- For which c is G_c discrete and not free? What is the topology of $\mathbb{H}^3/G_c$?

Ian Agol (2002) announced a complete classification of all non-free discrete groups on 2 parabolic generators. We will come to this at the end of the talk.

Riley's computer plot

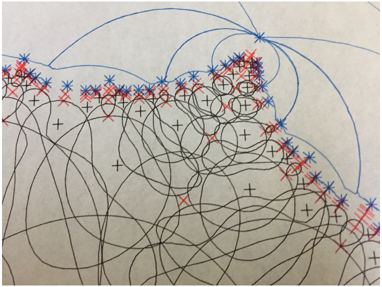


This computer plot illustrating the positive quadrant in the c-plane. was made by Robert Riley in 1979.* It was a remarkable achievement. Points above the dark scalloped "curve" $\mathcal{S}$ all represent free discrete groups.

* It is is easy to see the slice has a four-fold symmetry: G_c and G_{-c} are the same group and G_c and $G_{\bar{c}}$ are complex conjugates.

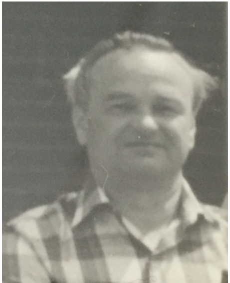
Riley's discoveries

Here is a blow up of part of the picture. The scalloped 'curve' $\mathcal{S}$ is picked out by $*$'s representing so-called cusp groups. He found many other discrete non-free groups: $\times$ is a Heckoid group; $+$ is a knot group. We will explain these different types later.



The value $c = \dfrac{1+i\sqrt{3}}{2}$, is the famous complement of the figure of eight knot K. This means that $S^3 - K$ is homeomorphic to $\mathbb{H}^3/G_c$ which, for this c, is a metrically complete finite volume hyperbolic manifold.

It was largely Riley's discovery of this particular group in early 1976 which inspired Thurston to make his astonishing hyperbolisation conjecture. Riley wrote an account of his discovery which can be found on the web.

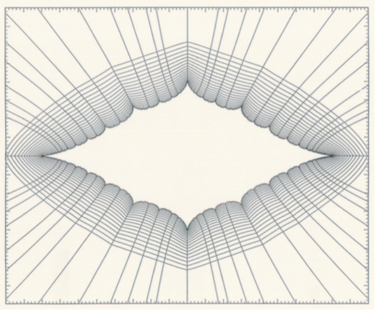


Robert Riley (1935-2000) at the 1984 Warwick symposium.

Another picture of the Riley slice

This picture gives a completely different view of the slice. Made by David Wright around 1992, it illustrates the Keen-Series pleating rays in the c-plane.

There is one ray for each p/q mod 2. The rays appear to radiate out from the central 'eye' in an extremely orderly regular way, the lines $(-\infty, 4)$ and $(4, \infty)$ corresponding to $0/1, 1/1$ respectively. The rays are dense in the subset of the c-plane corresponding to points for which G_c is both free and discrete.



The irregular boundary of the region filled out by the rays looks similar to Riley's scalloped 'curve' $\mathcal{S}$. (The 'equipotential' lines surrounding the 'eye' also have a geometrical meaning but we won't go into this here.)

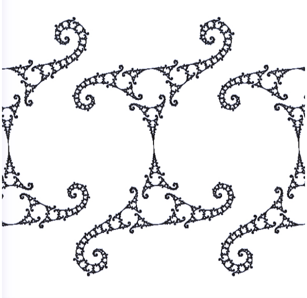
We will first look at the two pictures separately and then put them together. First however we need a bit of background about subgroups of $SL(2, \mathbb{C})$.

Background I: $SL(2, \mathbb{C})$ acting on $\hat{\mathbb{C}} = \mathbb{C} \cup \infty$

Consider the action of a group $G \subset SL(2, \mathbb{C})$ by linear fractional transformations (Möbius maps) on $\hat{\mathbb{C}}$. Its limit set $\Lambda(G)$ is the set of accumulation points of G-orbits. G acts properly discontinuously on the regular set $\Omega(G) = \hat{\mathbb{C}} \setminus \Lambda(G)$. If $\Omega(G) \neq \emptyset$ then G is automatically a discrete subgroup of $SL(2, \mathbb{C})$. (The converse is not always true). A discrete subgroup of $SL(2, \mathbb{C})$ is called a Kleinian group.

