2. Some linear PDE theory

Let us review what the basic problem that we are treating in PDE theory is: we are looking for a function f, defined on an open subset $U \subset \mathbb{R}^n$, such that at each point $x \in U$, a certain combination of partial derivatives and values of the function itself gives a certain value. On the boundary ∂U, we may or may not prescribe values for the function or its derivatives. A (random) example would be

$$D = B_1 := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leqslant 1\},$$

and

$$x\partial_x^2 f(x, y) + y\partial_y^2 f(x, y) = f(x, y)^2 \quad \text{on } D, \qquad f(x, y) = xy \quad \text{on } \partial D.$$

PDE do not always have a solution, and the solution may not always be unique. Even among those that do have a unique solution, there are precious few where it is possible to find the solution in the form of a closed formula. In this section, we will study some of these few cases.

2.1. **First order PDE.**

These PDE contain only first order derivatives. Recall the notation

$$\nabla f(x_1, \ldots x_n) = (\partial_{x_1} f(x_1, \ldots, x_n), \ldots, \partial_{x_n} f(x_1, \ldots, x_n))$$

for the gradient of a function, and the notation

$$\boldsymbol{b} \cdot \nabla f = b_1 \partial_{x_1} f + \ldots + b_n \partial_{x_n} f$$

for the scalar product with a vector $\boldsymbol{b} = (b_1, \ldots, b_n)$.

The transport equation. This is the simplest type of PDE there is.
Definition: Consider $D \subset \mathbb{R}^n \times \mathbb{R}$. We say that $u : D \to \mathbb{R}$ solves a *transport equation* (with constant coefficients) if

$$\partial_t u + \boldsymbol{b} \cdot \nabla_{\boldsymbol{x}} u = 0. \tag{2.1}$$

Above, we have $u = u(\boldsymbol{x}, t) = u(x_1, \ldots, x_n, t)$.
Can we solve this equation for u? Yes, and it is easy. The key is to see that although we have separated 'time' t and 'space' $\boldsymbol{x}$, what we really have is a gradient of a function of $n+1$ variables being perpendicular to a certain vector. More explicitly, define $\boldsymbol{c} = (b_1, \ldots, b_n, 1)$. Then (2.1) becomes

$$\boldsymbol{c} \cdot \nabla_{(\boldsymbol{x},t)} u = 0.$$

So, u is constant in the direction $(b_1, \ldots, b_n, 1)$. In other words,

$$u(\boldsymbol{x} + s\boldsymbol{b}, t + s) = u(\boldsymbol{x}, t) \tag{2.2}$$

for all $\boldsymbol{x} \in \mathbb{R}^n$ and all $s, t \in \mathbb{R}$. Whenever we know u on one point of such a line, we know it on the whole line. To know it at one point, we need the boundary condition.
Even in this simple example, we can see clearly that both existence and uniqueness can fail easily. We have established that the solution is constant on straight lines. Now, if D is e.g. a ball, then each straight line will intersect its boundary twice.

So, when prescribing values for u on ∂D, we have to be very careful in this case, otherwise we will prescribe two different values on the same straight line, and the PDE has no solution. On the other hand, we can also lose uniqueness. Before we see this, let us look at a case where we do have existence and uniqueness. This is the full space initial value problem, where (2.1) is on $D = \mathbb{R}^n \times (0, \infty)$, and $u(\boldsymbol{x}, 0) = h(\boldsymbol{x})$. This is natural, as we claim that we know u to be h at time $t = 0$ and want to see how it evolves in time. Now (2.2) gives us

$$u(\boldsymbol{x} + s\boldsymbol{b}, s) = u(\boldsymbol{x}, 0) = h(\boldsymbol{x}),$$

and so by putting $\boldsymbol{y} = \boldsymbol{x} + s\boldsymbol{b}$ we find

$$u(\boldsymbol{y}, s) = h(\boldsymbol{y} - s\boldsymbol{b}).$$

Note that the values of h are indeed 'transported' along the straight lines. Non-uniqueness now occurs if we look at more general (maybe less natural) boundary conditions. We could e.g. study (2.1) on the half space $D = \{(\boldsymbol{x}, t) : t + \boldsymbol{a} \cdot \boldsymbol{x} \geqslant 0\}$. This will still give a unique solution, except when $(\boldsymbol{b}, 1)$ is parallel to the boundary of D. Then, we will have no solution unless the boundary condition h is a constant, and infinitely many solutions if h is indeed constant, as the values along all other straight lines will not have been prescribed.
To summarize, in the very simple case of the transport equation we have found the solution by finding a coordinate direction (namely $(\boldsymbol{b}, 1)$) in which the solution is constant. Let us try this strategy for more complicated first order PDE.

