MA908 Partial Differential Equations in Finance

Exercise Sheet 2: Some basic PDE theory

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1. **Fundamental solution of the Laplace equation**
Check that the function $F : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ with

$$F(x) = \begin{cases} -\frac{1}{2\pi} \ln |x| & \text{if } n = 2, \\ \frac{1}{n(n-2)\alpha(n)} \frac{1}{|x|^{n-2}} & \text{if } n \geqslant 3 \end{cases}$$

solves the Laplace equation $\Delta F = 0$ on its domain of definition.

2. **Running payoff without time constraints**
Assume that a group of assets, modelled at time s by $\boldsymbol{y}_s \in \mathbb{R}^n$, solves the stochastic differential equation

$$\mathrm{d}y_s^{(i)} = f_i(\boldsymbol{y}_s, s)\,\mathrm{d}s + \sum_{i,j=1}^{n} g_{ij}(\boldsymbol{y}_s, s)\,\mathrm{d}\mathcal{W}_s^{(j)}.$$

Let $D \subset \mathbb{R}^n$ be a bounded open set. Assume that we get an (infinitesimal) payout of $\Psi(\boldsymbol{x}, t)\,\mathrm{d}t$ for each moment that the asset vales are $\boldsymbol{x}$, and a final payout $\Phi(\boldsymbol{x})$ at the moment that the asset value process hits the boundary ∂D. Derive a PDE for the expected payout given that we start at $\boldsymbol{x}$, i.e. for

$$u(\boldsymbol{x}) = \mathbb{E}_{\boldsymbol{y}_0 = \boldsymbol{x}} \left(\int_0^{\tau(\boldsymbol{x})} \Psi(\boldsymbol{y}_s)\,\mathrm{d}s + \Phi(\boldsymbol{y}_{\tau(x)}) \right).$$

Here $\tau(\boldsymbol{x})$ is the first time that $\boldsymbol{y}_s$ leaves D.

3. **A scaling relation**
Sometimes it is possible to tell some properties of the solution of a PDE, even though one cannot obtain it explicitly. As an example, show that any solution $u : \mathbb{R}^n \to \mathbb{R}$ of the PDE

$$x \cdot \nabla u(x) = \alpha u(x)$$

(with $\alpha > 0$) satisfies, for all $\lambda > 0$,

$$u(\lambda x) = \lambda^\alpha u(x).$$

Hint: differentiate with respect to λ.

4. (a) Solve the following initial value problem

$$\text{(I1)} \quad \begin{cases} \partial_x u + \partial_y u = u \\ \quad u(x, 0) = f(x)\,, \quad f \in C^1(\mathbb{R})\,,\ u \in C^1(\mathbb{R}^2)\,. \end{cases}$$

(b) Show that, and explain why, the problem

$$\text{(I2)} \quad \begin{cases} \partial_x u + \partial_y u = u \\ \quad u(x, x) = 1\,, \end{cases}$$

has no solutions.
`Hint:` The solution obtained in part (a) can be useful.

(c) Show that, and explain why, the problem

$$\text{(I3)} \quad \begin{cases} \partial_x u + \partial_y u = u \\ \quad u(x, x) = 0\,, \end{cases}$$

has infinitely many solution.