By a famous theorem of Ahlfors (1964) , $\Omega(G)/G$ is a finite union of Riemann surfaces with finite genus and finitely many punctures.

The limit set for G_c with $c = 0.05 + 0.93i$. Note that $\Omega(G_c)$ is connected but not simply connected. As we will see, in this case $\Omega(G_c)/G_c$ is a sphere with 4 punctures.

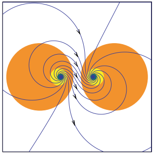


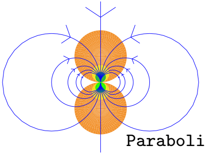
Background II: Classification of Möbius maps

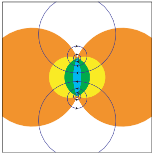
$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C})$ is classified $\operatorname{Tr} T = a + d$.

T is either:

- loxodromic: 2-fixed points on $\hat{\mathbb{C}}$ joined by an $\mathbb{H}^3$-line called its axis. In $\mathbb{H}^3$ acts by translation & rotation along the axis. Trajectories spiral in to the two fixed points. $\operatorname{Tr} T \notin [-2,2]$.
- parabolic: 1-fixed point on $\hat{\mathbb{C}}$. $\operatorname{Tr} T = \pm 2$ (and $T \neq \mathrm{id}$).
- elliptic: 2-fixed points on $\hat{\mathbb{C}}$. In $\mathbb{H}^3$ acts by pure rotation about its axis. $\operatorname{Tr} T \in (-2,2)$.
- purely hyperbolic: (special case of loxodromic). Acts by pure translation along its axis. $\operatorname{Tr} T \in (-\infty,-2) \cup (2,\infty)$.


Loxodromic


Parabolic

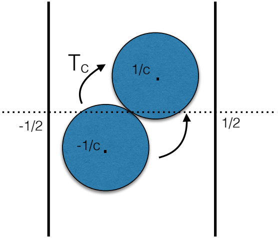

Elliptic

Back to the Riley slice: A fundamental domain for G_c

Let us start by trying to find a fundamental domain for G_c acting on $\hat{\mathbb{C}}$.

The isometric circle of an element $M \in SL(2,\mathbb{C})$ is the circle on which $|M'(z)| = 1$. It is defined for all M other than translations $z \mapsto z + t, t \in \mathbb{R}$.

The blue circles in the figure are the isometric circles of T_c and $T_c^{-1} = T_{-c}$. T_c maps the outside of the lower circle onto the inside of the upper. Provided that these two circles lie within a strip of width 1, it is easy to show by Klein's ping-pong theorem, that the white region within the strip is a fundamental domain R for the G_c action and that G_c is both free and discrete.

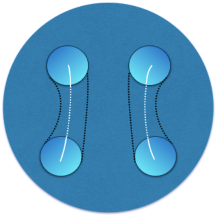


More generally, if G_c is discrete, the region within the strip and outside all its isometric circles is a fundamental domain called its Ford domain. It is analogous to a Dirichlet domain with the base point at ∞.

The quotient manifold $\mathbb{H}^3/G_c$

It is not hard to see that R glues up to form a $4\times$-punctured sphere $S_{0,4}$ This is the quotient Riemann surface $\Omega(G)/G$.

The G_c action extends to $\mathbb{H}^3$. The region $\hat{R} \subset \mathbb{H}^3$ within a slab of width 1 and outside the two hemispheres above the blue circles is a fundamental domain. $\hat{R}$ glues up to form a solid ball which has two parabolic cylinders drilled out. We denote this manifold $\mathcal{B}$. Its boundary $\partial\mathcal{B}$ at infinity is identified with $\Omega(G_c)/G_c$.



The white lines are missing in the solid sphere. They are at infinite hyperbolic distance from points in $\mathcal{B}$. Their endpoints are the 4 punctures on $\partial\mathcal{B}$. Notice that the endpoints are naturally are matched in pairs.

