MA 474

THE UNIVERSITY OF WARWICK

FOURTH YEAR EXAMINATION: JUNE 2002

REPRESENTATION THEORY

Time Allowed: 3 hours

Read carefully the instructions on the answer book and make sure that the particulars required are entered.

ANSWER 4 QUESTIONS.

If you have answered more than the required 4 questions in this examination, you will only be given credit for your 4 best answers.

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1. a) Give the definition of a division algebra over a field K. **[3]**

b) Show that if K is an algebraically closed field then the only division algebra over K is K itself. **[9]**

c) Show that the quaternions are a division algebra over the field of real numbers. **[3]**

d) Let D be a division algebra over a field K and let n be a positive integer. Show that the algebra of $n \times n$ matrices with entries in D is a simple algebra. **[5]**

e) Let A be the algebra of 2×2 matrices with entries in the quaternions. For each elementary 2×2 matrix find the elements of A which commute with the elementary matrix. Hence find the centre of the algebra A. **[5]**

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2. Let A be a finite dimensional algebra and let J be the intersection of the maximal right ideals. Using Nakayama's lemma, or otherwise;

a) Prove that J is a nilpotent ideal. **[9]**

b) Prove that a minimal right ideal is either projective or else is a submodule of J. **[8]**

Let A be the algebra over the real numbers, $\mathbb{R}$, defined by

$$A = \mathbb{R}[x]/ < (x^2 + 4)^3 >$$

c) Find the dimension of A. **[2]**

d) Find the irreducible representations, and the radical, of the algebra A. **[6]**

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3. Let A be a finite dimensional algebra.

a) Define a projective envelope of an A-module M. **[5]**

b) Show that any two projective envelopes of M are isomorphic. **[5]**

c) Show that every finite dimensional A-module has a projective envelope. **[15]**

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4. Let A be a finite dimensional algebra.

a) Define the Cartan matrix of A. **[5]**

b) Let A be the algebra over $\mathbb{Q}$ with basis $1, u, v, uv, vu$ and multiplication determined by

$$\begin{aligned} uu &= u & vv &= v \\ uvu &= u & vuv &= v \end{aligned}$$

(i) Find the irreducible representations of A. **[6]**

(ii) Find a non-trivial central idempotent in A. **[6]**

(iii) Find the Cartan matrix of A. **[8]**

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5. Let A be a finite dimensional K-algebra and M a module over A.

a) Explain what it means to say that M is finitely generated. **[4]**

b) Show that M is finite dimensional if and only if M is finitely generated. **[4]**

c) State and prove Fitting's lemma for finite dimensional modules over A. **[5]**

d) State the Krull-Schmidt theorem. **[4]**

e) Give an example of an algebra A, a finitely generated module M, and two inequivalent decompositions of M into indecomposable modules. **[8]**