Linear first order PDE. These are of the same shape as the transport equation, but now the coefficients $\boldsymbol{b}$ may depend on $\boldsymbol{x}$ and t, and we also allow a term proportional to u to appear.
Definition: Consider $D \subset \mathbb{R} \times \mathbb{R}^n$. We say that $u : D \to \mathbb{R}$ solves a *linear first order PDE* if

$$\boldsymbol{b}(\boldsymbol{x}, t) \cdot \nabla_{\boldsymbol{x},t} u(\boldsymbol{x}, t) + c(\boldsymbol{x}, t) u(\boldsymbol{x}, t) = 0 \tag{2.3}$$

on D.
Like in the transport equation, we look for a coordinate direction in which the solution is easy. Here, $\boldsymbol{b}(\boldsymbol{x}, t)$ seems promising. But note that now the 'easy' direction depends on where we are in $(\boldsymbol{x}, t)$-space! This means that we are dealing with an 'easy curve' instead of an 'easy straight line'. More precisely: Let $\gamma : \mathbb{R} \to \mathbb{R}^{n+1}$ be a curve in coordinate space. We write $\gamma(s) = (\boldsymbol{x}(s), t(s))$. Assume that γ is parallel to the vector $\boldsymbol{b}(\boldsymbol{x}, t)$ at every point $(\boldsymbol{x}, t)$ through which γ passes. In symbols, assume that

$$\dot{\gamma}(t) \equiv \frac{\mathrm{d}}{\mathrm{d}t} \gamma(t) = \boldsymbol{b}(\gamma(s)) \equiv \boldsymbol{b}(\boldsymbol{x}(x), t(s)). \tag{2.4}$$

The $\equiv$ signs mean (as always) that the two terms are the same by definition, i.e. there are just two different ways of writing them.
Let us now define $v(s) = u(\gamma(s)) \equiv u(\boldsymbol{x}(s), t(s))$. So, v is just what you get when you evaluate u along γ. The interesting point is that if u solves (1.3), then by the

chain rule,

$$\frac{\mathrm{d}}{\mathrm{d}s} v(s) = \boldsymbol{b}(\gamma(s)) \cdot \nabla_{\boldsymbol{x},t} u(\gamma(s)) = -c(\gamma(s)) u(\gamma(s)) = -c(\gamma(s)) v(s).$$

This is now an ordinary differential equation for v, which is easily solved: you can check that

$$v(s) = v(0)\, \mathrm{e}^{-\int_0^s c(\gamma(r))\,\mathrm{d}r}. \tag{2.5}$$

Even though it might not seem so, we have in some sense already solved (2.1). Namely, if we need to know the solution of (2.1) at a point $(\boldsymbol{x}, t)$ in a domain D, we start a curve γ at $(\boldsymbol{x}, t)$ and make sure that it fulfils (2.4). This means we have to solve the corresponding system of ordinary differential equations, which may or may not pose a problem in itself. Assuming that this goes well, however, we then follow γ until it hits the boundary. We take the prescribed value at the boundary as $v(0)$, re-parametrize γ so that $\gamma(0)$ is on the boundary, and apply (2.5) to get the value of u along the full curve γ. In particular, since γ contains $(\boldsymbol{x}, t)$, we get the value at that point.

Of course, this is not a closed form solution. To get the latter, we need to be able to solve the ODE defining γ, and to invert the coordinates to get from the 'curve coordinates' $\gamma(s)$ to the space coordinates $(\boldsymbol{x}, t)$. This is in general hard (and has nothing whatsoever to do with PDE theory), but sometimes it can be done. Here is an example.