What is a parabolic cylinder? A fundamental region in $\mathbb{H}^3$ for the parabolic translation $S(z) = z + 1$ is the slab above the lines $|\Re z| = \pm 1/2$. These lines glue together to form as cylinder, so that $\hat{\mathbb{C}}/ < S >$ is a cylinder. Likewise the part of each horizontal plane above $\mathbb{C}$ between the slides of the slab glues to a cylinder. Thus $\mathbb{H}^3/ < S >$ is a solid cylinder with the central line missing.

The region of free discrete groups

Even when the isometric circles of T_c, T_c^{-1} are not contained in the strip of width 1, it is still possible that G_c is free and discrete.

Let $\mathcal{D} \subset \mathbb{C}$ be the region for which G_c is free discrete and so that $\Omega(G) \neq \emptyset$.

An easy computation shows that when $|c| > 4$ the isometric circles of T_c, T_c^{-1} are contained in the strip of width 1. Hence $\mathcal{D}$ contains the region $|c| > 4$. One can show using Teichmüller theory that $\mathcal{D}$ is open and is the conformal image of a punctured disk.

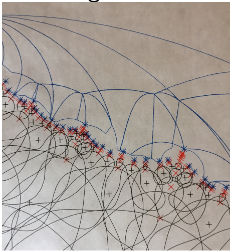
By general theory, there is a quasi-conformal homeomorphism between the regular sets of any two groups in $\mathcal{D}$, which extends to a quasi-isometry (controlled homeomorphism) between the quotient 3-manifolds. In other words, all the manifolds $\mathbb{H}^3/G_c$ for $c \in \mathcal{D}$ are homeomorphic to $\mathcal{B}$ and have homeomorphic boundaries $\partial\mathcal{B} = S_{0,4}$.

However $\mathcal{D}$ is not defined by any nice equations. Classically this was considered an almost impossible problem. In fact $\partial\mathcal{D}$ is the same as Riley's scalloped 'curve' $\mathcal{S}$.

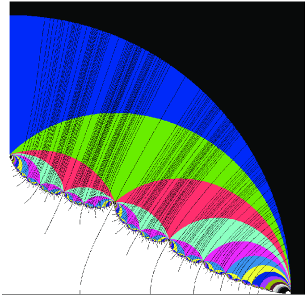
Locating $\mathcal{D}$

There are two ways to go about locating $\mathcal{D}$, both computational.

Riley's method was to use Ford fundamental domains – the region outside all the isometric circles of the group.



The cells with blue boundaries in Riley's plot are regions in which the Ford domain is bounded by the isometric hemispheres of the same set of group elements and hence has the same combinatorial shape.



The second method is to use the pleating rays discovered by Keen and myself. These we explain next.

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This image by Yamashita is a modern rendering of a variant of Riley's cells, superimposed with the rays. The cells get smaller and harder to detect as they approach 𝒮.
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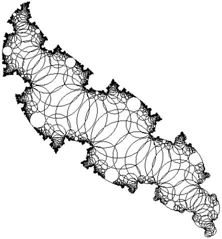
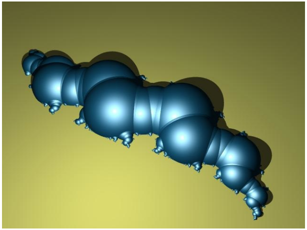
Pleated surfaces and bending lines

Let $\mathcal{C}(G)$ be the union of all hyperbolic lines in $\mathbb{H}^3$ joining pairs of points in the limit set $\Lambda(G)$.



Pictures by
Yair Minsky



There is a natural map β between the regular set $\Omega(G)$ and the boundary $\partial\mathcal{C}(G)$ of $\mathcal{C}(G)$.

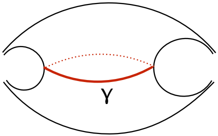
This boundary is an example of what Thurston called a pleated surface. It is made up of pieces of hyperbolic plane (hemispheres) meeting along hyperbolic lines called bending lines.

β passes to the quotient and induces a homeomorphism between $\Omega(G)/G$ and $\partial\mathcal{C}(G)/G$. Thus the convex hull boundary $\partial\mathcal{C}(G)/G$ is also the 4-punctured sphere $S_{0,4}$.

