PDE for Finance, Solutions to Sheet 4.

1) a) u is the price of an option on an underlying performing Brownian motion, which pays 1 if at maturity, the underlying was neither below 0 nor above 1, and 0 otherwise.

b) By the formula from the lecture,

$$u(x,t) = \sum_{k=1}^{\infty} a_k(t) \sin(k\pi x) \quad \text{with}$$

$$a_k(t) = 2\int_0^1 1\cdot \sin(k\pi y)\,dy\; e^{-k^2\pi^2 t}$$

$$= \frac{2}{k\pi}\Big[-\cos(k\pi y)\Big]_{y=0}^{y=1} e^{-k^2\pi^2 t}$$

$$= \begin{cases} 0 & \text{if } k \text{ is even} \\ \frac{4}{k\pi}\exp(-k^2\pi^2 t) & \text{if } k \text{ is odd.}\end{cases}$$

So, $$u(x,t) = \sum_{k=1}^{\infty} \frac{4}{(2k-1)\pi} \exp(-(2k-1)^2\pi^2 t)\, \sin((2k-1)\pi x)$$

c) By the maximum principle, $0 \le u(x,t) \le 1$.

So 1% accuracy means we have to find $N \in \mathbb{N}$ such that

$$\sum_{k=N}^{\infty} \frac{4}{(2k-1)\pi} \exp\left(-(2k-1)^2\pi^2 \frac{1}{100}\right) \sin((2k-1)\pi x) < \frac{1}{100}$$

for all $x \in [0,1]$. Since the sin takes the value 1 somewhere, we will need

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$$\sum_{k=N}^{\infty} \underbrace{\frac{4}{(2k-1)\pi} \exp\left(-(2k-1)^2\pi^2/100\right)}_{=: a_k} < \frac{1}{100}.$$

Now

$a_1 \approx 1.15$, $a_2 \approx 0.17$, $a_3 \approx 0.02$, $a_4 \approx 0.001$

Also, for $k > 4$, $\frac{a_k}{a_4} \leq \exp\left(\left[-(2k-1)^2 + (2\cdot 4-1)^2\right]\pi^2/100\right)$

$$= \exp\left((7-2k+1) \underbrace{(7+2k-1)}_{\geq 16} \underbrace{\pi^2/100}_{> 1/16} \right)$$

$$\leq \exp\left((8-2k)\right)$$

$$\Rightarrow \sum_{k=5}^{\infty} a_k \leq a_4 \sum_{k=5}^{\infty} e^{8-2k} = a_4 \sum_{k=0}^{\infty} e^{-2-2k}$$

$$= \frac{a_4}{e^2} \frac{1}{1-e^{-2}} = \frac{a_4}{e^2-1} < a_4$$

$$\Rightarrow \sum_{k=4}^{\infty} a_k < 2a_4 \approx 0.002$$

We need only 3 terms even for such small t!

2) a) $\partial_t \left[u(cx, c^2t)\right] = c^2(\partial_t u)(cx, c^2t)$

$\partial_{x_j}^2 \left[u(cx, c^2t)\right] = (c^2 \partial_x^2 u)(cx, c^2t)$ $\Big\} \Rightarrow \partial_t u = "\tfrac{1}{2}" \Delta u$

b) $u(x,t) = U\left(\frac{|x|^2}{t}\right)$; $\partial_{x_j} u(x,t) = U'\left(\frac{|x|^2}{t}\right)\frac{2x_j}{t}$;

$\partial_{x_j}^2 u(x,t) = U'\left(\frac{|x|^2}{t}\right)\frac{2}{t} + U''\left(\frac{|x|^2}{t}\right)\frac{4x_j^2}{t^2}$; $\partial_t u(x,t) = -U'\left(\frac{|x|^2}{t}\right)\frac{|x|^2}{t^2}$

$$\Rightarrow \partial_t u - \Delta u = -U'\left(\frac{|x|^2}{t}\right)\frac{|x|^2}{t^2} - U'\left(\frac{|x|^2}{t}\right)\frac{2n}{t} - U''\left(\frac{|x|^2}{t}\right)\frac{4|x|^2}{t^2}.$$

So, $\partial_t u - \Delta u = 0 \Leftrightarrow 4z\,U''(z) + (2n+z)\,U'(z) = 0$ $\quad (*)$

with $z = \frac{|x|^2}{t}$.

c) $(*)$ means $\underbrace{\frac{U''(z)}{U'(z)}}_{=(\ln U')'} = -\frac{n}{2z} - \frac{1}{4}$.

