MA908L Partial Differential Equations in Finance

Exercise Sheet 1: deriving PDE from SDE

Volker Betz January, 2012.

For problems 1) and 2), assume that the value y_t of an asset is given by

$$\mathrm{d}y_t = F(y_t, t)\,\mathrm{d}t + G(y_t, t)\mathrm{d}\mathcal{W}_t,$$

where $\mathcal{W}_t$ is standard Brownian motion. Let $\Phi : \mathbb{R} \to \mathbb{R}$ be a payoff function, and assume that $F, G, and \Phi$ are at least twice continuously differentiable.

1. **Payoff and discounting:**
   Derive the PDE satisfied by

$$u(x,t) = \mathbb{E}_{y_t=x}\left(\mathrm{e}^{-\int_t^T b(y_s,s)\,\mathrm{d}s}\,\Phi(y_T)\right),$$

   where $b : \mathbb{R}^2 \to \mathbb{R}$ is a discounting function.

2. **Running payoff:**
   Let $\Psi : \mathbb{R}^2 \to \mathbb{R}$ be a *running payoff* function, i.e. the (infinitesimal) payoff for an asset of value y_t at time t is given by $\Psi(y_t, t)\,\mathrm{d}t$. Derive the PDE satisfied by

$$u(x,t) = \mathbb{E}_{y_t=x}\left(\int_t^T \Psi(y_s, s)\,\mathrm{d}s + \Phi(y_T)\right).$$

   Then, derive the boundary value problem satisfied by the knockout option with running payoff, i.e.

$$u(x,t) = \mathbb{E}_{y_t=x}\left(\int_t^{\tau(x)} \Psi(y_s, s)\,\mathrm{d}s + \Phi(y_{\tau(x)})\right),$$

   where $\tau(x)$ is the first time that y_t leaves the open interval (a, b) (lifetime of the option). Which values of Φ do we need to specify for the problem to be uniquely determined?

3. **Expected time to leave an interval**

   (a) Use the results of the previous section to give an ordinary differential equation satisfied by the expected time that y_t takes to leave the open interval (a, b). In other words, give an ODE for the quantity

$$u(x) = \mathbb{E}_{y_0=x}\left(\tau(x)\right).$$

   (b) Specializing the results of (a) to the case of geometric Brownian motion,

$$\mathrm{d}y_t = \mu y_t\,\mathrm{d}t + \sigma y_t\,\mathrm{d}\mathcal{W}_t,$$

   should give you that $u(x)$ solves

$$\mu x \partial_x u + \frac{1}{2}\sigma^2 x^2 \partial_x^2 u = -1 \qquad \text{for } a < x < b,$$

   with boundary condition $u(a) = u(b) = 0$. (Check this!)
   For $\mu \neq \frac{1}{2}\sigma^2$, show that a solution to the above equation without taking into account the boundary conditions is given by

$$u(x) = \frac{1}{\sigma^2/2 - \mu}\ln x + c_1 + c_2 x^\gamma,$$

   with $\gamma = 1 - 2\mu/\sigma^2$, and c_1 and c_2 arbitrary constants.

   (c) Now use the boundary conditions to find the solution to the boundary value problem.