Bending lines and pleating rays

The bending lines project to disjoint and simple (=non-self intersecting) geodesics on $\partial\mathcal{C}(G)/G$. In particular if the line is an axis of an element $g \in G$, then it projects to a closed simple loop on $S_{0,4}$. In this situation all the bending lines are equivalent under G_c and the projected loop γ divides $S_{0,4}$ into two parts, each a sphere with two punctures and a hole.

The two halves lift to hemispheres which intersect along the axis $\mathrm{Ax}\, g$. Note that g maps the overlapping hemispheres to themselves. Since g maps these hemispheres to themselves, it has no rotational part so must be purely hyperbolic with real trace.



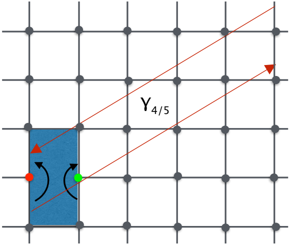
As c varies in $\mathcal{D}$, so does the limit set $\Lambda(G_c)$ and hence $\partial\mathcal{C}(G_c)$ and its bending lines.

Definition (Keen-S.) The pleating ray $\mathcal{P}_\gamma \subset \mathcal{D}$ of a loop γ on $S_{0,4}$ is the set of points $c \in \mathcal{D}$ for which the bending lines all project to γ.
If $\mathrm{Ax}\, g$ projects to γ then $\mathcal{P}_\gamma \subset \{c \in \mathcal{D} : \mathrm{Tr}\, g \in \mathbb{R}\}$.

Pleating rays and finding $\mathcal{D}$

To use pleating rays to find $\mathcal{D}$, one has to enumerate all the non-trivial (= not a loop round a single puncture) simple closed curves γ on $\partial\mathcal{C}/G$, up to homotopy in $\mathcal{B}$. Each such loop is the projection of a line of rational slope in the plane $\mathbb{R}^2$ with punctures at $\mathbb{Z}^2$. Thus curves on $\partial\mathcal{B}$ are indexed by $p/q, q \geq 0, p \in \mathbb{Z}$.

The blue region projects to $S_{0,4}$. There are punctures at each lattice point $\mathbb{Z}^2$. The red and green dots are the fixed points of S, T_c respectively. The vertical line $\gamma_{1,0}$ projects to a loop which is homotopically trivial in $\mathcal{B}$. Moreover $\gamma_{p/q}$ is homotopic in $\mathcal{B}$ to $\gamma_{(p+2q)/q}$.



It turns out that $\gamma_{p/q}$ is also homotopic in $\mathcal{B}$ to $\gamma_{-p/q}$.

Computing the rays

Let $\gamma_{p/q}$ be a non-trivial simple closed curve on $\partial\mathcal{B}$ represented by a word $W_{p/q} \in G$. Then $W_{p/q}$ is a product of S and $T = T_c$. $\operatorname{Tr} W_{p/q}$ is a polynomial $Q_{p/q}$ of degree q in c which can be read off from the punctured plane diagram.

Example For $p/q = 1/2$, $W_{1/2} = STS^{-1}T^{-1}$ and $\operatorname{Tr} W_{1/2} = Q_{1/2}(c) = 2 + c^2$.

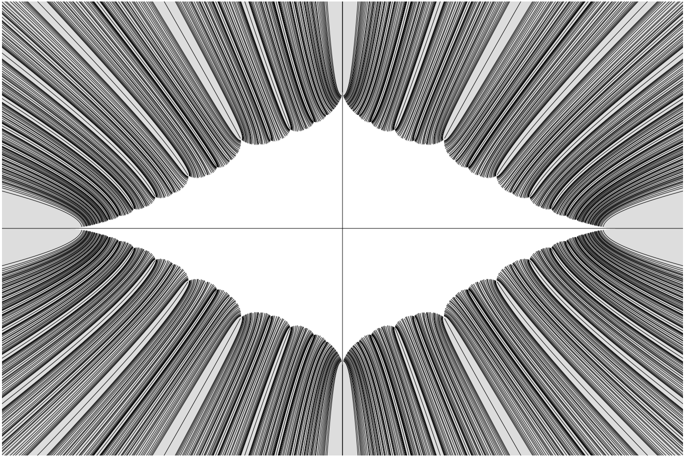
Recall that $\mathcal{P}_{p/q} = \{c \in \mathcal{D} : \partial\mathcal{C}(G_c)/G_c$ is bent along $\gamma_{p/q}\}$,
and $\mathcal{P}_{p/q} \subset Q^{-1}_{p/q}(\mathbb{R})$. However it is not true that $\mathcal{P}_{p/q} = Q^{-1}_{p/q}(\mathbb{R})$.