$\Rightarrow \ln U'(z) = -\frac{n}{2}\ln z - \frac{1}{4}z + c$

$\Rightarrow U'(z) = z^{-n/2} e^{-z/4} e^{c} \Rightarrow U(z) = \underbrace{e^{c}}_{\equiv a}\int_c^z s^{-n/2} e^{-s/4}\,ds + b$

d) $\partial_x U(x^2/t)$ is still a solution by interchanging derivatives:

$\partial_t (\partial_x U(\frac{x^2}{t})) - \partial_x^2 (\partial_x U(\frac{x^2}{t})) = \partial_x (\partial_t U(\frac{x^2}{t}) - \partial_x^2 U(\frac{x^2}{t})) = \partial_x 0 = 0.$

Now $\underset{(n=1)}{\partial_x U(\frac{x^2}{t})} = a\frac{2x}{t} \cdot U'(\frac{x^2}{t}) = a\frac{2x}{t}\left(\frac{t}{x^2}\right)^{\frac{1}{2}} e^{-\frac{x^2}{4t}} = a\frac{2}{\sqrt{t}} e^{-\frac{x^2}{4t}}.$

For $a = \frac{1}{4\sqrt{\pi}}$ this gives the fundamental solution.

3) a) $\partial_t E(t) = \int_D \partial_t u(x,t)\,dx = \int_D 2u(x,t)\,\partial_t u(x,t)\,dx =$

$= 2\int_D u(x,t)\,\Delta u(x,t)\,dx = -\int_D |\nabla u|^2 dx + 2\underbrace{\int_{\partial D} u\,(\nabla u\cdot\vec{n})\,dS(x)}_{=0 \text{ since } u=0 \text{ on } \partial D}$

≤ 0 since $\int_D |\nabla u|^2 dx \geq 0$.

b) Let u, v solve the heat eqn. with identical initial conditions.

Then $u(x,0) - v(x,0) =: w(x,0)$ solves it.

Now $E_w(t) = \int_0^t \underbrace{\partial_s E_w(s)}_{\leq 0} ds \leq 0$, but obviously

$$E_w(t) = \int w^2(x,t)dx \geq 0 \Rightarrow E_w(t) = 0$$

$$\Rightarrow \int w^2(x,t)dx = 0 \text{ and } w^2(x,t) \geq 0 \Rightarrow w^2(x,t) = 0. \quad \blacksquare$$

c) As in a),

$$\partial_t E(t) = 2\int_D u\, \partial_t u\, dx = \underbrace{2\int_D u\, \Delta u\, dx}_{\leq 0 \text{ by a)}} + 2\lambda \int u^2 dx \leq$$

$$\leq 2\lambda E(t).$$

$$\overset{\text{Gronwall}}{\Longrightarrow} \quad E(t) \leq E(0)\, e^{2\lambda t}.$$

Proof of Gronwall: $f'(x) \leq a f(x) \Leftrightarrow \underbrace{\frac{f'(x)}{f(x)}}_{=(\ln f(x))'} \leq a \Leftrightarrow \ln f(x) \leq ax$

$$\Leftrightarrow f(x) \leq e^{ax}.$$

d) Exact solution: $u(x,t) = \sin(\pi x)\, e^{(\lambda - \pi^2)t}$

$$\Rightarrow E(t) = e^{(\lambda-\pi^2)t} \int_0^1 \sin^2(\pi x)\, dx = \frac{1}{2} e^{(\lambda-\pi^2)t}.$$

Estimate from c) : $E(t) \leq e^{2\lambda t}$

So the estimate is not very good, grows much faster than the real $E(t)$.