PDE for finance Sheet 0

$\boxed{1}$ a) $\left(e^{-x^2}\right)' = -2x\, e^{-x^2}$

b) $\left(a^x\right)' = \left(e^{x \ln a}\right)' = \ln a \; e^{x \ln a} = \ln a \; a^x$

c) $\left(x^{(x^2)}\right)' = \left(e^{x \ln x^2}\right)' = \left(e^{2x \ln x}\right)' = \left(2 \ln x + 2x \cdot \frac{1}{x}\right) e^{2x \ln x}$

$= (2 \ln x + 2)\, x^{(x^2)}$

d) $\left(\frac{1}{1+x^2}\right)' = \frac{-2x}{(1+x^2)^2}$

b) $\frac{d}{dt} F(f(t), g(t)) = f'(t)\, \partial_1 F(f(t), g(t)) + g'(t)\, \partial_2 F(f(t), g(t))$.

With $h(t) = \int_0^t \phi(s,t)\, ds \equiv F(t,t)$ (where $F(x,y) = \int_0^x \phi(s,y)\, ds$),

this gives $\frac{d}{dt} h(t) = \phi(t,t) + \int_0^t \partial_2 \phi(s,t)\, ds$, and with

$\phi(s,t) = e^{-st} \cos(s)$, this is $h'(t) = e^{-t^2} \cos(t) - \int_0^t s\, e^{-st} \cos(s)\, ds$.

With $F(x,y) = \int_0^x ds \int_0^y ds'\, \phi(s-s')$, we get

$$\frac{d}{dt} \int_0^t ds \int_0^t ds'\, \phi(s-s') = \int_0^t \phi(t-s')\, ds' + \int_0^t \phi(s-t)\, ds =$$

$$= \int_0^t \phi(r)\, dr + \int_{-t}^0 \phi(r)\, dr = \int_{-t}^t \phi(s)\, ds .$$

$r = t - s' \Rightarrow$
$ds' = -dr$
$s' = 0 \Rightarrow r = t$
$s' = t \Rightarrow r = 0$

$u = s - t$
$\Rightarrow ds = du$
$s = 0 \Rightarrow u = -t$
$s = t \Rightarrow u = 0$

c) $\gamma : \mathbb{R} \to \mathbb{R}^n$ with $\gamma(0) = \vec{x} \in \mathbb{R}^n$. Slope of f

along γ is $\frac{d}{dt} f(\gamma(t)) = \sum_{j=1}^n \partial_j f(\gamma(t))\, \frac{d}{dt} \gamma_j(t) =$

$= \nabla f(\gamma(t)) \cdot \gamma'(t)$. So, $f(\gamma(t)) \equiv \text{const} \Leftrightarrow \nabla f(\gamma(t)) \cdot \gamma'(t) = 0$.

d) $\partial_x f = -\frac{x}{t} f \; ; \; \partial_x^2 f = \left(-\frac{1}{t} + \frac{x^2}{t^2}\right) f.$

$$\Rightarrow \underbrace{\partial_x^2 f + \partial_y^2 f}_{=\Delta f.} = \left(-\frac{2}{t} + \frac{|\vec{x}|^2}{t^2}\right) f. \qquad (|\vec{x}|^2 = x^2 + y^2$$

Now $\frac{d}{dt} h(t) f(x,t) = \left(h'(t) + h(t)\frac{|\vec{x}|^2}{2t^2}\right) f(x,t)$

For this to be $= \frac{1}{2} \Delta_{\vec{x}} \left[h(t) f(x,t)\right]$, we need

$$h'(t) + h(t)\frac{|x|^2}{2t^2} = \frac{1}{2}\left(-\frac{2}{t} + \frac{|x|^2}{t^2}\right) h(t)$$

$\Rightarrow h'(t) = \frac{2}{t} h(t)$, so $h(t) = t^{-\frac{1}{2}}$ does the job!

e) Easy solutions: $f(x,y) = xy$, or $f(xy) = x+y$ etc...

For $f(x) = \frac{1}{|x|}$ we have

$$\partial_{x_i} f(\vec{x}) = \partial_{x_i} \frac{1}{\sqrt{x_1^2 + \ldots + x_n^2}} = \frac{-1}{2\sqrt{x_1^2+\ldots+x_n^2}^3} \cdot 2x_i = -\frac{x_i}{|x|^3}.$$

$$\partial_{x_i}^2 f(\vec{x}) = -\frac{1}{|x|^3} + \frac{3}{2} \frac{2x_i}{(x_1^2 + \ldots + x_n^2)^{5/2}} = -\frac{1}{|x|^3} + 3\frac{x_i^2}{|x|^5}$$

$$\Rightarrow \Delta f(\vec{x}) = -n\frac{1}{|x|^3} + 3\frac{x_1^2 + \ldots + x_n^2}{|x|^5} = (-n+3)\frac{1}{|x|^3}$$

This is $= 0$ for $n = 3$ only.

$\boxed{2}$ a) $dh(y_t) = h'(y_t)\,dy_t + \frac{1}{2} h''(y_t)(dy_t)^2 =$

$$= e^{y_t}\left(F(y_t)dt + G(y_t)\,dW_t\right) + \frac{1}{2} e^{y_t} G^2(y_t)\,dt$$

$$= \underbrace{e^{y_t}}_{=h_t} \underbrace{\left(F(y_t)dt + \frac{1}{2}G^2(y_t)\right)}_{\overset{!}{=}\mu \text{ for gBM}} dt + \underbrace{e^{y_t}}_{=h_t} \underbrace{G(y_t)}_{\overset{!}{=}\sigma \text{ for gBM}} dW_t.$$

$\Rightarrow \; G = \sigma$ and $F = \mu - \frac{1}{2}\sigma^2$.

b) $dy_t^{(i)} = \sigma \, dW_t^{(i)}$

$$\Rightarrow dh(\vec{y}_t) = \sum_{i=1}^{n} \partial_{x_i} h(\vec{y}_t)\, dy_t^{(i)} + \frac{1}{2} \sum_{i,j=1}^{n} \partial_{x_i}\partial_{x_j} h(y_t) \underbrace{dy_t^{(i)}\, dy_t^{(j)}}_{\substack{= dt \text{ if } i=j \\ 0 \text{ otherwise}}}$$

$$= \sum_{i=1}^{n} \partial_{x_i} h(\vec{y}_t)\, dy_t^{(i)} + \frac{1}{2} \underbrace{\sum_{i=1}^{n} \partial_{x_i}^2 h(y_t)}_{=(\Delta h)(y_t)}$$

So h needs to be harmonic!

c) $$dh(y_t) = h'(y_t)\, dy_t + \frac{1}{2} h''(y_t) (dy_t)^2 =$$

$$= -\frac{1}{y_t^2} \left(\mu y_t \, dt + \sigma y_t \, dW_t \right) + \frac{1}{2} \frac{2}{y_t^3} \sigma^2 y_t^2 \, dt$$

$$= \left(-\mu \frac{1}{y_t} + \sigma^2 \frac{1}{y_t} \right) dt - \frac{\sigma}{y_t} dW_t$$

$$= (\sigma^2 - \mu)\, h_t \, dt - \sigma h_t \, dW_t$$

$\Rightarrow$ Again gBM with drift $\sigma^2 - \mu$, diffusion σ.