In general $Q^{-1}_{p/q}(\mathbb{R})$ is a complicated set of branched one manifolds with many singular points. In particular, it has $2q$ branches going off to infinity in directions $e^{i\pi r/q}$ for $r = 1, \dots, 2q$.

Theorem (Keen-S.) $\mathcal{P}_{p/q}$ is the unique branch which goes to infinity in direction $e^{i\pi(q-p)/q}$, and this branch has no singularities in $\mathcal{D}$. Along it $\operatorname{Tr} W_{p/q}$ increases monotonically from $-\infty$ to -2.

For example, $\mathcal{P}_{1/2}$ is the part of the imaginary axis with $\Im c > 2$.

There is a ray for each p/q mod 2, starting from $0/1$ along the negative real axis and moving around to $1/1$ on the positive axis. The two vertical rays correspond to $p/q = \pm 1/2$. The $\mathcal{P}_{p/q}$ ray has direction $\pi(q-p)/q$ at infinity.



Plot by Yamashita. The rays are dense in $\mathcal{D}$.

Rays: The main theorem

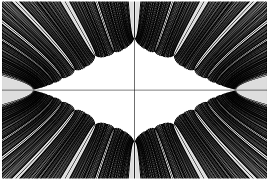
Theorem (Keen-S. PLMS 1994, with improvements by Parker-S. 1995)

- For $p/q \mod 2$, $\mathcal{P}_{p/q}$ is a connected* non-singular branch of $\mathrm{Tr}\, W_{p/q} \in \mathbb{R}$ in direction $e^{i\pi(q-p)/q}$ as $|c| \to \infty$. $\mathrm{Tr}\, W$ increases along $\mathcal{P}_{p/q}$ from $-\infty$ to -2.
- $\mathcal{P}_{p/q}$ ends on $\partial\mathcal{D}$ when $\mathrm{Tr}\, W_{p/q} = -2$. $W_{p/q}$ is parabolic and $\gamma_{p/q}$ has length zero. $\partial\mathcal{B}$ degenerates into a two triply punctured spheres $S_{0,3}$ joined by paired parabolic cusps. (Groups with an 'extra' parabolic like this are called cusp groups.)
- (Choi-S.) 2006 $\mathrm{Tr}\, W_{p/q} \in \mathbb{R}$ is also non-singular at the cusp group on $\partial\mathcal{D}$.
- The rays are pairwise disjoint and their union is dense in $\mathcal{D}$. They can be interpolated by 'irrational' rays on which the bending locus is a family of pairwise disjoint non-closed geodesics called a geodesic lamination.

* (Komori - S.) 1999 In fact $\mathcal{P}_{p/q} = \mathcal{P}_{-p/q}$ and each ray actually has two complex conjugate branches corresponding to conjugate groups.

Some deep results

The picture of the Riley slice with rays can be used to illustrate a number of very deep results proved by the Thurston school.



Theorem (Proved by Canary, Hersonsky and Shalen (2003) based on McMullen (1991)) Cusp groups are dense in ∂D.

Theorem (Bers density conjecture, proved by Ohshika (2005) based on Brock-Bromberg (2004)) All free discrete groups in the c-plane lie on $\mathcal{D} \cup \partial\mathcal{D}$.

Theorem (Ending lamination conjecture, proved by Ohshika-Miyachi (2010) based on Minsky (1999) and requiring Tameness conjecture proved by Agol and independently Calegari-Gabai, 2004.) For each irrational $\lambda \in (0, 1)$ there is a unique group on $\partial\mathcal{D}$ corresponding to the unique end point of the 'irrational' ray $\mathcal{P}_\lambda$.

