PDE in Finance Sheet 1

1) $u(x,t) = \mathbb{E}_{y_t = x}\left(e^{-\int_t^T b(y_s,s)\,ds}\,\Phi(y_T)\right).$

Since $u(x,T) = \Phi(x)$ for all x, we find

$u(y_T, T) = \Phi(y_T)$ almost surely (wrt the law of y_t)

So, $0 = \mathbb{E}_{y_t = x}\left(e^{-\int_t^T b(y_s,s)\,ds}\,\Phi(y_T)\right) - u(x,t) =$

$$= \mathbb{E}_{y_t = x}\left(e^{-\int_t^T b(y_s,s)\,ds}\,u(y_T,T) - u(y_t,t)\right)$$

$$= \mathbb{E}_{y_t = x}\left(\int_t^T d\left(e^{-\int_t^r b(y_s,s)\,ds}\,u(y_r,r)\right)dr\right)$$

$r \to \int_t^r b(y_s,s)\,ds$ is C^1, so

$$d\left(e^{-\int_t^r b(y_s,s)\,ds}\,u(y_r,r)\right) = e^{-\int_t^r b(y_s,s)\,ds}\Big(-b(y_r,r)\,u(y_r,r)\,dr +$$

$$+ du(y_r,r)\Big),$$ and

$$du(y_r,r) = \partial_y u(y_r,r)\,dy_r + \partial_t u(y_r,r)\,dr + \tfrac{1}{2}\partial_y^2 u(y_r,r)\,(dy_r)^2$$

$$= \partial_y u(y_r,r)\left(F(y_r,r)\,dr + G(y_r,r)\,dW_r\right) + \partial_t u(y_r,r)\,dr +$$

$$+ \tfrac{1}{2}\partial_y^2 u(y_r,r)\,G^2(y_r,r)\,dr$$

$$= \left(\partial_y u(y_r,r)\,F(y_r,r) + \partial_t u(y_r,r) + \tfrac{1}{2}\partial_y^2 u(y_r,r)\,G^2(y_r,r)\right)dr$$

$$+ \partial_y u(y_r,r)\,G(y_r,r)\,dW_r$$

The second term vanishes when integrating wrt r and taking expectation.

The dr-terms must be zero, so the PDE for u is

$$\partial_t u + F \partial_x u + \frac{1}{2} G^2 \partial_x^2 u - bu = 0.$$

$\boxed{2}$
$$u(x,t) = \mathbb{E}_{Y_t = x}\left(\int_t^T \psi(Y_s,s)\,ds + \Phi(Y_T) \right)$$

As above, $u(Y_T, T) = \Phi(Y_T)$, and $\mathbb{E}_{Y_t=x}\left(u(Y_t,t)\right) = u(x,t)$,

so
$$0 = \mathbb{E}_{Y_t=x}\left(\int_t^T \psi(Y_s,s)\,ds + u(Y_T,T) - u(Y_t,t) \right)$$
$$= \mathbb{E}_{Y_t=x}\left(\int_t^T \psi(Y_s,s)\,ds + \int_t^T du(Y_s,s) \right) \qquad (*)$$

Above we found

$$du(Y_s,s) = \left(\partial_y u(Y_s,s)\, F(Y_s,s) + \partial_t u(Y_s,s) + \frac{1}{2}\partial_y^2 u(Y_s,s)\, G^2(Y_s,s) \right) ds$$
$$+ \partial_y u(Y_s,s)\, G(Y_s,s)\, dW_s.$$

All of the ds-terms in $(*)$ must vanish, leading to the PDE

$$\partial_t u + F \partial_y u + \frac{1}{2} G^2 \partial_y^2 u + \psi = 0. \qquad (**)$$

For the knockout option, we have in addition the condition that

for $y=a$ or $y=b$,
$$u(y,t) = \mathbb{E}_{Y_t=y}\left(\underbrace{\int_t^{J(y)=t} \psi(Y_s,s)\,ds}_{=0} + \Phi(Y_{J(y)}) \right), \quad J(y) = t$$
$$= \Phi(y).$$

Only a,b matter for Φ!

This gives the boundary conditions $u(a,t) = \Phi(a)$, $u(b,t) = \Phi(b)$.

3) a) $u(x) = \mathbb{E}_{Y_0 = x}(J(x))$ is achieved by setting $\Psi = 1$ and $\Phi = 0$ in 2).

This gives $\partial_t u + F\partial_x u + \frac{1}{2} G^2 \partial_x^2 u = -1$, with boundary conditions $u(a) = u(b) = 0$. Since u is independent of t, also $\partial_t u = 0$, and we are left with

$$F\partial_x u + \frac{1}{2} G^2 \partial_x^2 u = -1 \quad , \quad u(a) = u(b) = 0.$$

b) Now, $F = \mu x$ and $G = \sigma x$, leading to

$$\mu x \partial_x u + \frac{1}{2}\sigma^2 x^2 \partial_x^2 u = -1, \quad u(a) = u(b) = 0$$

With $u(x) = \frac{1}{\sigma^2/2 - \mu} \ln x + c_1 + c_2 x^\gamma \quad \left(\gamma = 1 - \frac{2\mu}{\sigma^2}\right)$,

we have $\partial_x u = \frac{1}{\sigma^2/2 - \mu} \frac{1}{x} + \gamma c_2 x^{\gamma - 1}$

$$\partial_x^2 u = \frac{-1}{\sigma^2/2 - \mu} \frac{1}{x^2} + \gamma(\gamma - 1) c_2 x^{\gamma - 2},$$

and $\mu x \partial_x u + \frac{1}{2}\sigma^2 x^2 \partial_x^2 u = \frac{\mu}{\sigma^2/2 - \mu} - \frac{\sigma^2/2}{\sigma^2/2 - \mu} + c_2 \gamma \mu x^\gamma + c_2 \gamma(\gamma - 1) \frac{\sigma^2}{2} x^\gamma$

$$= -1 + c_2 \gamma x^\gamma \underbrace{\left(\mu + \frac{\sigma^2}{2}(\gamma - 1)\right)}_{=0} = -1.$$

c) $u(a) = 0$ needs $\frac{1}{\sigma^2/2 - \mu} \ln a + c_1 + c_2 a^\gamma = 0 \quad (1)$

$u(b) = 0$ needs $\frac{1}{\sigma^2/2 - \mu} \ln b + c_1 + c_2 b^\gamma = 0 \quad (2)$

(1) - (2) gives $\frac{1}{\sigma^2/2 - \mu}(\ln a - \ln b) = -c_2(a^\gamma - b^\gamma)$, determining c_2.

Plugging this into (1) or (2) determines c_1.