Example: $D = \{(x,t) : x > 0, t > 0\} \subset \mathbb{R}^2$. The initial condition is $u(x, 0) = g(x)$ for some function g, and the PDE is

$$x \partial_x u(x,t) - t \partial_x u(x,t) = u(x,t).$$

So in the framework above, we have $\boldsymbol{b}(x,t) = (-t, x)$. We seek γ with $\dot\gamma(s) = b(\gamma(s))$. Writing $\gamma(s) = (\gamma_1(s), \gamma_2(s))$, this means we have $\boldsymbol{b}(\gamma(s)) = (-\gamma_2(s), \gamma_1(s))$, so we want γ to fulfil

$$\dot\gamma_1(s) = -\gamma_2(s), \quad \dot\gamma_2(s) = \gamma_1(s).$$

The solution is given by

$$\gamma_1(s) = c\cos(s), \quad \gamma_2(s) = c\sin(s),$$

where c can be arbitrary and determines the points through which γ runs: namely, γ describes circles of radius c. Since in the context of (2.3), we have $c(x,t) = -1$, equation (2.5) now becomes

$$v(s) = v(0)\, \mathrm{e}^{-\int_0^s 1\,\mathrm{d}s} = v(0)\, \mathrm{e}^{-s}$$

Now we put it all together and invert the coordinates. If we want (x,t) to lie on γ, it means we want $(x,t) = (c\cos(s), c\sin(s))$. Resolving this for c and s, we find that $c = \sqrt{x^2 + t^2}$ (by squaring both components above, adding and taking the square root after using $\sin^2 + \cos^2 = 1$). Also $s = \arctan(t/x)$ (by dividing the components, and taking the arctan). We arrive at

$$u(x,t) = v(\arctan(t/x)) = g(\sqrt{x^2 + t^2})\, \mathrm{e}^{\arctan(t/x)}.$$

Characteristic curves. The method of seeking 'simple directions' can be extended to even more difficult first order PDE. In these cases, often the ODE for γ and for the solution along γ become coupled, and one has to solve the whole system at one go. While it is not conceptually much more difficult than what we had, it is considerably more messy, and we will not pursue it further.

2.2. **Laplace and Poisson equations.**

Motivation: Exit times from a domain. Recall the knock-out option example from the previous section. Here we study a similar problem, but with positive payout at the boundary. Consider the SDE

$$\mathrm{d}y_s^{(i)} = \mathrm{d}\mathcal{W}_i(s) \qquad \text{for } \boldsymbol{y}_s \in D, \qquad \boldsymbol{y}_0 = \boldsymbol{x} \in D,$$

with bounded $D \subset \mathbb{R}^n$, and the utility function

$$u(\boldsymbol{x}) = \mathbb{E}_{\boldsymbol{y}_0=\boldsymbol{x}}(\Phi(\boldsymbol{y}_{\tau(\boldsymbol{x})})),$$

where τ is the first time that $\boldsymbol{y}_t$ hits ∂D. Then in the same way as may times before, we find that u is the solution to the PDE

$$\Delta u \equiv \sum_{i=1}^{n} \partial_x^2 u = 0 \text{ on } D, \qquad u(x) = \Phi(x) \text{ on } \partial D. \tag{2.6}$$

This is the *Laplace equation.* If we include the running payoff f, i.e. if

$$u(\boldsymbol{x}) = \mathbb{E}_{\boldsymbol{y}_0=\boldsymbol{x}}\Big(\Phi(\boldsymbol{y}_{\tau(\boldsymbol{x})}) + \int_0^{\tau(\boldsymbol{x})} f(\boldsymbol{y}_s)\,\mathrm{d}s\Big),$$

then we obtain instead (by the familiar argument) that u solves the *Poisson equation*

$$\Delta u(\boldsymbol{x}) + f(\boldsymbol{x}) = 0 \text{ on } D, \qquad u(\boldsymbol{x}) = \Phi(\boldsymbol{x}) \text{ on } \partial D. \tag{2.7}$$