Corollary ∂D is a Jordan curve.

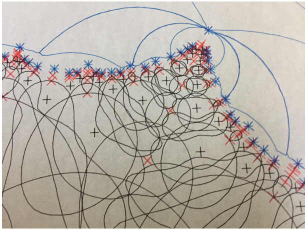
The main content of these results is that things are just as the picture indicates.

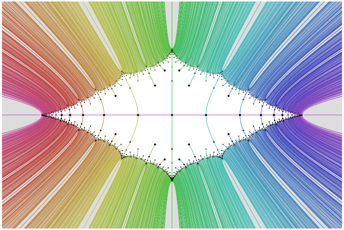
The groups in the Riley slice are a very special case. Most of these results extend to much wider classes of groups. The Keen-S. rays can also be generalised to much more general situations (Choi-S. 2006), although it is an open problem to prove that they are dense in the relevant parameter space.

Discrete groups outside $\mathcal{D}$.

The last part of my talk is largely inspired by the work of Makoto Sakuma and his many collaborators: H. Akiyoshi, M. Wada, Y. Yamashita, D. Lee, J. Parker et al.

Here again is a small part of Riley's plot. The scalloped 'curve' $\mathcal{S}$ is picked out by $*$'s representing cusp groups (ends of pleating rays). There are also discrete groups beneath $\mathcal{S}$ which as we have now seen is the boundary of the free discrete region $\mathcal{D}$. Being outside $\mathcal{D}$, they are not free. $+$ represents a knot complement and $\times$ a closely related Heckoid group. Each discrete group is enclosed by black loop marking a region of non-discreteness. Riley found all this by algebraic methods.

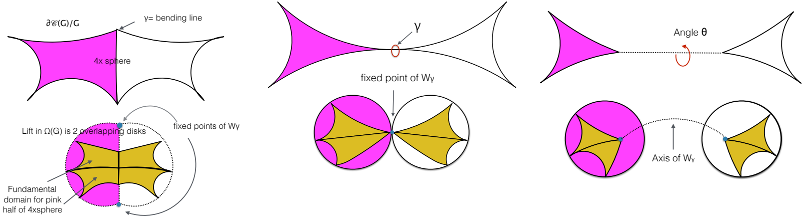




Compare this to the very recent picture made by Yamashita. The black dots mark discrete groups outside $\mathcal{D}$. As you can see they all lie on what appear to be extensions of the pleating rays.

Extending the rays outside $\mathcal{D}$

A pleating ray is a subset of a certain branch of the real locus $\operatorname{Tr} W_g \in \mathbb{R}$. This branch can be analytically continued outside $\mathcal{D} \cup \partial\mathcal{D}$, entering a region where $\operatorname{Tr} W_\gamma \in (-2, 2)$.



- When $\operatorname{Tr} W_\gamma(c) < -2$, we are in $\mathcal{D}$ and on $\mathcal{P}_\gamma$. γ is a bending line for $\partial\mathcal{C}/G$. It divides $\partial\mathcal{C}/G$ into 2 spheres each with 2 punctures and a boundary loop γ. The spheres lift to overlapping disks whose boundaries meet in the fixed points of W_γ.
- When $\operatorname{Tr} W_\gamma = -2$ then $W_\gamma(c)$ is parabolic. $\partial\mathcal{C}/G$ becomes 2 spheres with 3 punctures. The lifted disks are tangent and G_c is a cusp group. This point is on $\partial\mathcal{D}$.
- When $\operatorname{Tr} W_\gamma \in (-2, 2)$ then W_γ is elliptic so rotates about its axis. $\partial\mathcal{C}/G$ degenerates to 2 spheres with 2 punctures and one cone point. These lift to two disks, now disjoint but connected by $\operatorname{Ax} W_\gamma$.

On a ray outside $\mathcal{D}$: Heckoid groups and knots

Suppose that as above $\operatorname{Tr} W_\gamma \in (-2, 2)$ so W_γ is elliptic. For G_c to be discrete, must have finite order: $W_\gamma^n = \mathrm{id}$ for some $n \in \mathbb{N}$. This occurs when $\operatorname{Tr} W_\gamma = -2 \cos \pi / n$. This is called a Heckoid group, marked $\times$ in Riley's picture.

Otherwise $\mathbb{H}^3 / G_c$ is a cone manifold – it looks everywhere like a hyperbolic manifold, except that the angle round the projection of the elliptic axis is not 2π.

Finally we arrive at $\operatorname{Tr} W_\gamma = 2$ and $W_\gamma = \mathrm{id}$. At this point the lifted disks have zero diameter so the fixed points of the two parabolics defining the punctures coincide. These two parabolics will be conjugates of the original two generators. The relation $W_\gamma = \mathrm{id}$ says they commute.

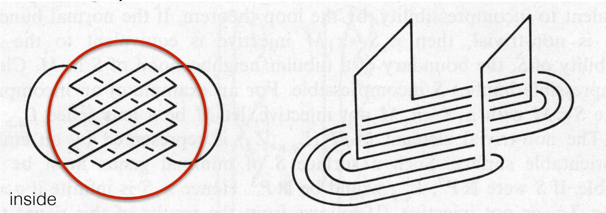
Claim The manifold $\mathbb{H}^3 / G_c$ where $\operatorname{Tr} W_\gamma = 2$ is the complement of the two-bridge knot or link associated to p/q[†], marked $+$ in Riley's picture.

To see why this is so we need to take a closer look at $\gamma_{p/q}$.

† (p, q) are not exactly the same as the Schubert parameters (α, β).

Connection to knots

Corresponding to every line of slope p/q one can form a two-bridge knot or link as in these pictures:



Picture for $p/q = 5/3$ from Hatcher and Thurston (1985).

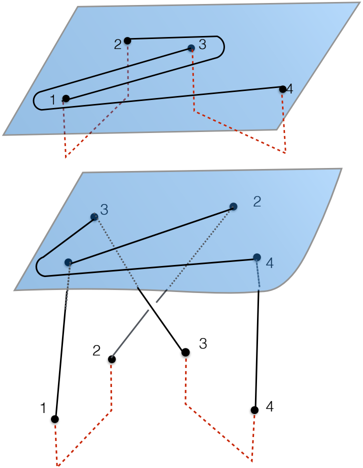
The red circle represents the boundary $S_{0,4}$ of $\mathcal{B}$. In the picture, the solid interior of $\mathcal{B}$ is *outside* the red circle. The sloping lines are on $\partial\mathcal{B}$ (inside the red circle).

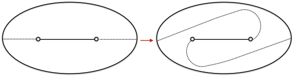
Up to homotopy, there is a unique closed loop on $S_{0,4}$ which is disjoint from the knot. It is nothing other than our curve $\gamma_{p/q}$.

$\gamma_{p/q}$ can be untwisted by applying homeomorphisms of $S_{0,4}$. Each basic move (half Dehn twist) moves one of the punctures around the other, and in doing so twists up the parts of the knot outside the red circle.

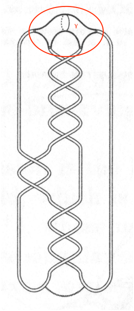
Unravelling $\gamma_{p/q}$ with half twists

A half twist interchanges two punctures and is the identity outside a neighbourhood.





A half-twist interchanging 2 and 3 induces twisting (braiding) of the cusp tubes ‘inside’ the ball.

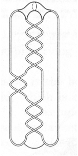


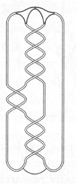
Applying repeated half-twists ‘unravels’ $\gamma_{p/q}$.

Rays and extended rays

This beautiful sequence of pictures taken from ASWY illustrates the same sequence as we move down a ray, first inside $\mathcal{D}$, then meeting $\partial\mathcal{D}$, and finally on the extended ray.

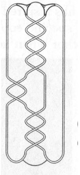
(A) $\gamma_{p/q}$ separates the two punctures with the hollow $S_{0,4}$ (the boundary of the convex hull) inside. This is a point on $\mathcal{P}_{p/q} \cap \mathcal{D}$.