Now, how can we solve equations (2.6) and (2.7)? Let us start with the seemingly strange case $D = \mathbb{R}^n \setminus \{0\}$. Then it is possible (with some experience) to guess a solution. It is given by

$$F(x) = \begin{cases} -\dfrac{1}{2\pi} \ln |x| & \text{for } n = 2, \\ \dfrac{1}{n(n-2)\alpha(n)} \dfrac{1}{|x|^{n-2}} & \text{for } n \geqslant 3. \end{cases} \tag{2.8}$$

Above, $\alpha(n)$ is the volume of the unit ball, and all the x-independent prefactors are normalisations that will be useful later on. The function F in (2.8) may seem very special, but it is of great importance and is called the *fundamental solution.* We will verify as an exercise that indeed $\Delta F = 0$ on D. As a quick challenge, try to think what the situation would be in $d = 1$: What solutions to the Laplace equation are there?

The key to solving (2.7) is now to observe that not only F, but also the function $x \mapsto F(x-y)$ solves (2.6). This leads to the following

Theorem 2.1. *Let F be the fundamental solution to the Laplace equation, and assume that f in* (2.7) *is nice enough, e.g. twice continuously differentiable with compact support. Put*

$$u(x) = -\int F(x-y)f(y)\,\mathrm{d}y$$

Then u solves (2.7).

The proof goes via careful integration by parts, where it is important to pay special attention to the region of space where F diverges. Here, it is also important to use the precise normalisation for F, otherwise the theorem would not hold. The proof is given e.g. in Evans.
Solving (2.6) and (2.7) on a domain $D \subset \mathbb{R}^n$ subject to boundary conditions is more difficult, but there is a general recipe that can be followed. Here it is:
Step 1: Instead of solving (2.6) for the given boundary function Φ, solve it (in the variable y) for the special boundary function $y \mapsto F(y-x)$ for all x. I.e., solve

$$\Delta\phi^x(y) = 0 \text{ on } D, \qquad \phi^x(y) = F(y-x) \text{ on } \partial D. \tag{2.9}$$

This seems not much easier than the original problem, but for some nice D it actually is. However, this is in general the hard step.
Step 2: Define

$$G(x,y) = F(y-x) - \phi^x(y) \qquad \text{for } x \in D, x \neq y.$$

G is called *Greens function* of the Laplace equation for the domain D.
Step 3: The Poisson equation is now solved by

$$u(x) = \int_{\partial D} \Phi(y)\frac{\partial G}{\partial \nu}(x,y)\,\mathrm{d}S(y) - \int_D f(y)G(x,y)\mathrm{d}y,$$

where ν is the vector of length one that is perpendicular to ∂D at the point y and points outward, and $\mathrm{d}S$ is the surface measure on ∂D. This theorem is again proved by careful integration by parts.
Example: Greens function on the half space
Consider the half space $\mathbb{R}^n_+ = \{x = (x_1,\dots,x_n) : x_n > 0\}$. The task of step 1 above is to find the solution to (2.9) for all $x \in \mathbb{R}^n_+$. Note that $y \mapsto F(y-x)$ trivially fulfils the boundary condition part of (2.9), but not the PDE part since $\Delta_y F(y-x) \neq 0$ for $y = x$; indeed, the function is not even defined there. But F only depends on $|y-x|$, and if we could somehow force its singularity to lie outside of the half plane, we would be in business. These two things suggest that we may try $\phi^x(y) = F(y-\bar{x})$, where $\bar{x} = (x_1,\dots,x_{n-1},-x_n)$. This now solves (2.9). So, $G(x,y) = F(y-x) - F(y-\bar{x})$. The derivative $\frac{\partial G}{\partial \nu}$ is just the derivative in the direction of the n-th coordinate, so on ∂D,

$$\frac{\partial G}{\partial \nu}(x,y) = -\partial_{y_n} G(x,y) = -\frac{2x_n}{n\alpha(n)|x-y|^n}.$$

Here, we have used that on ∂D, $x_n = 0$. Then,

$$u(x) = \frac{2x_n}{n\alpha(n)} \int_{\{y \in \mathbb{R}^n : y_n = 0\}} \frac{\Phi(y)}{|x-y|^n} \, \mathrm{d}y.$$

solves the Laplace equation (2.6). The solution to the Poisson equation (2.7) follows from step 3 above in the same way.