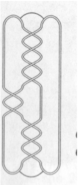




(B) $\gamma_{p/q}$ has been pinched to become parabolic, separating the convex hull into two three punctured spheres each of which lift to a pair of tangent circles in Λ.This is $\mathcal{P}_{p/q} \cap \partial\mathcal{D}$.

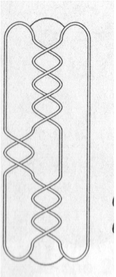
(C) $\gamma_{p/q}$ is elliptic, rotating around the axis joining the remnants of the two 3-punctured spheres. This is a Heckoid group.





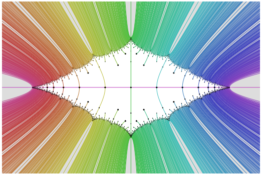
(D) Finally $\gamma_{p/q} = \mathrm{id}$. The remnants of the initial surface completely disappear and we are left with a knot or link (as q is odd or even). This is the knot complement.

The knot group at the end of the ray



At the end of the ray where $\mathrm{Tr}\, W_{p/q} = 2$ we have $W_{p/q} = \mathrm{id}$. The remnants of the initial surface completely disappear and we are left with a knot or link (depending on whether q is odd or even).
One can check that $W_{p/q}$ is the unique relation in the Wirtinger presentation of the knot, so that $W_{p/q} = \mathrm{id}$ represents the knot group.

By results of Thurston, $\mathbb{H}^3/G_{p/q}$ has a hyperbolic structure, unless the knot or link is a torus knot. This happens when $p/q = 1/n$ or $(n-1)/n$, $n \in \mathbb{N}$.



If $p/q = 1/n$, then $\mathcal{P}_{1/n}$ extends to the real axis. It ends at a Fuchsian group G which has an extra 'accidental symmetry'. G is a Hecke group. The quotient surface Σ_n a sphere with one puncture and two cone points of order n and 2. The knot complement fibres over the circle and is a Seifert fibre space over Σ_n.

Completing the picture

Theorem 1. (Announced and sketched by Agol 2002; recently completed by Akiyoshi, Ohshika, Parker, Sakuma & Yoshida.) Every non-free discrete group with two parabolic generators is either a two bridge knot or link complement, a Heckoid group, or a Hecke group.

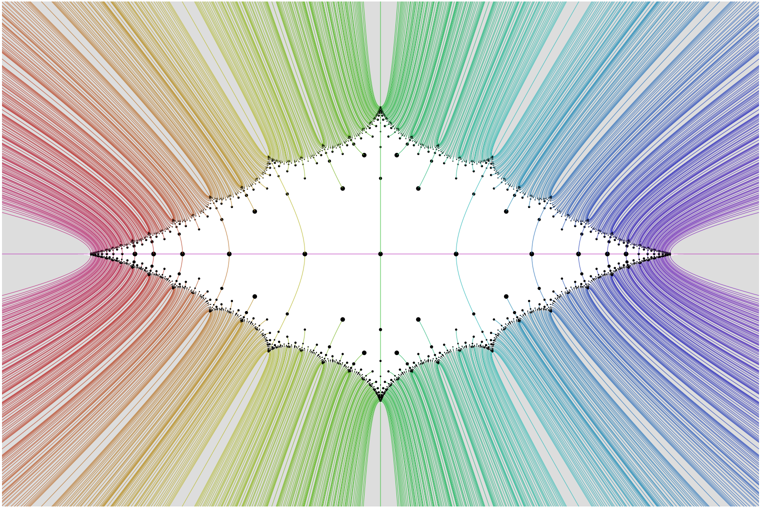
Theorem 2. (In progress). Akiyoshi, Sakuma, Yamashita, Wada.
The group corresponding to the relevant parameter on an extended ray is indeed either a Heckoid group or a knot complement as discussed.

Idea: Examine the patterns of isometric circles for specified c on the extended ray and show they give a fundamental domain for G_c.

Current project (S.) Find a proof of Theorem 2 by using the theory of deformations of cone manifolds and studying the degenerations which can occur.

Corollary of 1 & 2 All non-free discrete groups in the Riley slice lie on extended rays.

Conjecture A similar result is true in many other parameter spaces.



Thank you